On $\mathcal{D}$-equivalence class of complete bipartite graphs

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Abstract

Let G be a simple graph of order n. We mean by *dominating set*, a set $S \subseteq V(G)$ such that every vertex of G is either in S or adjacent to a vertex in S. The *domination polynomial* of G is the polynomial $\sum_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Two graphs G and H are said to be *$\mathcal{D}$-equivalent*, written $G \sim H$, if $D(G,x) = D(H,x)$. The *$\mathcal{D}$-equivalence class* of G is $[G] = \{ H \mid H \sim G \}$. Recently, the determination of $\mathcal{D}$-equivalence class of a given graph, has been of interest. In this paper, it is shown that for $n \geq 3$, $[K_{n,n}]$ has size two. We conjecture that the complete bipartite graph $K_{m,n}$ for $n - m \geq 2$, is uniquely determined by its domination polynomial.

Keywords: Domination polynomial, Dominating set.

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1 Introduction

Throughout this paper, G denotes a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of the vertices of G. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and the *degree* of v, denoted by $d(v)$ is $|N(v)|$. The *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. We denote $min\{d(v) \,|\, v \in V(G)\}$ by $\delta(G)$. We mean by *dominating set*, a set S of vertices of G that every vertex is either in S or adjacent to a vertex in S. An *i-dominating set* of G is a dominating set of size i and the number of i-dominating sets of G is denoted by $d(G,i)$. The cardinality of the minimum dominating set of G is denoted by $\gamma(G)$. A dominating set with cardinality $\gamma(G)$ is called a *γ-set* and the set of all γ-sets is denoted by $\Gamma(G)$. The *domination polynomial* of G, $D(G,x)$, is a polynomial of degree $|V(G)| = n$ in which the coefficient of x^i is $d(G,i)$ for each i, $1 \leq i \leq n$. For the further information about the properties of domination polynomials, see [3]. Two graphs G and H are said to be *dominating equivalent* or simply, *$\mathcal{D}$-equivalent*, written $G \sim H$, if $D(G,x) = D(H,x)$. It is evident that the relation $\sim$ of being $\mathcal{D}$-equivalent is an equivalence relation on the the family $\mathcal{G}$ of graphs and thus $\mathcal{G}$ is partitioned into equivalence classes, called the *$\mathcal{D}$-equivalence classes*. Given $G \in \mathcal{G}$, let $[G] = \{H \in \mathcal{G} \,|\, H \sim G\}$. We call $[G]$ the $\mathcal{D}$-equivalence class of G. A graph G is said to be *dominating unique*, simply *$\mathcal{D}$-unique*, if $[G] = \{G\}$. In [1], it is proved that all cycles are $\mathcal{D}$-unique. Also in [2], the $\mathcal{D}$-equivalence class of a path of order n, for $n \equiv 0 \,(mod\, 3)$, is completely determined. There are two interesting problems on $\mathcal{D}$-equivalence classes:

(i) Which graphs are $\mathcal{D}$-unique?

(ii) Determine the $\mathcal{D}$-equivalence class of some families of graphs.

As usual, for two natural numbers m and n, we denote the complete bipartite graph with part sizes m and n and the complete graph of order n by $K_{m,n}$ and K_n, respectively. The *cartesian product* of two graphs G and H, $G \boxtimes H$, is a graph whose vertex set is $V(G) \times V(H)$ such that two vertices (u, v) and (x, y) are adjacent in $G \boxtimes H$ if and only if $u = x$, v and y are adjacent in H or $v = y$, u and x are adjacent in G. In this paper, we show that the $\mathcal{D}$-equivalence class of $K_{n,n}$ has just two elements and we study the $\mathcal{D}$-equivalence class of $K_{m,n}$ for two natural numbers $m < n$. Furthermore, we conjecture that if $n - m \geq 2$, then $K_{m,n}$ is $\mathcal{D}$-unique.

2 The $\mathcal{D}$-equivalence class of $K_{n,n}$

In this section we prove that the $\mathcal{D}$-equivalence class of $K_{n,n}$, for $n \geq 3$, has cardinality 2 and it contains $K_{n,n}$ and $K_n \boxtimes K_2$. Before proving our theorems, we need two following lemmas.

