On the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ and other related topics

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Abstract

Consider a labeled and strongly oriented cycle $\overrightarrow{C_m}$ and a set $\Gamma = \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$, where $\overrightarrow{C_n}, \overleftarrow{C_n}$ are two labeled and strongly oriented cycles with the same underlying graph and opposite orientations. Let $h : E(\overrightarrow{C_m}) \longrightarrow \Gamma$ be any function that sends to every edge of $\overrightarrow{C_m}$ either $\overrightarrow{C_n}$ or $\overleftarrow{C_n}$. The main goal of this paper is to study the underlying graph of the product $\overrightarrow{C_m} \otimes_h \Gamma$, where the product is defined as follows:

$$V(\overrightarrow{C_m} \otimes_h \Gamma) = V(\overrightarrow{C_m}) \times V(\overrightarrow{C_n})$$

and

$$((a,b),(c,d)) \in E(\overrightarrow{C_m} \otimes_h \Gamma)$$
$$\Leftrightarrow (a,c) \in E(\overrightarrow{C_m}) \wedge (b,d) \in h(a,b).$$

This product is of interest since it preserves many different types of labelings. For instance, edge-magic and super edge-magic labelings. In this paper, we also study the algorithmic complexity of determining when a diagraph $\overrightarrow{D}$ can be factored using the product $\otimes_h$ in terms of a given set of diagraphs Γ. This is the main topic of the third section of the paper.

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1 Introduction

In this paper, we will follow for the general graph theory terminology and notation Chartrand and Lesniak [3]. In particular, we will use simple graphs i.e., graphs without loops nor multiple edges, unless otherwise specified. By a (p,q)-graph, we mean a graph of order p and size q.

In 1970, Kotzig and Rosa introduced the concept of edge-magic labeling of graphs under the name "magic valuation" [6]. These were later rediscovered by Ringel and Lladó [8], who coined one of the now more popular terms for them: edge-magic labeling. More recently, they have been referred to as edge-magic total labeling by Wallis [9]. For a (p,q)-graph G, a bijective function $f : V(G) \cup E(G) \longrightarrow \{1,2,\ldots,p+q\}$ is an edge-magic labeling of G if for each $uv \in E(G)$, $f(u)+f(uv)+f(v)$ is a constant k called the valence of f. If such a labeling exists then G is said to be an edge magic-graph. In [4], Enomoto *et al.*, defined an edge-magic labeling of a graph G of order p to be a super edge-magic labeling if $f(V(G)) = \{1,2,\ldots,p\}$. Super edge-magic graphs and labelings have been called strong edge-magic total graphs and labelings respectively by Wallis [9]. It is worth to mention that the concept of super edge-magic labeling was already known by Acharya and Hegde. They introduced this same concept in [1] under the name of strongly indexable graphs. However instead of thinking about constant sums, Acharya and Hegde where thinking on arithmetic progressions. In their effort to study labelings of the "magic type", Figueroa-Centeno *et al.* [5] introduced the concept of edge-magic and super edge-magic digraphs as follows. A digraph D is said to be (super) edge-magic if the underlying graph is (super) edge-magic. Also they extended the concept of (super) edge-magicness to graphs with loops. In the same paper the following operation involving digraphs was introduced. Let $D=(V,E)$ be a digraph with $V \subset \mathbb{N}$, and let $\Gamma = \{\gamma_i\}_{i=1}^{k}$ be a family of 1-regular digraphs (loops allowed) all of them with the same vertex set $V' = \{1,2,\ldots,n\}$. Let $h : E \longrightarrow \Gamma$ be a function that assigns to every arc of E an element of Γ. Then the product $\overrightarrow{D} \otimes_h \Gamma$ is defined as follows:

$$V(\overrightarrow{D} \otimes_h \Gamma) = V \times V'$$

and

$$((a,b),(c,d)) \in E(\overrightarrow{D} \otimes_h \Gamma)$$
$$\Leftrightarrow (a,c) \in E \wedge (b,d) \in E(h(a,c)).$$

Notice that the adjacency matrix of $\overrightarrow{D} \otimes_h \Gamma$, denoted by $A(\overrightarrow{D} \otimes_h \Gamma)$, can be obtain by multiplying every 0-entry of $A(\overrightarrow{D})$, where $A(\overrightarrow{D})$ denotes the adjacency matrix of $\overrightarrow{D}$, by the $n \times n$ null square matrix, and every 1-entry of $A(\overrightarrow{D})$ by $A(h(a,c))$, where $A(h(a,c))$ denotes the adjacency matrix of

$h(a,c)$. It is worthwhile mentioning that when the function h is constant then the previous product coincides with the classical Kronecker product of matrices.

Some definitions, terminology and operations involving diagraphs are given in [7] F.A. Muntaner-Batle defined that a super edge-magic labeling f of a bipartite graph $G = (V_1 \cup V_2, E)$ is called an special super edge-magic labeling of G (and G is called an special super edge-magic graph) if it has the extra property that $f(V_1) = \{i\}_{i=1}^{p_1}$ and $f(V_2) = \{i\}_{i=p_1+1}^{p_1+p_2}$, where $|V_1| = p_1$ and $|V_2| = p_2$. In [5] Figueroa-Centeno *et al.* introduced the following result involving (special) (super) edge-magic labeling and the previous product.

