Restricted arc-connectivity of digraphs*

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Abstract: Let $D = (V, A)$ be a strongly connected digraph. D is called λ'-optimal, if its restricted arc-connectivity equals the minimum arc degree. In this paper, we give some sufficient conditions for digraphs to be λ'-optimal.

Key words : Fault tolerance; Restricted arc-connectivity; λ'-optimal

1 Introduction

Throughout this paper only undirected connected graphs (strongly connected digraphs) without loops and multiple edges (arcs) are considered. Unless stated otherwise, we follow Bondy and Murty [1] for terminology and definitions.

Let $G = (V, E)$ be a connected graph. The *edge − connectivity* $\lambda(G)$ of G is the cardinality of a minimum edge cut S of G. The edge connectivity $\lambda(G)$ is an important feature for determining reliability and fault-tolerance of networks [4, 5, 6]. In the definition of $\lambda(G)$, no restrictions are imposed on the components of $G - S$. Hence, restricted edge connectivity was proposed in [7, 8].

An edge set $S \subset E$ is said to be a *restricted edge cut*, if $G - S$ is disconnected and every component of $G - S$ has at least two vertices. The

*The research is supported by NSFC (No. 11301440, 11171279,11101345), Xiamen University of Technology (YKJ12030R), the Foundation to the Educational Committee of Fujian (JA13240).

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restricted edge connectivity of G, denoted by $\lambda'(G)$, is the cardinality of a minimum restricted edge cut of G. Esfahanian and Hakimi proved the existence of restricted edge cuts and an upper bound for the restricted edge connectivity:

Theorem 1.1. *[7] For any connected graph G with at least four vertices which is not isomorphic to the star $K_{1,n-1}$, $\lambda'(G)$ is well defined. Furthermore, $\lambda'(G) \leq \xi(G)$.*

G is said to be *λ'-optimal*, if $\lambda'(G) = \xi(G)$, where $\xi(G) = \min\{d(x) + d(y) - 2 : xy \in E\}$ is the minimum edge degree of G. The restricted edge connectivity is the generalization of the classical edge connectivity and has received much attention in recent years (see, for example, [13, 2, 14, 18, 9, 10, 11]).

We consider $D = (V, A)$ to be a digraph without loops and parallel arcs. For a vertex $v \in V$ we denote the indegree, the outdegree of v, the minimum indegree and the minimum outdegree in D by $d_D^-(v), d_D^+(v)$ (simply $d^-(v), d^+(v)$), $\delta^-(D), \delta^+(D)$, respectively. We denote the minimum degree of D by $\delta(D) = min\{\delta^-(D), \delta^+(D)\}$, and $d(v) = min\{d^-(v), d^+(v)\}$. Moreover, for $S \subset V$, $D - S$ denotes the subdigraph of D induced by the vertex set of $V \setminus S$. For $v \in V$ let $N_D^+(v)$ be the set of out-neighbors and $N_D^-(v)$ the set of in-neighbors of v. For $X, Y \subseteq V$, (X, Y) denotes the set of all arcs with tail in X and head in Y, and $[X, Y]$ the set of edges of G with one end in X and the other in Y. If $X = V \backslash Y$, then denote $\partial^+(X) = (X, Y)$ or $\partial^-(Y) = (X, Y)$ A triangle is the subgraph of D which is isomorphic to K_3 in the underlying graph of D.

The concept of restricted arc-connectivity was introduced by Volkmann [15]. Let D be a strongly connected digraph. An arc set S of D is a *restricted arc cut* of D if $D - S$ has a non-trivial strong component D_1 such that $D - V(D_1)$ contains an arc. The *restricted arc connectivity* $\lambda'(D)$ is the minimum cardinality over all restricted arc cuts S. A strongly connected digraph D is called λ'-connected, if $\lambda'(D)$ exists. Let D be a digraph with finite girth $g = g(D)$. If $C_g = u_1u_2 \cdots u_gu_1$ is a g-cycle, then let $\xi(C_g) = min\{\sum_{i=1}^{g} d^+(u_i) - g, \sum_{i=1}^{g} d^-(u_i) - g\}$ and $\xi(D) = min\{\xi(C_g) : C_g$ is a g-cycle$\}$. Volkmann proved that each strong digraph D of order $n \geq 4$ and girth $g = 2$ or $g = 3$ except some families of digraphs is λ'-connected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$.

