ON THE PROPERTIES OF NEW FAMILIES OF PELL AND PELL-LUCAS NUMBERS

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ABSTRACT. In this paper, new families of Pell and Pell-Lucas numbers are introduced. In addition, we present the recurrence relations and the generating functions of the new families for $k = 2$.

1. INTRODUCTION

There are a lot of integer sequences such as Fibonacci, Pell, Lucas, etc. Pell and Pell-Lucas numbers are used by scientists for basic theories and their applications. For interest application of these numbers in science and nature, one can see [2, 4, 7, 12]. For instance, in science, authors gave sums of the generalized Pell numbers could be derived directly using a new matrix representation [4]. In [3], Horadam showed that some properties involving Pell numbers and gave Simpson formula

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

for the Pell numbers. Also Ercolano [5], found the matrix M for generating the Pell sequence as follows

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

In [2], Horadam and Mahon obtained Simpson formula for the Pell-Lucas numbers as follows

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8\,(-1)^{n+1}.$$

For the rest of this paper, for $n \geq 2$, the well known Pell $\{P_n\}$ and Pell-Lucas numbers $\{Q_n\}$ are defined by

$$P_n = 2P_{n-1} + P_{n-2} \text{ and } Q_n = 2Q_{n-1} + Q_{n-2}$$

with initial conditions given by $P_0 = 0, P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively.

In [1, 3], Horadam gave some equations related to Pell numbers and generating functions for powers of a certain generalized sequences of numbers.

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Falcon, in [11], introduced the k-Lucas sequences by using a special sequence of squares of k-Fibonacci numbers. Recently, Catarino and Vasco have considered the k-Pell numbers, k-Pell Lucas numbers and have presented some properties involving these number sequences [7, 9, 10]. In [8], Catarino and Vasco obtained an explicit formula for the term of order n of the k-Pell numbers sequence and Cassini's formula by using a matrix method.

The aim of this study is to introduce new families of Pell and Pell-Lucas numbers. The recurrence relations and the generating functions of the new families for $k = 2$ are also considered. In [6], authors gave similar results for Fibonacci numbers.

2. Main Results

In this section, we define the new generalized Pell and Pell-Lucas numbers and obtain some results related to these numbers by using [6].

Definition 1. *Let n and $k \neq 0$ be natural numbers. There exist unique numbers m and r such that $n = mk+r$ $(0 \leq r < k)$. Using these parameters, we define new generalized Pell numbers $P_n^{(k)}$ and Pell-Lucas numbers $Q_n^{(k)}$ by*

$$P_n^{(k)} = \frac{1}{(\alpha-\beta)^k}\left(\alpha^{m+1}-\beta^{m+1}\right)^r\left(\alpha^m-\beta^m\right)^{k-r} \tag{2.1}$$

and

$$Q_n^{(k)} = \left(\alpha^{m+1}+\beta^{m+1}\right)^r\left(\alpha^m+\beta^m\right)^{k-r} \tag{2.2}$$

where $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, respectively. The first few numbers of the new Pell and Pell-Lucas numbers as follows

$$\left\{P_n^{(1)}\right\}_{n=0}^{4} = \{0,1,2,5,12\}, \quad \left\{Q_n^{(1)}\right\}_{n=0}^{4} = \{2,2,6,14,34\},$$

$$\left\{P_n^{(2)}\right\}_{n=0}^{4} = \{0,0,1,2,4\}, \quad \left\{Q_n^{(2)}\right\}_{n=0}^{4} = \{4,4,4,12,36\},$$

$$\left\{P_n^{(3)}\right\}_{n=0}^{4} = \{0,0,0,1,2\} \quad \left\{Q_n^{(3)}\right\}_{n=0}^{4} = \{8,8,8,8,24\}$$

for $k = 1,2,3$. $P_n^{(1)}$ is the ordinary Pell numbers P_n and $Q_n^{(1)}$ is the ordinary Pell-Lucas numbers Q_n.

In the following we give some properties of the new generalized Pell and Pell-Lucas numbers.

Theorem 1. *Let $k, m \in \{1,2,3,...\}$ be fixed numbers. The generalized Pell numbers and the ordinary Pell numbers satisfy*

(i) $\sum_{i=0}^{k-1} \binom{k-1}{i} (-2)^{-i} P_{mk+i}^{(k)} = (-2)^{1-k} P_m P_{(m-1)(k-1)}^{(k-1)}$,

(ii) $\sum_{i=0}^{k-1} \binom{k-1}{i} 2^i P_{mk+i}^{(k)} = P_m P_{(m+2)(k-1)}^{(k-1)}$,

(iii) $\sum_{i=0}^{k-1} 2^{-i} P_{mk+i}^{(k)} = 2^{1-k} \frac{P_m}{P_{m-1}} \left[P_{(m+1)k}^{(k)} - 2^k P_{mk}^{(k)} \right]$.

