Super-λ Connectivity of Bipartite Graphs *

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Abstract: Let $G = (V, E)$ be a connected graph. G is super-λ if every minimum edge cut of G isolates a vertex. Moreover, An edge set $S \subseteq E$ is a restricted edge cut of G if $G - S$ is disconnected and every component of $G - S$ has at least 2 vertices. The restricted edge connectivity of G, denoted by $\lambda'(G)$, is the minimum cardinality of all restricted edge cuts. Let $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$, we say G is λ'-optimal if $\lambda'(G) = \xi(G)$. In this paper, we give a sufficient condition for bipartite graphs to be both super-λ and λ'-optimal.

Key words: restricted edge connectivity; λ'–optimal; super-λ; bipartite graph

1 Introduction

It is well-known that an interconnection network can be modeled by a graph with vertices representing sites and edges representing links between sites of the network. One fundamental consideration in the design of networks is reliability. Let $G = (V, E)$ be a simple undirected connected graph with vertex set $V(G)$, edge set $E(G)$ and the order of G denoted by $n(G)$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v in G and $d_G(v) = |N_G(v)|$ is the *degree* of v in G. Let $N_G[v] = N_G(v) \cup \{v\}$. For an edge $e = uv \in E(G)$, let $\xi_G(e) = d_G(u) + d_G(v) - 2$ be the *edge-degree* of e. We denote the *minimum degree* and the *minimum edge degree*

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of G by $\delta(G)$ and $\xi(G)$ respectively. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X. We denote $V(G)\backslash X$ by $\overline{X}$ for simplicity. For two disjoint vertex sets of G, namely X and Y, let $[X, Y]_G$ denote the set of edges with one endpoint in X and the other one in Y. The subscript G is often omitted when G is understood from the context. A *star* of order n is a complete bipartite graph $K_{1,n-1}$.

An edge set S of G is an *edge-cut* if $G - S$ is disconnected. The *edge-connectivity* $\lambda(G)$ of G is the minimum cardinality of all edge-cuts of G. A classic measure of network reliability is the edge-connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is. Obviously, $\lambda(G) \leq \delta(G)$. A graph G with $\lambda(G) = \delta(G)$ is naturally said to be *maximally edge connected*, or *λ-optimal* for simplicity. A graph G is *super-λ* if every minimum edge cut of G isolates a vertex. Sufficient conditions for graphs to be λ-optimal and super-λ were given by several authors, for example, in the papers by Balbuena et al.[2], Dankelmann et al.[5], Hellwig[8], Meng[12]. The more results on it refer to the survey by Hellwig and Volkmann [9].

Esfahanian and Hakimi [6, 7] propose the concept of restricted edge connectivity. An edge set $S \subseteq E(G)$ is a *restricted edge cut* if $G - S$ is disconnected and every component of $G - S$ has at least 2 vertices. The *restricted edge connectivity* of G, denoted by $\lambda'(G)$, is the minimum cardinality of all restricted edge cuts. If $|S| = \lambda'(G)$, we call S a λ'-cut of G. The restricted edge connectivity $\lambda'(G)$ also has been used to measure the reliability of a network. In [14], Wang and Li proved that the larger $\lambda'(G)$ is, the more reliable the network is. In order to more accurately measure the reliability, λ'-optimal was proposed. In [6], the authors proved that $\lambda'(G) \leq \xi(G)$ holds for any graph G of order at least 4, which is not isomorphic to the star $K_{1,n-1}$. A graph G is *λ'-optimal* if $\lambda'(G) = \xi(G)$. For many results on λ'-optimal of graphs, please refer to [1, 3, 10, 11, 13, 15, 16, 17] and the survey [9].

For a non-complete graph G, let $NC2(G) = \min\{|N(x) \cup N(y)| : x, y \in V(G)$ and $d(x, y) = 2\}$, where $d(x, y)$ is the distance of x and y in G. In [5], Dankelmann and Volkmann obtained a sufficient condition for bipartite graphs to be λ-optimal by condition $NC2(G)$. Inspired by it, in this paper, we want to prove that bipartite graph is both super-λ and λ'-optimal by the condition $NC2(G)$. The idea in our proof follows the lines of the proof of Theorem 4.1 in [11] with modifications needed because of the complexity and with a result of Hellwig and Volkmann, we obtain a sufficient condition for bipartite graphs to be both super-λ and λ'-optimal. The graph terminology and notation not defined here, we follow [4].