Wang and Lin presented the concept of minimum arc degree [17]. If xy is an arc with $yx \notin A(D)$, then call $\xi'(xy) = min\{d^+(x)+d^+(y)-1, d^-(x)+d^-(y)-1, d^+(y)+d^-(x), d^+(x)+d^-(y)-1\}$ the arc degree of xy. If xy is an arc with $yx \in A(D)$, then call $\xi'(xy) = min\{d^+(x)+d^+(y)-2, d^-(x)+d^-(y)-2, d^+(y)+d^-(x)-1, d^+(x)+d^-(y)-1\}$ the arc degree of xy. The *minimum arc degree* of D is $\xi'(D) = min\{\xi'(xy) : xy \in A(D)\}$.

Theorem 1.2. *[17] Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-connected and $\lambda'(D) \leq \xi'(D)$.*

In the same paper the authors propose $\xi'(D) \leq \xi(D)$ for various digraphs. A λ'-connected digraph is called *λ'-optimal* if $\lambda'(D) = \xi'(D)$.

In this paper, we give some sufficient conditions for digraphs to be λ'-optimal.

2 Sufficient conditions for λ'-optimal digraphs

We start this section with a simple, but very useful lemma.

Lemma 2.1. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If there is a minimum restricted arc cut $S = \partial^+(X)$ such that there exists an arc xy in $D[X]$ and $yx \in A(D)$ with the property that*

$$|(X\backslash\{x,y\},\overline{X})| \geq |(N^+(x)\cap X)\backslash(N^+(y)\cup\{y\})| + 2|N^+(x)\cap N^+(y)\cap X| + |(N^+(y)\cap X)\backslash(N^+(x)\cup\{x\})|,$$

then D is λ'-optimal.

Proof. The hypotheses implies

$$\begin{aligned}
\xi'(G) &\leq d^+(x)+d^+(y)-2 \\
&= |N^+(x)\backslash\{y\}| + |N^+(y)\backslash\{x\}| \\
&= |(N^+(x)\cap X)\backslash\{y\}| + |N^+(x)\cap\overline{X}| + |(N^+(y)\cap X)\backslash\{x\}| \\
&\quad + |N^+(y)\cap\overline{X}| \\
&= |(\{x,y\},\overline{X})| + |(N^+(x)\cap X)\backslash(N^+(y)\cup\{y\})| + \\
&\quad |(N^+(y)\cap X)\backslash(N^+(x)\cup\{x\})| + 2|N^+(x)\cap N^+(y)\cap X| \\
&\leq |(\{x,y\},\overline{X})| + |(X\backslash\{x,y\},\overline{X})| \\
&= |(X,\overline{X})| = \lambda'(G).
\end{aligned}$$

Since $\lambda'(G) \leq \xi'(G)$, we deduce that $\lambda'(G) = \xi'(G)$, and thus D is λ'-optimal. □

Remark 1. If yx is not an arc in D, then we also obtain a similar result. Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If there is a minimum restricted arc cut $S = \partial^+(X)$ such that there exists an arc xy in $D[X]$ and $yx \notin A(D)$ with the property that

$$\begin{aligned} |(X\backslash\{x,y\},\overline{X})| \geq & |(N^+(x) \cap X)\backslash(N^+(y) \cup \{y\})| + |(N^+(y) \cap X)\backslash N^+(x)| \cdot \\ & 2|N^+(x) \cap N^+(y) \cap X|, \end{aligned}$$

then D is λ'-optimal.

Proof. The hypotheses implies

$$\begin{aligned} \xi'(G) & \leq d^+(x) + d^+(y) - 1 \\ & = |N^+(x)\backslash\{y\}| + |N^+(y)| \\ & = |(N^+(x) \cap X)\backslash\{y\}| + |N^+(x) \cap \overline{X}| + |N^+(y) \cap X| + |N^+(y) \cap \overline{X}| \\ & = |(\{x,y\},\overline{X})| + |(N^+(x) \cap X)\backslash(N^+(y) \cup \{y\})| + \\ & \quad |(N^+(y) \cap X)\backslash N^+(x)| + 2|N^+(x) \cap N^+(y) \cap X| \\ & \leq |(\{x,y\},\overline{X})| + |(X\backslash\{x,y\},\overline{X})| \\ & = |(X,\overline{X})| = \lambda'(G). \end{aligned}$$

Since $\lambda'(G) \leq \xi'(G)$, we deduce that $\lambda'(G) = \xi'(G)$, and thus D is λ'-optimal.