Theorem 1. *Let $\overrightarrow{D} = (V,E)$ be a (special) (super) edge-magic digraph with a (special) (super) edge-magic labeling f for which each vertex takes the name of its (special) (super) edge-magic label. Also let $\Gamma = \{\gamma_i\}_{i=1}^{k}$ be the family of all super edge-magic 1-regular labeled digraph of order p where each vertex takes the name of its super edge-magic label. Consider any function $h : E \longrightarrow \Gamma$. Then the digraph $\overrightarrow{D} \otimes_h \Gamma$ is (super) edge-magic.*

A problem that Proposition 1 presents is that in many cases we do not know before hand, which one is the resulting product digraph. In this paper our main goal, although not the only one, is to study the product $\overrightarrow{C_m} \otimes_h \Gamma$ in the following case:

1. The digraph $\overrightarrow{C_m}$ is a cycle of length m oriented in a cyclic way. That is to say either clockwise or counterclockwise.

2. The set $\Gamma = \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ consists of two cyclically directed cycles of order n with $und(\overrightarrow{C_n}) = und(\overleftarrow{C_n})$ and with opposite orientations.

3. We consider any function $h : E(\overrightarrow{C_m}) \longrightarrow \Gamma$ that sends the arcs of $\overrightarrow{C_m}$ to the elements of Γ.

2 The Product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$

In this section we study the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$. We start by precisely defining every element involved in the product.

1. The oriented cycle $\overrightarrow{C_m}$ is the digraph with the vertex set

$$V(\overrightarrow{C_m}) = \{0, 1, 2, \ldots, m-1\}$$

and the arc set

$$E(\overrightarrow{C_m}) = \{(m-1, 0)\} \cup \{(i, i+1)\}_{i=0}^{m-2}.$$

Similarly for oriented cycle $\overrightarrow{C_n}$

2. The oriented cycle $\overleftarrow{C_n}$ is the digraph with the vertex set $V(\overleftarrow{C_n}) = \{0, 1, \ldots, n-1\}$ and with arc set $E(\overleftarrow{C_n}) = \{i \pmod n, i-1 \pmod n)\}_{i=0}^{n-1}$

3. The function $h : E(\overrightarrow{C_m}) \longrightarrow \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ is any function that assigns to every arc of $\overrightarrow{C_m}$ either $\overrightarrow{C_n}$ or $\overleftarrow{C_n}$.

Next, we will use the notation $N(h^-)$ in order to denote the number of edges for which the function h assigns the digraph $\overleftarrow{C_n}$.

Theorem 2. *Let m, n be two odd numbers with n prime. Then the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} = n\overrightarrow{C_m}$ if and only if $m - 2N(h^-) \equiv 0 \pmod n$.*

Proof. Let $(0,0) \in V(\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\})$. By the way the digraph $\overrightarrow{C_m}$ has been defined, since $(i, i+1) \in E(\overrightarrow{C_m})$ for $0 \leq i \leq m-1$, it follows that there exists $b_0, b_1, \ldots, b_{m-1}$ with $((i, b_i), (i+1, b_{i+1})) \in E(\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\})$. That is to say, the oriented path (may be closed oriented path)

$$((0,0),(1,b_1)),((1,b_1),(2,b_2)),((2,b_2),(3,b_3)),\ldots,$$
$$((m-2,b_{m-2}),(m-1,bm-1)),((m-1,b_{m-1}),(0,b_m))$$

is an oriented subpath of the product digraph $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$. Hence a cycle of length m will be closed if and only if $b_m = 0$. It is easy to observe that $b_m \equiv (m - 2N(h^-)) \pmod n$. Therefore, we need $m - 2N(h^-) \equiv 0 \pmod n$ in order to close a cycle of length m. A similar reasoning will give us the remaining cycles of length m. □

Theorem 3. *Let $m, n \in \mathbb{N}$ and consider the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$. Let g be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $< g >$, such that $| < g > | = k$. Also let $N_g(h^-) < m$ be a natural number that satisfies the following congruence relation*

$$m - 2N_g(h^-) \equiv g \pmod n.$$

Then, if the function h assigns to exact by $N_g(h^-)$ arcs of $\overrightarrow{C_m}$ the element $\overleftarrow{C_n}$, the product

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$$

consists of exactly $\frac{n}{k}$ disjoint copies of a strongly oriented cycle $\overrightarrow{C_{mk}}$. In particular if $gcd(g,n) = 1$, then $< g >= \mathbb{Z}_n$ and if the function h assigns to exactly $N_g(h^-)$ arcs of $\overrightarrow{C_m}$ the digraph $\overleftarrow{C_n}$, then