Lemma 1.(Theorem 13, [1]) *For two natural numbers m and n, $D(K_{m,n}, x) = ((x + 1)^m - 1)((x + 1)^n - 1) + x^m + x^n$.*

Lemma 2.(Lemma 4, [1]) *Let G be a graph of order n with domination polynomial $D(G, x) = \sum_{i=1}^{n} d(G, i)x^i$. If $d(G, j) = \binom{n}{j}$ for some j, then $\delta(G) \geq n - j$. More precisely, $\delta(G) = n - l$, where $l = min\{j|d(G, j) = \binom{n}{j}\}$, and there are at least $\binom{n}{n-1-\delta(G)} - d(G, n - 1 - \delta(G))$ vertices of degree $\delta(G)$ in G. Furthermore, if for every two* *vertices of degree $\delta(G)$, say u and v, we have $N[u] \neq N[v]$, then there are exactly $\binom{n}{n-1-\delta(G)} - d(G, n-1-\delta(G))$ vertices of degree $\delta(G)$.*

Corollary 1. *If G and H are two graphs and $D(G,x) = D(H,x)$, then $\delta(G) = \delta(H)$.*

Corollary 2.(Theorem 14, [1]) *Let H be a k-regular graph with $N[u] \neq N[v]$, for every $u, v \in V(H)$. If $D(G,x) = D(H,x)$, then G is also a k-regular graph.*

In the next theorem, we determine the $\mathcal{D}$-equivalence class of the complete bipartite graph $K_{n,n}$.

Theorem 1. *For every natural number n, $[K_{n,n}] = \{K_{n,n}, K_n \boxtimes K_2\}$.*

Proof. For $n = 1, 2$ the assertion is obvious. So assume that $n \geq 3$. Let G be a graph and $D(G,x) = D(K_{n,n}, x)$, for some natural number n. The equality $D(G,x) = D(K_{n,n}, x)$ implies that $|V(G)| = 2n$, $\gamma(G) = \gamma(K_{n,n}) = 2$ and $|\Gamma(G)| = |\Gamma(K_{n,n})| = d(K_{n,n}, 2) = n^2$. By Corollary 2, we deduce that G is an n-regular graph. Therefore; $|E(G)| = n^2$. So, the number of nonadjacent pairs of the vertices of G is $n^2 - n$. Since $|\Gamma(G)| = n^2$, there exist two adjacent vertices $u, v \in V(G)$ such that $\{u, v\}$ is a γ-set of G. Since $\{u, v\}$ is a γ-set and $d(u) = d(v) = n = |V(G)|/2$, we have $N(u) \cap N(v) = \varnothing$. Thus G has the following graph as a spanning subgraph,

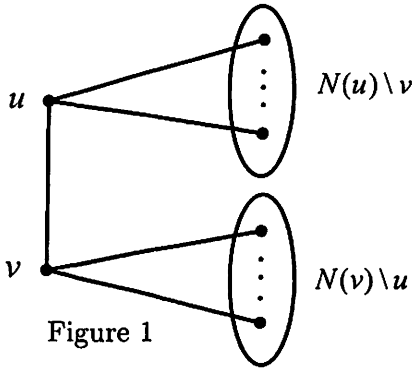

Figure 1

Now, we consider two cases:

(i) There is no γ-set $\{x, y\}$ such that $x = u$, $y \in N(v)$ or $x = v$, $y \in N(u)$. By this assumption, we deduce that $|\Gamma(G)| \leq (n-1)^2 + 2(n-1) + 1 = n^2$. Hence for every $x \in N(u)$, $\{x, u\}$ is a γ-set of G. This implies that every vertex of $N(u) \setminus \{v\}$ should be adjacent to the every vertex of $N(v) \setminus \{u\}$. Therefore; G has the complete bipartite graph $K_{n,n}$ as a subgraph. Since $|E(G)| = n^2$, we have $G = K_{n,n}$.