Proof. (i)

$$\begin{aligned} & \sum_{i=0}^{k-1} \binom{k-1}{i} (-2)^{-i} P_{mk+i}^{(k)} \\ = \; & (-2)^{1-k} P_m \sum_{i=0}^{k-1} \binom{k-1}{i} (-2)^{k-1-i} (P_m)^{k-1-i} (P_{m+1})^i \\ = \; & (-2)^{1-k} P_m \sum_{i=0}^{k-1} \binom{k-1}{i} (-2P_m)^{k-1-i} (P_{m+1})^i . \end{aligned}$$

Then by using the binomial theorem

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{k-1}{i} (-2)^{-i} P_{mk+i}^{(k)} &= (-2)^{1-k} P_m (P_{m+1} - 2P_m)^{k-1} \\ &= (-2)^{1-k} P_m (P_{m-1})^{k-1} \\ &= (-2)^{1-k} P_m P_{(m-1)(k-1)}^{(k-1)} \end{aligned}$$

is obtained and this completes the proof.

(ii)

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{k-1}{i} 2^i P_{mk+i}^{(k)} &= \sum_{i=0}^{k-1} \binom{k-1}{i} (P_m)^{k-i} 2^i (P_{m+1})^i \\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} (P_m)^{k-i} (2P_{m+1})^i \\ &= P_m \sum_{i=0}^{k-1} \binom{k-1}{i} (P_m)^{k-1-i} (2P_{m+1})^i . \end{aligned}$$

Then by using the binomial theorem

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{k-1}{i} 2^i P_{mk+i}^{(k)} &= P_m (2P_{m+1} + P_m)^{k-1} \\ &= P_m (P_{m+2})^{k-1} \\ &= P_m P_{(m+2)(k-1)}^{(k-1)} \end{aligned}$$

is obtained.

(*iii*)

$$\begin{aligned}
\sum_{i=0}^{k-1} 2^{-i} P_{mk+i}^{(k)} &= \sum_{i=0}^{k-1} (P_m)^{k-i} (P_{m+1})^i 2^{-i} \\
&= (P_m)^k \sum_{i=0}^{k-1} \left(\frac{P_{m+1}}{2P_m}\right)^i \\
&= (P_m)^k \frac{\left(\frac{P_{m+1}}{2P_m}\right)^k - 1}{\frac{P_{m+1}}{2P_m} - 1} \\
&= 2^{1-k} \frac{P_m}{P_{m-1}} \left[(P_{m+1})^k - (2P_m)^k\right] \\
&= 2^{1-k} \frac{P_m}{P_{m-1}} \left[P_{(m+1)k}^{(k)} - 2^k P_{mk}^{(k)}\right].
\end{aligned}$$

□

Theorem 2. *Let $k, m \in \{1, 2, 3, ...\}$ be fixed numbers. The generalized Pell-Lucas numbers and the ordinary Pell-Lucas numbers satisfy*

(*i*) $\sum_{i=0}^{k-1} \binom{k-1}{i} (-2)^{-i} Q_{mk+i}^{(k)} = (-2)^{1-k} Q_m Q_{(m-1)(k-1)}^{(k-1)}$,

(*ii*) $\sum_{i=0}^{k-1} \binom{k-1}{i} 2^i Q_{mk+i}^{(k)} = Q_m Q_{(m+2)(k-1)}^{(k-1)}$,

(*iii*) $\sum_{i=0}^{k-1} 2^{-i} Q_{mk+i}^{(k)} = 2^{1-k} \frac{Q_m}{Q_{m-1}} \left[Q_{(m+1)k}^{(k)} - 2^k Q_{mk}^{(k)}\right]$.

Proof. The proof can be given in a similar way to the proof of Theorem 1. □

Theorem 3. *We have the following relation for the new family of Pell numbers with $k = 2$*

$$P_{2(n+s-1)}^{(2)} - P_{n+s} P_{n+s-2} = (-1)^{n+s}$$

where $0 \le n, 0 \le s$ and $1 \le n + s$.