Remark 2. By symmetry we have a similar result for $\overline{X}$. Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If there is a minimum restricted arc cut $S = \partial^+(\overline{X})$ such that there exists an arc xy in $D[\overline{X}]$ and $yx \in A(D)$ with the property that

$$\begin{aligned} |(\overline{X}\backslash\{x,y\},X)| \geq & |(N^-(x) \cap \overline{X})\backslash(N^-(y) \cup \{y\})| + 2|N^-(x) \cap N^-(y) \cap \overline{X}| \\ & +|(N^-(y) \cap \overline{X})\backslash(N^-(x) \cup \{x\})|, \end{aligned}$$

then D is λ'-optimal. Analogue to Remark 1 the statement holds for $yx \notin A(D)$ as well.

If G is a graph with $\delta(G) \geq 3$, then we have

Corollary 2.2. *[13] Let G be a λ'-connected graph with $\delta(G) \geq 3$. If there is a λ'-cut S with the vertex sets X and $\overline{X}$ of the two components of $G - S$ such that there exists an edge xy in $G[X]$ with the property that*

$$|[X\backslash\{x,y\},\overline{X}]| \geq |(N(x)\cap X)\backslash(N(y)\cup\{y\})| + 2|N(x)\cap N(y)\cap X| + |(N(y)\cap X)\backslash(N(x)\cup\{x\})|,$$

then G is λ'-optimal.

Corollary 2.3. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If there is a minimum restricted arc cut $S = \partial^+(X)$ such that there exists an arc xy in $D[X]$ and $yx \in A(D)$ with the property that each vertex in $[(N^+(x)\cap X)\backslash(N^+(y)\cup\{y\})]\cup[(N^+(y)\cap X)\backslash(N^+(x)\cup\{x\})]$ has at least one out-neighbor in $\overline{X}$, and each vertex in $N^+(x)\cap N^+(y)\cap X$ has at least two out-neighbors in $\overline{X}$, then D is λ'-optimal.*

Corollary 2.4. *[13] Let G be a λ'-connected graph with $\delta(G) \geq 3$. If there is a λ'-cut S with the vertex sets X and $\overline{X}$ of the two components of $G - S$ such that there exists an edge xy in $G[X]$ with the property that each vertex in $[(N(x) \cap X)\backslash(N(y) \cup \{y\})] \cup [(N(y) \cap X)\backslash(N(x) \cup \{x\})]$ has at least one neighbor in $\overline{X}$, and each vertex in $N(x) \cap N(y) \cap X$ has at least two neighbors in $\overline{X}$, then G is λ'-optimal.*

By combining Lemma 2.1 and Remark 2, we obtain

Corollary 2.5. *[17] Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$, and let $S = \partial^+(X)$ be a minimum restricted arc cut. If there is an arc xy in $D[X]$ such that $|N^+(z)\cap\overline{X}| \geq 2$ for any $z \in X\setminus\{x,y\}$, or there is an arc xy in $D[\overline{X}]$ such that $|N^-(z)\cap X| \geq 2$ for any $z \in \overline{X}\backslash\{x,y\}$, then D is λ'-optimal.*

Let D be a strongly λ'-connected digraph and $S = \partial^+(X)$ be a minimum restricted arc cut. Let $X_i = \{x \in X : |N^+(x)\cap\overline{X}| = i\}$, $\overline{X_i} = \{y \in \overline{X} : |N^-(y)\cap X| = i\}, i = 0,1$, and $X_2 = \{x \in X : |N^+(x)\cap\overline{X}| \geq 2\}$, $\overline{X_2} = \{y \in \overline{X} : |N^-(y)\cap X| \geq 2\}$. The strong components of a digraph D can be labeled $D_1, \cdots, D_t$ such that there is no arc from D_j to D_i unless $j < i$ [3]. We call such an ordering an acyclic ordering of the strong components of D.

Lemma 2.6. *[17] Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ and let xy be an arc. Then $\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^-(x) \cup \partial^+(y)$ and $\partial^+(x) \cup \partial^-(y)$ are restricted arc cuts of D.*

Lemma 2.7. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Furthermore, let $S = \partial^+(X)$ be a minimum restricted arc cut, and for all vertices u, v with $uv \notin A(D)$ we have $|N^+(u) \cap N^-(v)| \geq 2$. If $D[X]$ contains no arcs, then D is λ'-optimal.*

Proof. Assume that D is not λ'-optimal and $S = \partial^+(X)$ is a minimum restricted arc cut.