2 Main results

Firstly, we give some previous results on λ'-connected graphs, which will be needed in our proof.

Lemma 2.1. *(Hellwig and Volkmann [10]). Let G be a λ'-connected graph. If there is a λ'-cut $[X, \overline{X}]$ such that there exists an edge uv in $G[X]$ with the property that $|[X\backslash\{u,v\}, \overline{X}]| \geq |(N(u)\backslash N[v]) \cap X| + |(N(v)\backslash N[u]) \cap X| + 2|N(u) \cap N(v) \cap X|$, then G is λ'-optimal.*

Let G be a bipartite graph and the two parts of G denoted by V' and V'' respectively, by Lemma 2.1 we can easily obtain the following lemma.

Lemma 2.2. *Let G be a λ'-connected bipartite graph. If there is a λ'-cut $[X, \overline{X}]$ such that there exists an edge uv in $G[X]$ with the property that $|[X\backslash\{u,v\}, \overline{X}]| \geq | (N(u)\backslash\{v\}) \cap X| + |(N(v)\backslash\{u\}) \cap X|$, then G is $\lambda'-$optimal.*

Lemma 2.3. *(Esfahanian and Hakimi [6]) Let G be a connected graph with $n(G) \geq 4$, except for the star $K_{1,n(G)-1}$, is λ'-connected and satisfies $\lambda(G) \leq \lambda'(G) \leq \xi(G)$.*

Lemma 2.4. *(N. Alon Exercise 16.2.13 of [4]) Let $G = G[V', V'']$ be a bipartite graph such that $d(x') \geq 1$ for all $x' \in V'$ and $d(x') \geq d(x'')$ for all $x'x'' \in E(G)$, where $x' \in V'$ and $x'' \in V''$. Then G has a matching covering every vertex of V'.*

Lemma 2.5. *(Dankelmann and Volkmann [5]) Let G be a connected bipartite graph of order $n(G) \geq 3$. If $NC2(G) \geq \lceil \frac{n(G)+1}{4} \rceil$, then $\lambda(G) = \delta(G)$.*

Lemma 2.6. *(Hellwig and Volkmann [11]) If a graph G is $\lambda'-$optimal and $\delta(G) \geq 3$, then G is super-λ.*

In the following, we give a sufficient condition for bipartite graph to be both super-λ and λ'-optmal by condition $NC2(G)$.

Theorem 2.7. *Let $G = G[V', V'']$ be a connected bipartite graph with $n(G) = n \geq 12$ and $\delta(G) \geq 3$. If $NC2(G) \geq \lceil \frac{n+1}{4} \rceil + 2$, then G is both super-λ and λ'-optimal.*

Proof. Since $\delta(G) \geq 3$, G cannot be isomorphic to the star $K_{1,n-1}$. By Lemma 2.3, G is λ'-connected. By Lemma 2.6, we only need to prove that G is λ'-optimal. We may assume that G is not λ'-optimal and let $[X, \overline{X}]$ be a λ'-cut with $3 \leq |X| \leq |\overline{X}|$. This implies that $|X| \leq \lfloor \frac{n}{2} \rfloor$. Let $X' = X \cap V'$ and $X'' = X \cap V''$. Without loss of generality, we may assume that $|X'| \leq |X''|$. This leads to $|X'| \leq \lfloor \frac{n}{4} \rfloor$.

Case 1. $|X'| = \lfloor \frac{n}{4} \rfloor$.

Since $|X| \leq \lfloor \frac{n}{2} \rfloor$, we have that $\lfloor \frac{n}{4} \rfloor \leq |X''| \leq \lfloor \frac{n}{4} \rfloor + 1$ and we consider the following cases.

Subcase 1.1. There exists an edge $e = x'x''$, where $x' \in X'$ and $x'' \in X''$ such that $|N(x') \cap X''| \geq 3$ and $|N(x'') \cap X'| \geq 3$.