(ii) Without loss of generality, we assume that there exists $\omega \in N(u) \setminus \{v\}$ that $\{v, \omega\}$ is a γ-set. So, $N(u) \setminus \{v, \omega\} \subseteq N(\omega)$. Since $|N(u) \setminus \{v\}| = n - 1$, we conclude that there is a unique vertex $\omega' \in N(v) \setminus \{u\}$ that $\omega\omega' \in E(G)$. Since $n \geq 3$, $N(u) \setminus \{v, \omega\} \neq \varnothing$. Thus for every $x \in N(u) \setminus \{v\}$, $N(u) \cap N(x) \neq \varnothing$ and so $|N[u] \cup N[x]| \leq 2n-1$. This implies that $\{u, x\}$ is not a γ-set, for every $x \in N(u) \setminus \{v\}$. Also for every $y \in N(v) \setminus \{u, \omega'\}$, $\omega \notin N(y) \cup N(v)$. Thus $\{y, v\}$ is not a γ-set. Now, we claim that there are at least $n - 2$ vertices of $N(u) \setminus \{v\}$, say $u_1, \ldots, u_{n-2}$, such that for every $1 \leq i \leq n-2$, $\{v, u_i\}$ forms a γ-set. By contradiction, suppose that there are at most $n - 3$ such vertices. This implies that

$$|\Gamma(G)| \leq (n-1)^2 + (n-3) + n + 1 = n^2 - 1,$$

where $(n-1)^2$ is counted for the maximum number of γ-sets with one element in $N(u) \setminus \{v\}$ and another in $N(v) \setminus \{u\}$. Also, n is counted for the maximum number of γ-sets $\{u, x\}$, where $x \in N[v]$. Finally, we add 1, because $\{v, \omega'\}$ may be a γ-set. So, the claim is proved. Similarly, one can see that there are at least $n-2$ vertices of $N(v)\setminus\{u\}$, say $v_1, \dots, v_{n-2}$, such that for every $1 \leq i \leq n-2$, $\{u, v_i\}$ forms a γ-set. Therefore, the subgraphs induced on $N[u] \setminus \{v\}$ and $N[v] \setminus \{u\}$ are isomorphic to K_n. Since the degree of each vertex of G is n, so G is isomorphic to $K_n \boxtimes K_2$. □

3 The $\mathcal{D}$-equivalence class of $K_{m,n}$

In this section we would like to determine the $\mathcal{D}$-equivalence class of $K_{n,n+1}$. Let n be a natural number. Consider the graph $K_{n+1} \boxtimes K_2$ and remove one of its vertices. Call this graph by H_n. It is not hard to see that $D(H_n, x) = D(K_{n,n+1}, x)$. It seems that $[K_{n,n+1}] = \{K_{n,n+1}, H_n\}$. We will show that if G is a graph, $D(G, x) = D(K_{n,n+1}, x)$ and G contains a γ-set containing two vertices of the minimum degree, then $G \cong H_n$. Among other results, we prove that if G is a triangle-free graph and $D(G, x) = D(K_{m,n}, x)$, then $G = K_{m,n}$.

Lemma 3. *Let G and H be two graphs whose domination polynomials are the same. If G contains k vertices of degree $\delta(G) = \delta(H)$ (see Corollary 1) whose closed neighborhoods are distinct, then H contains such k vertices too.*

Proof. Let $S \subseteq V(G)$, $|S| = |V(H)| - \delta(H) - 1$ and S is not a dominating set. It suffices to prove that $S = V(H) \setminus N[u]$, for some $u \in V(H)$, where $d(u) = \delta(H)$. There exists $u \in V(H)$ such that $S \cap N[u] = \varnothing$. So, $N[u] \subseteq V(H) \setminus S$. Since $|V(H) \setminus S| = \delta(H) + 1$, we have $d(u) = \delta(H)$. Since there exist k such subsets in $V(G)$ and $D(G, x) = D(H, x)$, the proof is complete. □

Theorem 2. *Let G be a graph and n be a natural number. If $D(G, x) = D(K_{n,n+1}, x)$ and there exist two vertices of degree n such that these vertices form a γ-set of G, then $G \cong H_n$.*

Proof. By Lemma 2, there are at least $n+1$ vertices $X = \{x_1, \ldots, x_{n+1}\}$ of degree $\delta(G) = n$ such that for every $1 \leq i < j \leq n+1$, $N[x_i] \neq N[x_j]$. Let $\{a_1, b_1\}$ be a γ-set containing two vertices of degree n. We can assume that $a_1, b_1 \in X$. To see this, by Lemma 3, we note that there exist $1 \leq i, j \leq n+1$ such that $N[a_1] = N[x_i]$ and $N[b_1] = N[x_j]$. Since $\{a_1, b_1\}$ is a γ-set, we conclude that $i \neq j$. Two cases may be considered:

