# A note on the thickness of $K_{l,m,n}$

**Yan Yang**

Department of Mathematics
Tianjin University, Tianjin 300072, P.R.China
yanyang@tju.edu.cn

**Abstract** The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar spanning subgraphs into which $G$ can be decomposed. In this note, we obtain the thickness of the complete tripartite graph $K_{l,m,n} (l \leq m \leq n)$ for the following cases: (1) $l+m \leq 5$; (2) $l+m$ is even and $n > \frac{1}{2}(l+m-2)^2$; (3) $l+m$ is odd and $n > (l+m-2)(l+m-1)$.

**Keywords** thickness; complete tripartite graph; planar subgraphs decomposition.

## 1 Introduction

The *thickness* $\theta(G)$ of a graph $G$ is the minimum number of planar spanning subgraphs into which $G$ can be decomposed. It is a classical topological invariant of a graph and also has important applications to VLSI design [1]. Determining the thickness of a graph is NP-hard [5], so the results about thickness are few, the only types of graphs whose thicknesses have been determined are hypercubes [4], complete graphs [2] and complete bipartite graphs [3]. It is natural to ask, what are the thicknesses for the complete tripartite graphs. On this problem, as far as the author know, the only result was shown in [7], Poranen proved that $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$. The reader is referred to [6] for more background on the thickness problems.

In this note, we study the thickness of complete tripartite graph $K_{l,m,n}$ (we assume $l \leq m \leq n$ throughout this note), the main result of this note is the following theorem.

**Theorem** *The thickness of $K_{l,m,n}$ is $\lceil \frac{l+m}{2} \rceil$ when $l+m$ is even and $n > \frac{1}{2}(l+m-2)^2$; or $l+m$ is odd and $n > (l+m-2)(l+m-1)$.*

From this theorem and some constructions, we determine the thickness of $K_{l,m,n}$ for $l+m \leq 5$, as follows.

**Corollary** *The thicknesses of $K_{1,1,n}$, $K_{1,2,2}$ and $K_{2,2,2}$ are all one; the thicknesses of $K_{1,2,n}$, $K_{1,3,n}$ and $K_{2,2,n}$ are all two when $n \geq 3$; the thickness of $K_{1,4,n}$ is two when $4 \leq n \leq 12$, three when $n \geq 13$; the thickness of $K_{2,3,n}$ is two when $3 \leq n \leq 12$, three when $n \geq 13$.*

## 2 The proofs of the theorem and the corollary

**Prove the theorem** For the complete tripartite graph $K_{l,m,n}$ with the vertex partition $(X, Y, Z)$, where $X = \{x_1, \ldots, x_l\}$, $Y = \{y_1, \ldots, y_m\}$ and $Z =$

$\{z_1, \ldots, z_n\}$, firstly, we prove

$$\theta(K_{l,m,n}) \leq \lceil \frac{l+m}{2} \rceil \tag{1}$$

by constructing a planar subgraphs decomposition of it with $\lceil \frac{l+m}{2} \rceil$ planar subgraphs as follows:

(1) Arrange all vertices from $Y$ and $Z$ to a line, place a vertex from $X$ on each side of the line, and join both vertices from $X$ to all vertices from $Y$ and $Z$, then we will get a planar subgraph of $K_{l,m,n}$.

(2) Repeat this proceduce with different vertices from $X$, until all of them have been used, then we will get $\lceil \frac{l}{2} \rceil$ planar subgraphs of $K_{l,m,n}$.

(3) Arrange all vertices from $Z$ to a line, place a vertex from $Y$ on each side of the line, and join both vertices from $Y$ to all vertices from $Z$, then we will get the $(\lceil \frac{l}{2} \rceil + 1)$th planar subgraph of $K_{l,m,n}$.

(4) Repeat this procedure with different vertices from $Y$, until all of them have been used, if $l$ and $m$ are both odd, then place the last vertex from $Y$ in the $\lceil \frac{l}{2} \rceil$th planar subgraph such that the last vertex from $X$ on one side of the line and the last vertex from $Y$ on the other side. A planar decomposition of $K_{l,m,n}$ with $\lceil \frac{l+m}{2} \rceil$ planar subgraphs is obtained.