In this case, we may assume that $(N(x') \cap X'') \backslash \{x''\} = \{a_1, a_2, \cdots, a_k\}$ where $k \geq 2$.

If $k = 2p$ is even, there exist p pairs of vertices, say $(a_1, a_{2p}), (a_2, a_{2p-1}), \cdots, (a_p, a_{p+1})$ satisfies that

$$\begin{aligned} |N(a_i) \cap \overline{X}| + |N(a_{2p+1-i}) \cap \overline{X}| &\geq |(N(a_i) \cup N(a_{2p+1-i})) \cap \overline{X}| \\ &\geq |N(a_i) \cup N(a_{2p+1-i})| - |X'| \\ &\geq \lceil \frac{n+1}{4} \rceil + 2 - \lfloor \frac{n}{4} \rfloor \\ &\geq 3 \qquad\qquad (1) \end{aligned}$$

for $i = 1, 2, \cdots, p$ by $NC2(G) \geq \lceil \frac{n+1}{4} \rceil + 2$ and $|X'| = \lfloor \frac{n}{4} \rfloor$.

If $k = 2p + 1$ is odd, by $NC2(G) \geq \lceil \frac{n+1}{4} \rceil + 2$ and $|X'| = \lfloor \frac{n}{4} \rfloor$, there are at least one vertex in $\{a_1, a_2, \cdots, a_k\}$, say a_{2p+1}, satisfies that $|N(a_{2p+1}) \cap \overline{X}| \geq 2$. For other vertices $\{a_1, a_2, \cdots, a_{2p}\}$, similar to the above, we also obtain that

$$|N(a_i) \cap \overline{X}| + |N(a_{2p+1-i}) \cap \overline{X}| \geq 3, \qquad i = 1, 2, \cdots, p.$$

Hence, in both cases, we deduce that

$$|[N(x') \cap (X'' \backslash \{x''\}), \overline{X}]| \geq |N(x') \cap (X'' \backslash \{x''\})| \qquad (2)$$

By analogous method, let $N(x'')\cap(X'\backslash\{x'\}) = \{b_1, b_2, \cdots, b_l\}$, where $l \geq 2$. If $|X''| = \lfloor\frac{n}{4}\rfloor$, the similar to the above, we have that

$$|[N(x'') \cap (X'\backslash\{x'\}), \overline{X}]| \geq |N(x'') \cap (X'\backslash\{x'\})| \tag{3}$$

We may assume that $|X''| = \lfloor\frac{n}{4}\rfloor + 1$. Since $NC2(G) \geq \lceil\frac{n+1}{4}\rceil + 2$, we have that for any two vertices b_i, b_j in $\{b_1, b_2, \cdots, b_l\}$

$$\begin{aligned}
|N(b_i) \cap \overline{X}| + |N(b_j) \cap \overline{X}| &\geq |(N(b_i) \cup N(b_j)) \cap \overline{X}| \\
&\geq |N(b_i) \cup N(b_j)| - |X''| \\
&\geq \lceil\frac{n+1}{4}\rceil + 2 - \lfloor\frac{n}{4}\rfloor - 1 \\
&\geq 2
\end{aligned}$$

If $l = 2q$ is even, let $b_j = b_{2q+1-i}$. If $l = 2q + 1$ is odd, there are at least one vertex in $\{b_1, b_2, \cdots, b_l\}$, say b_{2q+1}, satisfies that $|N(b_{2q+1}) \cap \overline{X}| \geq 1$. For other vertices $\{b_1, b_2, \cdots, b_{2q}\}$, the similar to the above, we also obtain that

$$|N(b_i) \cap \overline{X}| + |N(b_{2q+1-i}) \cap \overline{X}| \geq 2, \qquad i = 1, 2, \cdots, p.$$

Hence, in both cases, we deduce that (3) and for an edge $e = x'x'' \in E(G[X])$

$$\begin{aligned}
|[X\backslash\{x', x''\}, \overline{X}]| &\geq |[N(x') \cap (X''\backslash\{x''\}), \overline{X}]| + |[N(x'') \cap (X'\backslash\{x'\}), \overline{X}]| \\
&\geq |N(x') \cap (X''\backslash\{x''\})| + |N(x'') \cap (X'\backslash\{x'\})| \qquad (4)
\end{aligned}$$

By Lemma 2.2, G is λ'-optimal, a contradiction.