Case 1. There are three vertices $x_i, x_j, x_k \in X$ such that $\{x_i, x_j\}$ and $\{x_i, x_k\}$ are two γ-sets. With no loss of generality we assume that $i = 1, j = 2, k = 3$. Since $d(x_1) = d(x_2) = d(x_3) = n$, $x_2, x_3 \notin N(x_1)$. Let $W = V(G) \setminus N[x_1]$. We have $x_2, x_3 \in W$. Since $\{x_1, x_2\}$ and $\{x_1, x_3\}$ are γ-sets, $W \subseteq N[x_2]$ and $W \subseteq N[x_3]$. On the other hand $|W| = d(x_2) = d(x_3) = n$. This implies that $|N(x_1) \cap N(x_2)| = |N(x_1) \cap N(x_3)| = 1$ and so there are two vertices $z_2, z_3 \in N(x_1)$ such that $x_2z_2, x_3z_3 \in E(G)$. Since $N[x_2] \neq N[x_3]$, we conclude that $z_2 \neq z_3$. We note that

(1) If $\{u, v\} \subseteq N[x_1]$ is a γ-set, then $\{u, v\} = \{z_2, z_3\}$.
(2) No pair of W forms a γ-set, because $W \cap N[x_1] = \varnothing$.

We claim that the induced subgraph on W is a complete graph. If there are at least $n-1$ vertices of W, say $w_1, \ldots, w_{n-1}$, such that for every j, $1 \leq j \leq n-1$, $\{x_1, w_j\}$ forms a γ-set, then the induced subgraph on W is a complete graph and the claim is proved. So, we may assume that there are at most $n-2$ vertices of W with this property. Hence we find

$$|\Gamma(G)| \leq 1 + (n-2) + n^2 = n^2 + n - 1.$$

But for $n \neq 2$, $|\Gamma(G)| = d(K_{n,n+1}, 2) = n^2+n$, and so $|\Gamma(G)| \geq n^2+n$, for every $n \geq 2$, a contradiction, and the claim is proved. Now, we show that there are at least $n-1$ vertices of $N(x_1)$, $u_1, \ldots, u_{n-1}$, such that $\{x_2, u_j\}$ forms a γ-set, for $1 \leq j \leq n-1$. By contradiction assume that there are at most $n-2$ vertices of $N(x_1)$ with this property. By (1) and (2) no pair of W is a γ-set and at most one pair of $N[x_1]$, i.e. $\{z_2, z_3\}$ forms a γ-set. Hence we find

$$|\Gamma(G)| \leq n + 1 + (n-2) + n(n-1) = n^2 + n - 1,$$

a contradiction. Hence $\{x_2, v\}$ is a γ-set for at least $n-1$ vertices $v \in N(x_1)$. Thus the induced subgraph on $N[x_1] \setminus \{z_2\}$ is a complete graph. Similarly, the induced subgraph on $N[x_1] \setminus \{z_3\}$ is also a complete graph. By a similar counting as we did for the previous inequality, at least one of the sets $\{x_2, z_2\}$ and $\{x_3, z_3\}$ is a γ-set. Hence $z_2 z_3 \in E(G)$ and the induced graph on $N[x_1]$ is a complete graph. We claim that $X \cap N(x_1) = \varnothing$. Let $x \in X \cap N(x_1)$. We know that $d(x) = d(x_1) = n$. Since the induced subgraph on $N[x_1]$ is a complete graph of order $n+1$, we find that $N[x] = N[x_1]$, a contradiction. This yields that $X \setminus \{x_1\} \subseteq W$ and so $G \cong H_n$.

Case 2. For every $a \in X$, there exists at most one vertex $b \in X$ such that $\{a, b\}$ forms a γ-set. By the assumption and with no loss of generality, assume that $\{x_1, x_2\}$ forms a γ-set. Now, for every $3 \leq j \leq n+1$, we conclude that none of the $\{x_1, x_j\}$ and $\{x_2, x_j\}$ is a γ-set. Since $\{x_1, x_2\}$ is a γ-set, we deduce that $|N(x_1) \cap N(x_2)| = 1$.