Proof. From the matrix M and the definition of ordinary Pell numbers, we have

$$\begin{aligned}
\begin{bmatrix} P_{n+s} & P_{n+s-1} \\ P_{n+s-1} & P_{n+s-2} \end{bmatrix} &= M \begin{bmatrix} P_{n+s-1} & P_{n+s-2} \\ P_{n+s-2} & P_{n+s-3} \end{bmatrix} \\
&= M^{(n+s-2)} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \\
&= M^{(n+s-2)} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

If we take the determinant of the both sides we have the following

$$P^{(2)}_{2(n+s-1)} - P_{n+s}P_{n+s-2} = (-1)^{n+s}.$$

□

Theorem 4. *For the new family of Pell-Lucas numbers with $k = 2$ we have the following relation*

$$Q^{(2)}_{2(n+s-1)} - Q_{n+s}Q_{n+s-2} = 8\,(-1)^{n+s-1}$$

where $0 \le n, 0 \le s$ and $1 \le n+s$.

Proof. The proof can be obtained in a similar way to the proof of Theorem 3. □

Theorem 5. *A recurrence relation for $P^{(2)}_n$ is given by*

$$P^{(2)}_n = 2P^{(2)}_{n-1} + 2P^{(2)}_{n-3} + P^{(2)}_{n-4}, \quad n = 4, 5, \ldots$$

Also we can give a recurrence relation for $Q^{(2)}_n$ by

$$Q^{(2)}_n = 2Q^{(2)}_{n-1} + 2Q^{(2)}_{n-3} + Q^{(2)}_{n-4}, \quad n = 4, 5, \ldots$$

Proof. If we take $n = 2m$ we have the following equation

$$P^{(2)}_{2m} = 2P^{(2)}_{2m-1} + 2P^{(2)}_{2m-3} + P^{(2)}_{2m-4}.$$

Here we use two relations for the proof

$$\begin{aligned} P^{(2)}_{2m} &= (P_m)^2 \\ P^{(2)}_{2m+1} &= P_mP_{m+1}. \end{aligned}$$

Then by using these relations we have the following

$$\begin{aligned} P^{(2)}_{2m} &= (P_m)^2 \\ &= P_m\,(2P_{m-1} + P_{m-2}) \\ &= 2P_mP_{m-1} + (2P_{m-1} + P_{m-2})\,P_{m-2} \\ &= 2P^{(2)}_{2m-1} + 2P^{(2)}_{2m-3} + P^{(2)}_{2m-4}. \end{aligned}$$

Also we have the following equation for $n = 2m+1$

$$P^{(2)}_{2m+1} = 2P^{(2)}_{2m} + 2P^{(2)}_{2m-2} + P^{(2)}_{2m-3}.$$

Similarly,

$$\begin{aligned} P^{(2)}_{2m+1} &= P_mP_{m+1} \\ &= P_m\,(2P_m + P_{m-1}) \\ &= 2P_mP_m + P_{m-1}\,(2P_{m-1} + P_{m-2}) \\ &= 2P^{(2)}_{2m} + 2P^{(2)}_{2m-2} + P^{(2)}_{2m-3} \end{aligned}$$

is obtained. The proof is similar for $Q^{(2)}_n$. □

Here we note that for the case $n = 2m+1$, we can easily get a shorter recurrence relation

$$\begin{aligned} P_{2m+1}^{(2)} &= P_m P_{m+1} \\ &= P_m (2P_m + P_{m-1}) \\ &= 2P_m P_m + P_m P_{m-1} \\ &= 2P_{2m}^{(2)} + P_{2m-1}^{(2)}. \end{aligned}$$

Similarly, for the even case $n = 2m$, if m is odd we have

$$\begin{aligned} P_{2m}^{(2)} - P_{m-1}P_{m+1} &= (-1)^{m+1} \\ P_{2m}^{(2)} &= P_{m-1}(2P_m + P_{m-1}) + (-1)^{m+1} \\ P_{2m}^{(2)} &= 2P_m P_{m-1} + (P_{m-1})^2 + (-1)^{m+1} \\ P_{2m}^{(2)} &= 2P_{2m-1}^{(2)} + P_{2m-2}^{(2)} + 1. \end{aligned}$$

If m is even we have

$$\begin{aligned} P_{2m}^{(2)} - P_{m-1}P_{m+1} &= (-1)^{m+1} \\ P_{2m}^{(2)} &= P_{m-1}(2P_m + P_{m-1}) + (-1)^{m+1} \\ P_{2m}^{(2)} &= 2P_m P_{m-1} + (P_{m-1})^2 + (-1)^{m+1} \\ P_{2m}^{(2)} &= 2P_{2m-1}^{(2)} + P_{2m-2}^{(2)} - 1. \end{aligned}$$