Subcase 1.2. For any edge $e = x'x''$, where $x' \in X'$ and $x'' \in X''$, have that $|N(x')\cap X''| \leq 2$ or $|N(x'')\cap X'| \leq 2$. We consider the following cases.

Subcase 1.2.1. There exists an edge $e = x'x''$, where $x' \in X'$ and $x'' \in X''$, such that $|N(x') \cap X''| \geq 3$ and $|N(x'') \cap X'| \leq 2$.

If $|N(x'') \cap (X'\backslash\{x'\})| = 0$ or $|N(x'') \cap (X'\backslash\{x'\})| = |\{v'\}| = 1$ and $|N(v') \cap \overline{X}| \geq 1$, we can obtain the inequalities (3) and by the similar method to the Subcase 1.1, we also have the inequality (4) and we have done.

Now we may assume that $|N(x'') \cap (X'\backslash\{x'\})| = |\{v'\}| = 1$ and $|N(v') \cap \overline{X}| = 0$. If there is a vertex $v'' \in N(x') \cap (X''\backslash\{x''\})$ with $|N(v'') \cap X'| \geq 3$, then we consider the edge $x'v''$. By Subcase 1.1, G is λ'-optimal, a contradiction. Thus, $|N(v'') \cap X'| \le 2$ for all vertices in $v'' \in N(x') \cap (X''\backslash\{x''\})$. Since $|N(x') \cap X''| \ge 3$, there exists a pair of vertices $u, w \in N(x') \cap (X''\backslash\{x''\})$ such that

$$\begin{aligned} |N(u) \cap \overline{X}| + |N(w) \cap \overline{X}| &\ge |(N(u) \cup N(w)) \cap \overline{X}| \\ &= |N(u) \cup N(w)| - |(N(u) \cup N(w)) \cap X'| \\ &\ge |N(u) \cup N(w)| - 3 \\ &\ge \lceil \frac{n+1}{4} \rceil + 2 - 3 \\ &\ge 3. \end{aligned}$$

Hence,

$$\begin{aligned} |[X\backslash\{x', x''\}, \overline{X}]| &\ge |[(N(x') \cap X'')\backslash\{x''\}, \overline{X}]| \\ &\ge |(N(x') \cap X'')\backslash\{x''\}| + 1 \\ &= |(N(x') \cap X'')\backslash\{x''\}| + |(N(x'') \cap X')\backslash\{x'\}| \end{aligned}$$

and so to the desired result.

Subcase 1.2.2. There exists an edge $e = x'x'', x' \in X', x'' \in X''$ such that $|N(x') \cap X''| \le 2$ and $|N(x'') \cap X'| \ge 3$.

This case is similar to subcase 1.2.1.

Subcase 1.2.3. Both $|N(x') \cap X''| \le 2$ and $|N(x'') \cap X'| \le 2$ for all edges $e = x'x''$, where $x' \in X', x'' \in X''$.

Since $\delta \ge 3$, we have that $|N(v') \cap \bar{X}| \ge 1$ and $|N(v'') \cap \bar{X}| \ge 1$ where $v' \in N(x'') \cap (X'\backslash\{x'\})$ and $v'' \in N(x') \cap (X''\backslash\{x''\})$ respectively. Hence, both the inequalities (2) and (3) are hold and we have done.

Case 2. $|X'| \le \lfloor \frac{n}{4} \rfloor - 1$.

If there exists an edge $e = x'x'', x' \in X', x'' \in X''$ such that $|N(x') \cap X''| \ge |N(x'') \cap X'|$ and $|N(x') \cap X''| \ge 3$. Let $N(x') \cap X''\backslash\{x''\} = \{a_1, a_2, \cdots, a_k\}$.