Let $N(x_1) \cap N(x_2) = \{z\}$. Then G has the following graph as a spanning subgraph,

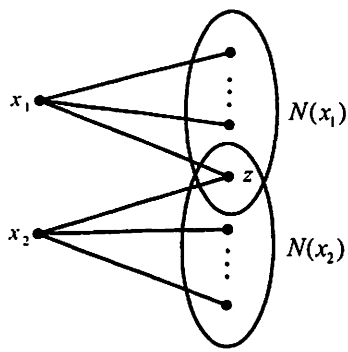

Figure 2

Note that, for $i = 1, 2$,

(i) no pair of $N(x_i) \setminus \{z\}$ forms a γ-set.
(ii) Also, for every $u \in N(x_i) \setminus \{z\}$, $\{x_i, u\}$ is not a γ-set.

We would like to find an upper bound for $|\Gamma(G)|$. Assume that $|N(x_1) \cap X| = t$. Thus $|X \cap (N(x_2) \setminus \{z\})| = n - 1 - t$. Therefore, by (i) and (ii), we find

$$|\Gamma(G)| \leq (n-1)^2 + 2(n-1) + 2n - (t + (n-1-t)) + 1 = n^2 + n + 1, \quad (*)$$

where $(n-1)^2$ is counted for the maximum number of γ-sets with one element in $N(x_1)\setminus\{z\}$ and another in $N(x_2)\setminus\{z\}$. Also, $2(n-1)$ counts the maximum number of γ-sets $\{z, f\}$, where $f \in (N(x_1) \cup N(x_2)) \setminus \{z\}$. We prove that the number of γ-sets which contains x_1 or x_2 is at most $2n - (t + (n-1-t)) + 1$. Let

$$\Omega = \{\{x_i, x_i'\} \mid x_i' \in N(x_{3-i}),\ i = 1, 2\}.$$

Clearly, $|\Omega| = 2n$. Since none of the $\{x_1, x_j\}$ and $\{x_2, x_j\}$ is a γ-set, for $3 \le j \le n$, and $|N(x_1) \cap X| = t$, the number of elements of Ω containing x_2 which is not a γ-set, is at least t. Moreover; since $|X \cap (N(x_2) \setminus \{z\})| = n - 1 - t$, the number of elements of Ω containing x_1 which is not a γ-set, is at least $n - 1 - t$. Finally, we add 1, because $\{x_1, x_2\}$ is a γ-set.

Remark 1. We note that if $z \in X$, one can replace $n^2 + n + 1$ with $n^2 + n$ in the inequality $(*)$. This follows from the fact that $\{x_1, z\}$ is not a γ-set, and so we obtain $2n - (t + (n - t))$ instead of $2n - (t + (n - 1 - t))$. Thus by counting in the inequality $(*)$, $\{z, v\}$ is a γ-set for every $v \in V(G) \setminus \{z, x_1, x_2\}$.

Since $|\Gamma(G)| = |\Gamma(K_{n,n+1})| = n^2 + n$, there is at most one vertex $t_1 \in N(x_1) \cap X$ and one vertex $t_2 \in X \cap (N(x_2) \setminus \{z\})$, such that $\{t_1, t_2\}$ is not a γ-set. Since by assumption, for every $a \in X$, there exists at most one vertex $b \in X$ such that $\{a, b\}$ forms a γ-set, we obtain $t = 0$ or $n - 1 - t = 0$ or $1 \le t, n - 1 - t \le 2$. Now, we consider four subcases:

Subcase 2.1. If $n - 1 - t = 2$ and $t = 2$, by the pigeonhole principle, there exist $x_i, x_j, x_k \in X$ such that $\{x_i, x_j\}$ and $\{x_i, x_k\}$ are γ-sets, a contradiction.