Finally,

$$P_{2m}^{(2)} = \begin{cases} 2P_{2m-1}^{(2)} + P_{2m-2}^{(2)} + 1 & \text{if } m \text{ is odd,} \\ 2P_{2m-1}^{(2)} + P_{2m-2}^{(2)} - 1 & \text{if } m \text{ is even.} \end{cases}$$

We can give a shorter recurrence relation for $Q_n^{(2)}$

$$Q_{2m+1}^{(2)} = 2Q_{2m}^{(2)} + Q_{2m-1}^{(2)}, \text{ for } n = 2m+1 \text{ and}$$

$$Q_{2m}^{(2)} = \begin{cases} 2Q_{2m-1}^{(2)} + Q_{2m-2}^{(2)} + 8 & \text{if } m \text{ is odd,} \\ 2Q_{2m-1}^{(2)} + Q_{2m-2}^{(2)} - 8 & \text{if } m \text{ is even.} \end{cases}, \text{ for } n = 2m.$$

Theorem 6. *The generating function $G_n^{(2)}(x)$ for $P_n^{(2)}$ is given by*

$$G_n^{(2)}(x) = \frac{x^2}{1 - 2x - 2x^3 - x^4}.$$

Proof. Let $G_n^{(2)}(x)$ be as the following function

$$G_n^{(2)}(x) = \sum_{n=0}^{\infty} P_n^{(2)} x^n. \tag{2.3}$$

Hence, we can write

$$2xG_n^{(2)}(x) = \sum_{n=1}^{\infty} P_{n-1}^{(2)} x^n, \tag{2.4}$$

$$2x^3 G_n^{(2)}(x) = \sum_{n=3}^{\infty} P_{n-3}^{(2)} x^n, \tag{2.5}$$

$$x^4 G_n^{(2)}(x) = \sum_{n=4}^{\infty} P_{n-4}^{(2)} x^n. \tag{2.6}$$

Operations (2.3) − {(2.4) + (2.5) + (2.6)} and using Theorem 5 give

$$\begin{aligned} & \left(1 - 2x - 2x^3 - x^4\right) G_n^{(2)}(x) \\ = & \left(P_0^{(2)} + P_1^{(2)} x + P_2^{(2)} x^2 + P_3^{(2)} x^3\right) - 2x\left(P_0^{(2)} + P_1^{(2)} x + P_2^{(2)} x^2\right) \\ & -2\left(P_0^{(2)} x^3\right) + \left(\sum_{n=4}^{\infty} P_n^{(2)} - 2P_{n-1}^{(2)} - 2P_{n-3}^{(2)} - P_{n-4}^{(2)}\right) x^n \\ = & \; x^2. \end{aligned}$$

Hence the generating function is

$$G_n^{(2)}(x) = \frac{x^2}{1 - 2x - 2x^3 - x^4}$$

which completes the proof. □

Theorem 7. *The generating function $G_n^{(2)}(x)$ for $Q_n^{(2)}$ is given by*

$$G_n^{(2)}(x) = \frac{4\left(1 - x - x^2 - x^3\right)}{1 - 2x - 2x^3 - x^4}.$$

Proof. Let $G_n^{(2)}(x)$ be as the following function

$$G_n^{(2)}(x) = \sum_{n=0}^{\infty} Q_n^{(2)} x^n. \tag{2.7}$$

From Theorem 5, we have

$$2xG_n^{(2)}(x) = \sum_{n=1}^{\infty} Q_{n-1}^{(2)} x^n, \tag{2.8}$$

$$2x^3 G_n^{(2)}(x) = \sum_{n=3}^{\infty} Q_{n-3}^{(2)} x^n, \tag{2.9}$$

$$x^4 G_n^{(2)}(x) = \sum_{n=4}^{\infty} Q_{n-4}^{(2)} x^n. \tag{2.10}$$

Operations (2.7) − {(2.8) + (2.9) + (2.10)} and using Theorem 5 give

$$\left(1-2x-2x^3-x^4\right) G_n^{(2)}(x) = 4\left(1-x-x^2-x^3\right).$$

Hence the generating function is

$$G_n^{(2)}(x) = \frac{4\left(1-x-x^2-x^3\right)}{1-2x-2x^3-x^4}$$

which ends the proof. □

3. Conclusion

In this study we mainly obtained new families of Pell and Pell-Lucas numbers. Also we gave some properties for these new families.

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