If $k = 2p$ is even, there exist p pairs of vertices, say $(a_1, a_{2p}), (a_2, a_{2p-1}), \cdots, (a_p, a_{p+1})$. Since $NC2(G) \ge \lceil \frac{n+1}{4} \rceil + 2$ and $|X'| \le \lfloor \frac{n}{4} \rfloor - 1$, we have that

$$\begin{aligned}|N(a_i)\cap\bar{X}|+|N(a_{2p+1-i})\cap\bar{X}| &\geq| \ (N(a_i)\cup N(a_{2p+1-i}))\cap\bar{X}| \\ &\geq |N(a_i)\cup N(a_{2p+1-i})| -| X'| \\ &\geq \lceil\frac{n+1}{4}\rceil+2-\lfloor\frac{n}{4}\rfloor+1 \\ &= 4 \ (i=1,2,\cdots,p).\end{aligned}$$

If $k=2p+1$ is odd, by condition $NC2(G)\geq\lceil\frac{n+1}{4}\rceil+2$ and $|X'|\leq\lfloor\frac{n}{4}\rfloor-1$, there are at least one vertex, say a_{2p+1} satisfies that $|N(a_{2p+1})\cap\bar{X}|\geq 2$, and as the above, we have that

$|N(a_i)\cap\bar{X}|+|N(a_{2p+1-i})\cap\bar{X}|\geq\lceil\frac{n+1}{4}\rceil+2-\lfloor\frac{n}{4}\rfloor+1\geq 4, \quad i=1,2,\cdots,p.$

Hence, in both cases, we deduce that

$$|[(N(x')\cap X'')\backslash\{x''\},\bar{X}]|\geq 2|(N(x')\cap X'')\backslash\{x''\}|.$$

Since $|N(x')\cap X''|\geq|N(x'')\cap X'|$, this leads to (4) and G is λ'-optimal, a contradiction.

In the following, we may assume that $|N(x')\cap X''|<|N(x'')\cap X'|$ or $|N(x')\cap X''|\leq 2$ for any edge $x'x''$, where $x'\in X', x''\in X''$ and consider the following cases.

Subcase 2.1. For any edge $e=x'x''$, where $x'\in X', x''\in X''$, have that $2\geq|N(x')\cap X''|\geq|N(x'')\cap X'|$.

If $|N(x')\cap X''|=1$, we have that $|X|=2$, a contradiction. Thus, $|N(x')\cap X''|=2$ and let $\{v''\}=N(x')\cap(X''\backslash\{x''\})$. If there is a vertex $v'\in N(x'')\cap(X'\backslash\{x'\})$ with $|N(v')\cap\overline{X}|=0$, we have that $|N(v')\cap X''|\geq 3$ by $\delta(G)\geq 3$, a contradiction. Thus, $|N(v')\cap\overline{X}|\geq 1$ and the inequality (3) holds. If $|N(v'')\cap\overline{X}|\geq 1$, this leads to inequality (2) and we have down.

Thus, we may assume that $|N(v'')\cap\overline{X}|=0$. But in this case, by $\delta(G)\geq 3$, we have that $|N(v'')\cap X'|\geq 3$, also a contradiction. Hence, inequality (4) holds and so the desire result.

Subcase 2.2. $|N(x')\cap X''|<|N(x'')\cap X'|$ for any edge $e=x'x'', x'\in X', x''\in X''$.

We consider $G[X]$, it is clear that $G[X]$ is connected by $[X,\overline{X}]$ by is a λ'−cut. By Lemma 2.4, $G[X]$ has a matching covering every vertex of X''. Since $\sum_{x'\in X'}d_X(x')=\sum_{x''\in X''}d_X(x'')$, we have that $|X''|<|X'|$, which contradicted with our hypothesis, and we have done.

We consider all the cases, and complete the proof. □

By the proof of Theorem 2.7, we can obtain the following results.

Corollary 2.8. *Let K_{n_1,n_2} be a complete bipartite graph with $n(G) = n \geq 12$. If $min\{n_1, n_2\} \geq \lceil \frac{n+1}{4} \rceil + 2$, then K_{n_1,n_2} is both super-λ and λ'-optimal*

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