Claim 1. $X_0 = \varnothing$ or $\overline{X_0} = \varnothing$. Suppose there is $x \in X_0$ and $y \in \overline{X_0}$. Then $xy \notin A(D)$ and $|N^+(x) \cap N^-(y)| \geq 2$. Since $N^+(x) \subseteq X$ and $N^-(y) \subseteq \overline{X}$, we have a contradiction.

Assume $X_0 = \varnothing$. Let $D_1, D_2, \cdots, D_t$ be an acyclic ordering of the strong components of $D[\overline{X}]$.

Claim 2. $t \geq 2$. Since $D[X]$ contains no arcs, by the definition of restricted arc cuts, we have $t \geq 2$.

Claim 3. $|X| \geq 2$. For any $y_1 \in D_1$, $y_t \in D_t$, we have $y_t y_1 \notin A(D)$ and $|N^+(y_t) \cap N^-(y_1)| \geq 2$. Since $y_1 \in D_1$, $y_t \in D_t$, $N^-(y_1) \subseteq V(D_1) \cup X$ and $N^+(y_t) \subseteq V(D_t) \cup X$, and so $N^+(y_t) \cap N^-(y_1) \subseteq X$, that is $|X| \geq 2$.

Claim 4. Both D_1 and D_t are not trivial. Suppose that D_1 is trivial and let $V(D_1) = \{y_1\}$. Since D is strong, there exists $x \in X$ such that $xy_1 \in A(D)$. Noting that $\partial^+(x) \cup \partial^-(y_1) \subseteq S$, we have that $\lambda'(D) = |S| \geq \xi'(xy_1) \geq \xi'(D)$, contrary to the assumption. Suppose that D_t is trivial and let $V(D_t) = \{y_t\}$. Since D is strong, there exists $x \in X$ such that $y_t x \in A(D)$. Let $S' = \partial^+(\{y_t, x\})$. By Lemma 2.6, S' is a restricted arc cut of D. For any $x', x'' \in X$, by assumption, we have $|N^+(x') \cap N^-(x'')| \geq 2$, and so $|N^+(x')| \geq 2$. It follows that $|S'| \leq d^+(x) + |X| - 1 < d^+(x) + 2(|X| - 1) \leq |S|$, which is contrary to the minimality of S. Therefore, both D_1 and D_t are not trivial.

This implies that $S'' = (V \backslash (V(D_1)), V(D_1)) = (X, V(D_1))$ is a restricted arc cut of D. Noting that $S'' \subseteq S$, it follows that $S'' = S$ from the minimality of S. Let $y_t \in V(D_t)$. Then for any $y \in V(D_1)$, we have $y_t y \notin A(D)$. By assumption, $|N^+(y_t) \cap N^-(y)| \geq 2$. Combining this with the fact that $N^+(y_t) \cap N^-(y) \subseteq X$, we have $|N^-(y) \cap X| \geq 2$ and so

$|N^-(y) \cap (V \setminus (V(D_1))| \geq 2$. Since D_1 is non-trivial, there exists an arc $y_1 y_1'$ in D_1. By Corollary 2.5, D is λ'-optimal, a contradiction. □

Lemma 2.8. *[17] Let D be a λ'-connected digraph with $\lambda'(D) \leq \xi'(D)$. If D has no minimum restricted arc cut of the form $\partial^+(X)$, where X is a subset of $V(D)$, then D is λ'-optimal.*

Theorem 2.9. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If for all vertices u, v with $uv \notin A(D)$ we have $|N^+(u) \cap N^-(v)| \geq 2$, and for each triangle T there is at least one vertex $v \in V(T)$ such that $d^+(v) \geq \lfloor n/2 \rfloor + 1$, then D is λ'-optimal.*

Proof. Clearly, $\delta(D) \geq 2$. Suppose D is not λ'-optimal. By Lemma 2.8, there is $X \subseteq V(D)$ such that $S = \partial^+(X)$ is a minimum restricted arc cut. Without loss of generality, we assume $|X| \leq \lfloor n/2 \rfloor$.

Similar to Lemma 2.7 we obtain $X_0 = \varnothing$ or $\overline{X_0} = \varnothing$. We assume $X_0 = \varnothing$.