Subcase 2.2. $t = 0$ or $n - 1 - t = 0$. First assume that $t = 0$. Thus $X \setminus \{x_1\} = N[x_2] \setminus \{z\}$. If both $\{x_1, z\}$ and $\{x_2, z\}$ are γ-sets, then $\{z\}$ is a γ-set, a contradiction. Thus at least one of them is not a γ-set. Since $z \notin X$, according to the counting in the inequality $(*)$, we conclude that for every $u \in V(G) \setminus \{z, x_1, x_2\}$, $\{u, z\}$ is a γ-set for G. Assume that there exists $v \in N(x_2) \setminus \{z\}$ such that $zv \notin E(G)$. Since $\{p, z\}$ is a γ-set, for every $p \in N(x_2) \setminus \{v, z\}$, we deduce that $pv \in E(G)$. Thus $\{x_1, v\} \subseteq X$ is a γ-set, a contradiction. Hence z is adjacent to every vertex in $N[x_2] \setminus \{z\}$. Since $\{z\}$ is not a dominating set, there exists $y \in N(x_1) \setminus \{z\}$ such that $yz \notin E(G)$. Thus $\{z, x_2\}$ is not a γ-set. Since $\{u, z\}$ is a γ-set for every $u \in N(x_2) \setminus \{z\}$, we conclude that $N(x_2) \setminus \{z\} \subseteq N(y)$. We claim that the induced subgraph on $N[x_2]$ is a complete graph. Suppose on the contrary that there exist two vertices $x_i, x_j \in X \setminus \{x_1, x_2\}$ such that $x_i x_j \notin E(G)$. Since $\{z, x_2\}$ is not a γ-set, according to the counting in the inequality $(*)$, we conclude that for every $h \in N(x_1)$, $\{x_j, h\}$ is a γ-set. Thus $x_i h \in E(G)$, for every $h \in N(x_1)$. This implies that $d(x_i) \geq n + 1$, a contradiction. Therefore the claim is proved. Let $b \in N[x_2] \setminus \{z\} = X \setminus \{x_1\}$. Since $y \in N(b)$ and the induced subgraph on $N[x_2]$ is a complete graph, we obtain $d(b) \geq n+1$, which contradicts $b \in X$.

Now, assume that $n-1-t = 0$. The proof of this case is similar to the case $t = 0$, if $z \notin X$. Assume that $n - 1 - t = 0$ and $z \in X$. If $n = 1$, then since $K_{1,2}$ is $\mathcal{D}$-unique and $H_1 = K_{1,2}$, we are done. If $n = 2$, then G is a cycle of order 5. But $D(K_{2,3}, x) \neq D(C_5, x)$, a contradiction. Thus assume that $n \geq 3$. So $t \geq 2$. Let $\{z, a\} \subseteq N(x_1) \cap X$. By Remark 1, for every $q \in (N(x_1) \cup N(x_2)) \setminus \{z\}$, $\{q, z\}$ is a γ-set. This yields that $az \in E(G)$, since otherwise $aq \in E(G)$ for each

$q \in (N(x_1) \cup N(x_2)) \setminus \{z, a\}$ and so $d(a) \geq n+1$, a contradiction. Hence $az \in E(G)$. Since $d(z) = d(a) = n$, thus $\{z, a\}$ is not a γ-set, which contradicts Remark 1.

Subcase 2.3. $t = 1$ and $n - 1 - t = 2$. With no loss of generality, suppose that $N(x_1) \cap X = \{x_3\}$ and $(N(x_2) \setminus \{z\}) \cap X = \{x_4, x_5\}$. If $x_3 = z$, then by Remark 1, $\{z, v\}$ is a γ-set, for every $v \in V(G) \setminus \{z, x_1, x_2\}$. Thus $\{z, x_4\}$ and $\{z, x_5\}$ are γ-sets, a contradiction to the assumption of Case 2. So let $x_3 \neq z$. Since at least one of the pairs $\{x_3, x_4\}$ and $\{x_3, x_5\}$ is not a γ-set, by taking note of the counting of γ-sets in the right hand side of the inequality $(*)$, we conclude that for every $h \in N(x_1) \setminus \{x_3\}$, $\{x_2, h\}$ is a γ-set. This implies that for every $h \in N(x_1) \setminus \{x_3, z\}$, $hx_3 \in E(G)$. Thus $\{x_2, x_3\} \subseteq X$ is a γ-set, a contradiction. The case $t = 2$ and $n - 1 - t = 1$ is similar.

Subcase 2.4. $t = 1$ and $n - 1 - t = 1$. Thus $n = 3$. With no loss of generality, suppose that $N(x_1) \cap X = \{x_3\}$ and $(N(x_2) \setminus \{z\}) \cap X = \{x_4\}$. If $x_3 = z$, then we claim that $zx_4 \in E(G)$. If $zx_4 \notin E(G)$, then by Remark 1, $\{z, v\}$ is a γ-set, for every $v \in V(G) \setminus \{z, x_1, x_2\}$ and so $vx_4 \in E(G)$. Thus $d(x_4) = 4$, a contradiction. This proves the claim. Since $d(z) = d(x_4) = 3$ and $zx_4 \in E(G)$, thus $\{z, x_4\}$ is not a γ-set, which contradicts Remark 1. So, assume that $x_3 \neq z$. According to the counting in the inequality $(*)$, either for every $u \in N(x_1) \setminus \{x_3\}$, $\{x_2, u\}$ is a γ-set or for every $v \in N(x_2) \setminus \{x_4\}$, $\{x_1, v\}$ is a γ-set. So, either $N(x_1) \subseteq N[x_3]$ or $N(x_2) \subseteq N[x_4]$. Hence, at least one of the $\{x_2, x_3\}$ and $\{x_1, x_4\}$ is a γ-set, a contradiction. □