Secondly, we show that $\theta(K_{l,m,n}) \geq \lceil \frac{l+m}{2} \rceil$ when $n$ is sufficiently large in comparison with $l+m$. Because the complete bipartite graph $K_{l+m,n}$ is a subgraph of $K_{l,n,m}$, we have

$$\theta(K_{l,n,m}) \geq \theta(K_{l+m,n}) \tag{2}$$

From [3], we have

$$\lceil \frac{l+m}{2} \rceil \geq \theta(K_{l+m,n}) \geq \lceil \frac{(l+m)n}{2(l+m+n-2)} \rceil \tag{3}$$

in which the upper bound comes from a planar subgraphs decomposition of the complete bipartite graph in [3] and the lower bound follows from the Euler's polyhedron formula. And from (3), we get that the thickness of $K_{l+m,n}$ is $\lceil \frac{l+m}{2} \rceil$ when $l+m$ is even and $n > \frac{1}{2}(l+m-2)^2$; or $l+m$ is odd and $n > (l+m-2)(l+m-1)$. Combine it with (1) and (2), the theorem follows. □

**Prove the corollary** It is trivial to see the graphs $K_{1,1,n}$, $K_{1,2,2}$ and $K_{2,2,2}$ are planar graphs, so their thicknesses are all one. From the theorem, we have $\theta(K_{1,2,n}) = \theta(K_{1,3,n}) = \theta(K_{2,2,n}) = 2$, when $n \geq 3$ and $\theta(K_{1,4,n}) = \theta(K_{2,3,n}) = 3$, when $n \geq 13$. From (2) and (3), we have $\theta(K_{1,4,n}) \geq 2$, when $4 \leq n \leq 12$, and $\theta(K_{2,3,n}) \geq 2$, when $3 \leq n \leq 12$. We construct a planar decomposition of $K_{1,4,12}$ as illustrated in Figure 1, which shows $\theta(K_{1,4,12}) \leq 2$, so we have $\theta(K_{1,4,n}) = 2$ for $4 \leq n \leq 12$. From the planar decomposition of $K_{1,4,12}$ as shown in Figure 1, we regard $y_4$ in $K_{1,4,12}$ as $x_2$ in $K_{2,3,12}$, delete edge $x_1y_4$ in $G_1$ and add edges $y_1y_4, y_2y_4$ and $y_3y_4$ in $G_2$, we will get a planar decomposition of $K_{2,3,12}$ with two planar subgraphs, which shows $\theta(K_{2,3,12}) \leq 2$, so we have $\theta(K_{2,3,n}) = 2$ for $3 \leq n \leq 12$. Summarizing the above, the corollary follows. □

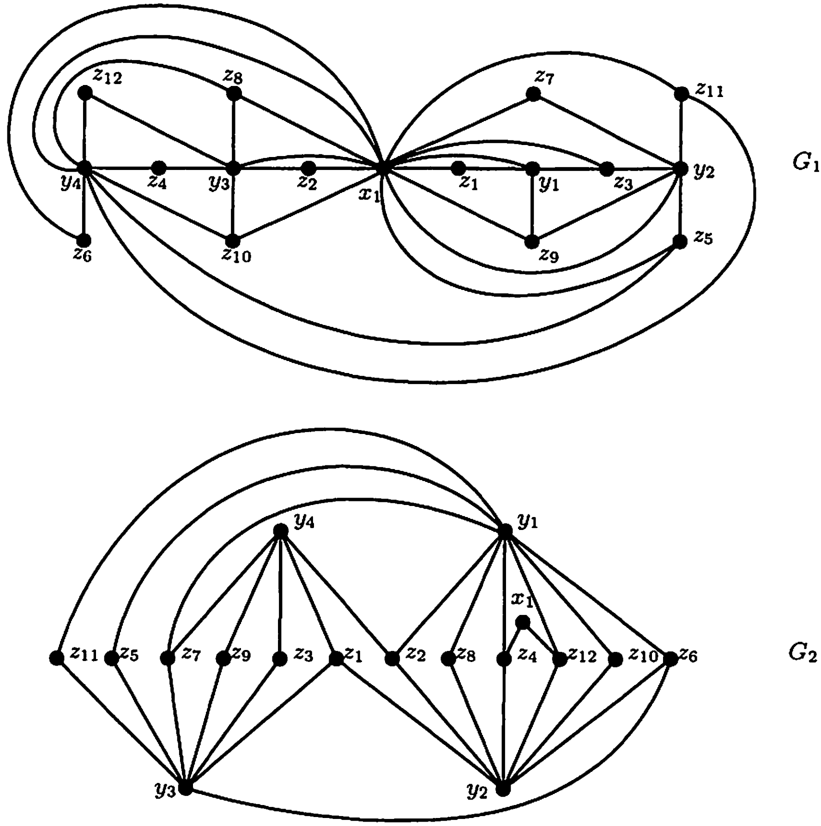


**Figure 1** A planar decomposition of $K_{1,4,12}$

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