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} = C_{mn}.$$

Proof. Assume the the function h assigns to exactly $N_g(h^-)$ arcs of $\overrightarrow{C_m}$ the diagraph $\overleftarrow{C_n}$. Also consider a directed subpath of $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ (possibly closed) that starts at $(0,0)$. Then using a similar reasoning to the one used in the proof of Theorem 2, the vertex that is in position mk in the path is a vertex of the form $(0, k[m-2N_g(h^-)] \pmod{n})$. Remember that k is the smallest number that makes $kg \equiv 0 \pmod{n}$ since $|\langle g \rangle| = k$ and therefore we cannot close an oriented cycle of smaller length than mk. A similar reasoning will give us the remaining cycles of length mk. □

3 NP-completeness

The goal of this section is to prove the following result:

Theorem 4. *The problem of deciding if a given digraph G can be factored into a product of the form $H \otimes_h \mathfrak{S}$ where*

1. *H is a digraph.*
2. *$\mathfrak{S}$ is a given family of digraphs all of them of the same order.*
3. *$h : E(H) \longrightarrow \mathfrak{S}$*

is at least NP-complete.

In order to prove this result, we state and prove the following results.

Theorem 5. *A diagraph G of order $2p$ contains a tournament of order p if and only if there exist:*

1. *A digraph H with a loop (λ, λ).*
2. *A family of digraphs $\mathfrak{S}$ with $|\mathfrak{S}| \leq 4$ and with a tournament of order p, $\vec{K}_p$, in $\mathfrak{S}$.*
3. *A function $h : E(H) \longrightarrow \mathfrak{S}$ with $h(\lambda, \lambda) = \longrightarrow K_p$ and $G = H \otimes_h \mathfrak{S}$*

Proof. ($\Rightarrow$) Let $[p] = \{1, 2, \ldots, p\}$ be the vertex set of the tournament.

- Let H be the digraph with $V(H) = \{0, 1\}$ and with adjacency matrix $A(H) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- Let $\mathfrak{S} = \{S_1, S_2, S_3, \longrightarrow K_p\}$ with $V(S_i) = [p]$ and
  - $E(S_1) = \{(a, b-p)$ with $(a, b) \in E(G)$ and $a, b-p \in [p]\}$
  - $E(S_2) = \{(a-p, b)$ with $(a, b) \in E(G)$ and $a-p, b \in [p]\}$

- $E(S_3) = \{(a-p, b-p) \text{ with } (a,b) \in E(G) \text{ and } a-p, b-p \in [p]\}$

- Let $h(0,0) = \longrightarrow K_p$, $h(0,1) = S_1$, $h(1,0) = S_2$ and $h(1,1) = S_3$. By construction,

$$G = H \otimes_h \mathfrak{S}.$$

($\Leftarrow$) It is easy to check that the set $\{\lambda\} \times [p] \subset V(G)$ is the vertex set of the tournament $\longrightarrow K_p$.

□

Remark 1. *Observe that it is easy to find a tournament in a digraph if we know how to find a clique of the same order of the tournament in a graph and viceversa.*

In [10] we found the half-clique problem that we state next. The half-clique problem asks: *Given a graph G with an even number of vertices, does there exist a clique of G consisting of exactly half the vertices of G?.* The following result is well known:

Theorem 6. *The half-clique problem is NP-complete.*

As a corollary of the above discussion, we obtain Theorem 4.

4 Conclusion

In [5] Figueroa-Centeno et al. introduced the product $\otimes_h$ as a tool to generate exponentally many labelings for many different graphs. However a problem exists: it is not always easy to determine before hand the output of the product $\otimes_h$.

The paper has been divided into two parts. The first part consists on studying the behavior of the product $\otimes_h$ when we deal with a cycle $\overrightarrow{C_m}$ oriented in a cyclic way, with a set of the form $\{\overrightarrow{C_n}, \overleftarrow{C_n}\}$, where $\overrightarrow{C_n}$ and $\overleftarrow{C_n}$ are two cyclic oriented cycles with opposite orientations with the same underlying graph and with any function $h : E(\overrightarrow{C_m}) \longrightarrow \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$. This question has been closed.

The second part of the paper has been devoted to show that the problem of factoring a given graph, by means of the product $\otimes_h$ can be in ocations an NP-complete problem. This has been proved by showing a relationship among the product $\otimes_h$ and the well known "half-clique problem".

At this point we suggest the following directions for further research.

1. What can be said about the product $H \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$, where H is a unicyclic digraph with the cycle oriented cyclically, $\overrightarrow{C_n}$ and $\overleftarrow{C_n}$ are

two cyclic oriented cycles with opposite orientation with the same underlying graph and with any function $h : E(\overrightarrow{C_m}) \longrightarrow \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$?

2. Let Γ be a set of unicyclic diagraphs with equal order and size, with the same vertex set V and with the cycle oriented cyclically. Let H be any diagraph and consider any function $h : E(H) \longrightarrow \Gamma$. What can we say about the product $H \otimes_h \Gamma$?

3. Under which conditions does the product $\otimes_h$ result in a diagraph G with connected underlying graph?

Finally, we will close this section mentioning that the product $\otimes_h$ has already given many results on the topic of graph labelings. However we feel that this diagraph operation, should not only be studied because of this relationship with graph labelings but also because it is interesting just as a diagraph operation itself.

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