Case 1. $X_1 = \varnothing$.

Subcase 1.1. $D[X]$ contains at least one arc. Then by Corollary 2.5, D is λ'-optimal, a contradiction.

Subcase 1.2. $D[X]$ contains no arcs. Then by Lemma 2.7 D is λ'-optimal, a contradiction.

Case 2. $X_1 \neq \varnothing$.

By Lemma 2.7 $D[X]$ contains at least one arc.

Subcase 2.1. If there is no arc in X_1, then take $u \in X_1$ and $|N^+(u) \cap \overline{X}| = 1$. Since $\delta(D) \geq 2$, there is $v \in X$ such that $uv \in A(D)$. If $vu \in A(D)$, then uv satisfies the condition of Lemma 2.1. Hence D is λ'-optimal, again a contradiction. If $vu \notin A(D)$, then by Remark 1 we also get the desired result.

Subcase 2.2. There is an arc uv in X_1. So $d^+(u), d^+(v) \leq \lfloor n/2 \rfloor$, and for any $y \in N^+(u) \cap N^+(v)$, $d^+(y) \geq \lfloor n/2 \rfloor + 1$. Thus, by Remark 1 on Lemma 2.1 we get the desired result. □

The proof of the following result is similar to the proof of Theorem 3.2 in [14].

Theorem 2.10. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If for all vertices u, v with $uv \notin A(D)$ we have $|N^+(u) \cap$*

$N^-(v)| \geq 2$, and $D[N^+(u) \cap N^-(v)]$ contains a 2-cycle, then D is λ'-optimal.

Proof. Suppose D is not λ'-optimal. By Lemma 2.8, there is $X \subseteq V(D)$ such that $S = \partial^+(X)$ is a minimum restricted arc cut.

Claim 1. $X_0 = \varnothing$ or $\overline{X_0} = \varnothing$. The proof is similar to Lemma 2.7.

We assume $X_0 = \varnothing$. Let $D_1, D_2, \cdots, D_t$ be an acyclic ordering of the strong components of $D[\overline{X}]$.

If $X = X_2$, then, by Corollary 2.5, D is λ'-optimal, a contradiction. Hence

Claim 2. $X_1 \neq \varnothing$. Let $x_1 \in X_1$ and $N^+(x_1) \cap \overline{X} = \{y_1\}$.

Claim 3. $D[X]$ contains at least one arc. By Lemma 2.7 we obtain the result.

Claim 4. $|X| \geq 3$. By Claim 3 $D[X]$ contains an arc. Hence $|X| \geq 2$. Assume $D[X]$ contains an arc xy. If $|X| = 2$, then $\lambda'(D) = |S| \geq \xi'(xy) \geq \xi'(D)$, a contradiction. So $|X| \geq 3$.

Claim 5. $\overline{X_0} = \varnothing$. By the definition of x_1, for any $y \in \overline{X} - y_1$, $x_1 y \notin A(D)$. Then $2 \leq |N^+(x_1) \cap N^-(y)| = |N^+(x_1) \cap N^-(y) \cap X| + |N^+(x_1) \cap N^-(y) \cap \overline{X}| \leq |N^-(y) \cap X| + |N^+(x_1) \cap \overline{X}| = |N^-(y) \cap X| + 1$, we have $|N^-(y) \cap X| \geq 1$.

Similarly, $|\overline{X}| \geq 3$, $D[\overline{X}]$ contains at least an arc.

Claim 6. $|X_1| \geq 2$ and $|\overline{X_1}| \geq 2$.

If $|X_1| = 1$ and $|\overline{X_1}| = 1$. Let $X_1 = \{u\}$ and since $\delta(D) \geq 2$, there is $v \in X$ such that $uv \in A(D)$. According to Corollary 2.5 D is λ'-optimal, a contradiction.

Claim 7. $A(D[X_1]) \neq \varnothing$ and $A(D[\overline{X_1}]) \neq \varnothing$.

If there is no arc in X_1, then for any $u \in X_1$ each vertex $v \in N^+(u) \cap X$ has at least two out-neighbors in $\overline{X}$. Hence, if $vu \in A(D)$, then uv satisfies the condition of Lemma 2.1; if $vu \notin A(D)$, then uv satisfies the condition of Remark 1 to Lemma 2.1, and D is λ'-optimal, a contradiction. Similarly, if there is no arc in $\overline{X_1}$, then for any $v \in \overline{X_1}$ each vertex $u \in N^-(v) \cap \overline{X}$ has at least two in-neighbors in X. So according to Remark 2 on Lemma 2.1 we are done.