Conjecture 1. $[K_{n,n+1}] = \{\, K_{n,n+1}, H_n \,\}$.

The following remark has a simple proof and we omit it.

Remark 2. Let G be a graph such that $D(G,x) = D(K_{m,n},x)$ and G contains $K_{m,n}$ as a subgraph, for some natural numbers m and n. Then $G = K_{m,n}$.

Theorem 3. *Let G be a graph and $n > m$ be two natural numbers. If $D(G,x) = D(K_{m,n},x)$ and G contains at most $m-3$ triangles, then $G = K_{m,n}$.*

Proof. If $m = 1$, then by Corollary 2 of [2], we are done. Thus assume that $m \geq 2$. First suppose that $m \geq 3$. By Corollary 1, there exists $x_1 \in V(G)$ such that $d(x_1) = m$. Assume that $N(x_1) = \{y_1,\ldots,y_m\}$. Set $V(G) \setminus N[x_1] = \{x_2,\ldots,x_n\}$. Note that if $m \geq 3$, then for every $1 \leq i < j \leq m$, $\{y_i,y_j\}$ is not a γ-set. Suppose on the contrary that there exist i,j, $1 \leq i < j \leq m$, such that $\{y_i,y_j\}$ is a γ-set. Thus for every $k \in \{1,\ldots,m\} \setminus \{i,j\}$, $y_k \in N(y_i) \cup N(y_j)$. This implies that G has at least $m-2$ triangles, a contradiction. Also for every $2 \leq i < j \leq n$, $x_1 \notin N(x_i) \cup N(x_j)$. Hence $\{x_i,x_j\}$ is not a γ-set for every $2 \leq i < j \leq n$. We claim that there exists at most one x_j, $j = 2,\ldots,n$ such that $\{x_1,x_j\}$ is a γ-set. With no loss of generality assume that both $\{x_1,x_2\}$ and $\{x_1,x_3\}$ are γ-sets. Therefore, for every k, $4 \leq k \leq n$, $x_2x_3x_k$ is a triangle, a contradiction, since $n > m$, and the claim is proved. If there is no γ-set of the form $\{x_1,x_j\}$, $2 \leq j \leq n$, then by noting that $|\Gamma(G)| = mn$, then we conclude that $\{x_i,y_j\}$ is a γ-set for every $1 \leq i \leq n$, $1 \leq j \leq m$. Then G contains $K_{m,n}$ as a subgraph. Now, by Remark 2, the proof in this case is complete. Now, with no loss of generality suppose that x_2 is the unique vertex of X such that $\{x_1,x_2\}$ is a γ-set. This yields that for every $3 \leq j \leq n$, $x_j \in N(x_2)$. Clearly, there exists y_i, $1 \leq i \leq m$, such that $\{x_1, y_i\}$ is a γ-set. Thus $\{x_2, \ldots, x_n\} \subseteq N(y_i)$ and G contains triangles $x_2 y_i x_j$ for every j, $3 \leq j \leq n$, a contradiction.

Now, assume that $m = 2$. If $n \geq 4$, then by a similar method as we did for the case $m \geq 3$ and using the equality $|\Gamma(G)| = 2n + 1$, we obtain a contradiction. Thus suppose that $n \leq 3$. If $n = 2$, then using Theorem 1, we obtain the result. If $n = 3$, then $D(G, x) = D(K_{2,3}, x)$. The uniqueness of G in this case is proved by considering all graphs of order 5 and minimum degree 2. □

Lemma 4. *Let G be a triangle-free graph and m, n be two natural numbers. If $D(G, x) = D(K_{m,n}, x)$, then $G = K_{m,n}$.*

Conjecture 2. *Let m and n be two natural numbers such that $n - m \geq 2$. Then $K_{m,n}$ is $\mathcal{D}$-unique.*

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