So $A(D[X_1]) \neq \varnothing$ and $A(D[\overline{X_1}]) \neq \varnothing$. Set $N_{X_1} = N^+(X_1) \cap \overline{X}$ and $N_{\overline{X_1}} = N^-(\overline{X_1}) \cap X$. Suppose that there exists a vertex $u \in N_{X_1} \cap \overline{X_1}$, then all pairs of vertices v, w, where $v \in X_1 \cap N^-(u)$ and $w \in \overline{X_1}, w \neq u$, $vw \notin A(D)$ and $D[N^+(v) \cap N^-(w)]$ does not contain a 2-cycle, a contradiction.

Consequently $N_{X_1} \cap \overline{X_1} = \varnothing$ and thus each vertex in N_{X_1} has at least two in-neighbors in X. Analogously, we obtain $N_{\overline{X_1}} \cap X_1 = \varnothing$.

If there exist two vertices $u \in N_{\overline{X_1}}$ and $v \in N_{X_1}$ such that $uv \notin A(D)$, then the vertices u', v', where $v' \in X_1, v'v \in A(D)$ and $u' \in \overline{X_1}, uu' \in A(D)$, $v'u' \notin A(D)$ and $D[N^+(v') \cap N^-(u')]$ does not contain a 2-cycle, which is a contradiction. Consequently, there exists each arc uv, where $u \in N_{\overline{X_1}}, v \in N_{X_1}$.

Thus

$$|(N_{\overline{X_1}}, \overline{X})| \geq |\overline{X_1}| + |N_{\overline{X_1}}||N_{X_1}| \tag{1}$$

since $N_{\overline{X_1}} \cap X_1 = \varnothing$. If

$$|X_1| - 2 \leq |\overline{X_1}| + |N_{\overline{X_1}}||N_{X_1}| - 2|N_{\overline{X_1}}|, \tag{2}$$

then we consider an arbitrary arc $uv \in A(D[X_1])$. Then by using (1) and (2) we obtain

$$\begin{aligned}
|(X\backslash\{u,v\}, \overline{X})| &= |(X, \overline{X})| - |(\{u,v\}, \overline{X})| = |(X, \overline{X})| - 2 \\
&\geq 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + |X_1| + |(N_{\overline{X_1}}, \overline{X})| - 2 \\
&\geq 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + |X_1| + \overline{X_1}| + |N_{\overline{X_1}}||N_{X_1}| - 2 \\
&\geq 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + |X_1| + |X_1| - 2 + 2|N_{\overline{X_1}}| - 2 \\
&= 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + 2|X_1| - 4 + 2|N_{\overline{X_1}}| \\
&= 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + 2|X_1\backslash\{u,v\}| + 2|\{u,v\}| - 4 + 2|N_{\overline{X_1}}| \\
&= 2|X\backslash(X_1 \cup N_{\overline{X_1}})| + 2|X_1\backslash\{u,v\}| + 2|N_{\overline{X_1}}| = 2|X\backslash\{u,v\}| \\
&\geq 2|(N^+(u) \cap X)\backslash(N^+(v) \cup \{v\})| + \\
&\quad 2|(N^+(v) \cap X)\backslash(N^+(u) \cup \{u\})| + 2|N^+(u) \cap N^+(v) \cap X| \\
&\geq |(N^+(u) \cap X)\backslash(N^+(v) \cup \{v\})| + 2|N^+(u) \cap N^+(v) \cap X| \\
&\quad + |(N^+(v) \cap X)\backslash(N^+(u) \cup \{u\})|.
\end{aligned}$$

If $vu \in A(D)$, then this is a contradiction to Lemma 2.1. In case $vu \notin A(D)$, we have a contradiction according to Remark 1.

If $|\overline{X_1}| - 2 \leq |X_1| + |N_{\overline{X_1}}||N_{X_1}| - 2|N_{X_1}|$, then we also can get the desired result.

Hence

$$|X_1| - 2 \geq |\overline{X_1}| + |N_{\overline{X_1}}||N_{X_1}| - 2|N_{\overline{X_1}}| + 1, \tag{3}$$

and

$$|\overline{X_1}| - 2 \geq |X_1| + |N_{\overline{X_1}}||N_{X_1}| - 2|N_{X_1}| + 1, \tag{4}$$

Then

$$|\overline{X_1}| + |N_{\overline{X_1}}||N_{X_1}| - 2|N_{\overline{X_1}}| + 3 \overset{(3)}{\leq} |X_1| \overset{(4)}{\leq} |\overline{X_1}| - 3 - |N_{\overline{X_1}}||N_{X_1}| + 2|N_{X_1}|,$$

which implies

$$\begin{aligned} &2|N_{\overline{X_1}}||N_{X_1}| - 2|N_{\overline{X_1}}| - 2|N_{X_1}| + 6 \leq 0 \\ &\Leftrightarrow |N_{\overline{X_1}}||N_{X_1}| - |N_{\overline{X_1}}| - |N_{X_1}| + 3 \leq 0 \\ &\Leftrightarrow (|N_{\overline{X_1}}| - 1)(|N_{X_1}| - 1) + 2 \leq 0, \end{aligned}$$

a contradiction to $|N_{X_1}|, |N_{\overline{X_1}}| \geq 1$. □

Theorem 2.11. *[17] Let D be a connected digraph with order $n \geq 4$. If for all vertices u, v with $uv \notin A(D)$ we have $|N^+(u) \cap N^-(v)| \geq 3$, then D is λ'-optimal.*

Corollary 2.12. *Let D be a connected digraph with order $n \geq 4$. If for all vertices u, v with $uv \notin A(D)$ we have $d^+(u) + d^-(v) \geq n + 1$, then D is λ'-optimal.*

Corollary 2.13. *[16] Let G be a connected graph with order $n \geq 4$. If for all vertices u, v with $uv \notin E(G)$ we have $d(u) + d(v) \geq n + 1$, then G is λ'-optimal.*

Corollary 2.14. *Let D be a connected digraph with order $n \geq 4$. If $\delta(D) \geq (n + 1)/2$, then D is λ'-optimal.*

The following example shows that Theorem 2.10 is independent of Theorem 2.9 and Theorem 2.11.

Example 1. Let H_1 be a copy of the complete graph $K_{4p-6}, p \geq 4$ with $V(H_1) = \{x_1, x_2, \cdots, x_{4p-6}\}$ and let H_2 be a copy of the complete graph K_3 with $V(H_2) = \{y_1, y_2, y_3\}$. We define the vertex set of graph G as the union of $V(H_1), V(H_2)$ and three additional vertices z_1, z_2, w. Apart from $E(H_1), E(H_2)$, the edge set of G contains the edges $z_1z_2, z_ix, i = 1, 2$

for all $x \in V(G) \setminus \{z_1, z_2\}$ and $wx_i, i = 2p-2, 2p-1, \cdots, 4p-6$, and $y_ix_j, i = 1,2,3; j = 1,2,\cdots,2p-4$. Then $n(G) = 4p$. We replace each edge of $E(G)$ by two arcs in opposite directions and denote D the new obtained digraph. $\xi'(G) = d^+(y_1) + d^+(y_2) - 2 = 2(2p) - 2$. In the triangle $y_1y_2y_3y_1$ there is no vertex y_i with $d^+(y_i) \geq \lfloor n/2 \rfloor + 1$ and the nonadjacent vertices w, y_1 only have two common neighbors z_1, z_2. Hence Theorem 2.9 and Theorem 2.11 do not show that D is λ'-optimal. But Theorem 2.10 shows that D is λ'-optimal. The example graph G is obtained by Hellwig [13]. We use it to get digraph D.

The following example shows that Theorem 2.9 is independent of Theorem 2.10 and Theorem 2.11.

Example 2. Let G' be the graph obtained by adding the edges $y_ix_{2p-3}, i = 1,2,3$ and by removal of the edge z_1z_2 in the graph G in Example 1. We replace each edge of $E(G')$ by two arcs in opposite directions and denote D' the new obtained digraph. It is $d(x) \geq \lfloor n(D')/2 \rfloor + 1$ for all $x \neq w, x \in V(D')$. w and y_3 have only two common neighbors and $D[N^+(w) \cap N^-(y_3)]$ does not contain an arc. But Theorem 2.9 shows that D' is λ'-optimal. The example graph G' is obtained by Hellwig [13]. We use it to get digraph D'.

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