Nowhere-zero 3-flows in Tensor Products of Graphs*

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Abstract

The tensor product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined to be the graph with vertex set $\{(x,y) : x \in V(G_1), y \in V(G_2)\}$ and edge set $\{(x_1,y_1)(x_2,y_2) : x_1x_2 \in E(G_1), y_1y_2 \in E(G_2)\}$. Very recently, Zhang, Zheng and Mamut showed that if $\delta(G_1) \geq 2$ and G_2 does not belong to a well-characterized class $\mathcal{G}$ of graphs, then $G_1 \times G_2$ admits a nowhere-zero 3-flow. However, it is unclear whether $G_1 \times G_2$ admits a nowhere-zero 3-flow if $\delta(G_1) \geq 2$ and G_2 does belong to $\mathcal{G}$, especially for the simplest case that $G_2 = K_2$. The main objective in this paper is to show that for any graph G with $2 \leq \delta(G) \leq \Delta(G) \leq 3$, $G \times K_2$ admits a nowhere-zero 3-flow if and only if either every cycle in G contains an even number of vertices of degree 2 or every cycle in G contains an even number of vertices of degree 3. We also extend the sufficiency of the above result to a result for graphs $G \times K_2$, where all odd vertices in G are of degree 3.

Keywords: graph, cycle, nowhere-zero flow, tensor product

*Supported by NIE AcRf funding (RI 5/06 DFM) of Singapore.
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1 Introduction

Let k be any positive integer and G a (simple) graph with vertex set $V(G)$ and edge set $E(G)$. We say that G admits a *nowhere-zero k-flow* if there exists an orientation D of G and a function $f : A(D) \longrightarrow \{\pm i : i = 1, 2, \cdots, k-1\}$ such that for every $x \in V(G)$,

$$\sum_{a \in A^+(x)} f(a) = \sum_{a \in A^-(x)} f(a), \tag{1}$$

where $A(D)$ is the arc set of D and $A^+(x)$ (resp. $A^-(x)$) is the set of arcs in D going out from x (resp. coming into x).

This paper focuses on the study of existence of nowhere-zero 3-flows of tensor products of graphs. For any two graphs G and H, the tensor product of G and H, denoted by $G \times H$, is defined to be the graph with vertex set $\{(x, y) : x \in V(G), y \in V(H)\}$ and edge set $\{(x_1, y_1)(x_2, y_2) : x_1x_2 \in E(G), y_1y_2 \in E(H)\}$. Let $\mathcal{G}$ be the family of graphs defined as follows:

(i) $K_2 \in \mathcal{G}$;

(ii) for any two graphs $G_1, G_2 \in \mathcal{G}$, the graph obtained by adding an edge joining a vertex in G_1 and a vertex in G_2 is also in $\mathcal{G}$.

Very recently, Zhang, Zheng and Mamut [4] showed that if G and H are two connected graphs such that $\delta(G) \geq 2$ and $H \notin \mathcal{G}$, then $G \times H$ admits a nowhere-zero 3-flow, where $\delta(G)$ is the minimum degree of G. They left behind the following problem.

Problem *Characterize G with $\delta(G) \geq 2$ and $H \in \mathcal{G}$ such that $G \times H$ admits nowhere-zero 3-flows.*

In this paper, we study the above problem for the case that $H \cong K_2$ and $\Delta(G) = 3$, where $\Delta(G)$ is the maximum degree of G. While this case looks simple apparently, it actually turns out to be non-trivial. We will characterize all connected graphs G with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ such that $G \times K_2$ admits a nowhere-zero 3-flow. For any integer $i \geq 0$, let $V_i(G) = \{x \in V(G) : d(x) = i\}$, where $d(x)$ is the degree of x. The following is our main result.

Theorem 1 *Let G be any connected graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$. Then $G \times K_2$ admits a nowhere-zero 3-flow if and only if either $|V_3(G) \cap$*

$V(C)|$ *is even for every cycle* C *in* G *or* $|V_2(G) \cap V(C)|$ *is even for every cycle* C *in* G.

The proof of Theorem 1 will be given in Sections 2 and 3, and a generalization of the sufficiency of this result will be presented in Section 4.

2 Sufficiency of Theorem 1

Let G be a simple graph and u, v any two vertices in G. An $u-v$ *walk* in G of length k is a sequence of vertices $u_0, u_1, u_2, \cdots, u_k$ in G, where $u_0 = u$ and $u_k = v$, such that $u_i u_{i+1}$ is an edge in G for all $i = 0, 1, \cdots, k-1$. We denote this walk by $u_0 u_1 u_2 \cdots u_k$. Note that an edge or a vertex may appear in a walk more than once. The walk $u_0 u_1 u_2 \cdots u_k$ is called a *path* if $u_i \neq u_j$ for all $0 \leq i < j \leq k$; a *closed walk* if $u_0 = u_k$; and a *cycle* if it is closed and $u_i \neq u_j$ for all $0 \leq i < j \leq k-1$.

Let x, y denote the vertices in K_2. Note that a walk between vertices (u_0, x) and (u_k, x) in $G \times K_2$ is of the form $(u_0, x)(u_1, y)(u_2, x) \cdots (u_k, x)$, and a walk between vertices (u_0, x) and (u_k, y) in $G \times K_2$ is of the form $(u_0, x)(u_1, y)(u_2, x) \cdots (u_k, y)$, where $u_0 u_1 \cdots u_k$ is a walk in G.

A walk is said to be *even* if its length is even, and *odd* otherwise. Let $\mathcal{W}(G)$ be the set of all walks in G and $\mathcal{W}_1(G)$ the set of walks $u_0 u_1 \cdots u_k$ in G such that $u_i \neq u_j$ whenever $j - i$ is even for any i, j with $0 < j - i < k$.

Lemma 1 *Let* G *be any graph and* $u_0 u_1 \cdots u_k$ *be any walk in* G. *Let* x, y *denote the vertices in* K_2. *Then* $(u_0, x)(u_1, y)(u_2, x) \cdots (u_k, x)$ *is a cycle in* $G \times K_2$ *if and only if* $k \geq 4$ *is even and* $u_0 u_1 u_2 \cdots u_k$ *is a closed walk in* $\mathcal{W}_1(G)$.

Proof. ($\Rightarrow$) Assume that $(u_0, x)(u_1, y)(u_2, x) \cdots (u_k, x)$ is a cycle in $G \times K_2$. It is clear that k is even, $k \geq 4$ and $u_0 u_1 u_2 \cdots u_k$ is a closed walk.

Suppose on the contrary that $u_0 u_1 \cdots u_k \notin \mathcal{W}_1(G)$. Then $u_i = u_j$ for some i, j with $j - i$ even and $0 < j - i < k$. This implies that (u_i, x) and (u_j, x) are the same vertex, and (u_i, y) and (u_j, y) are the same vertex in $G \times K_2$, contradicting the given condition that $(u_0, x)(u_1, y)(u_2, x) \cdots (u_k, x)$ is a cycle.

($\Leftarrow$) Assume that $u_0 u_1 \cdots u_k$ is a closed walk contained in $\mathcal{W}_1(G)$, where k

(≥ 4) is even. Then the following is a closed walk in $G \times K_2$:

$$(u_0, x)(u_1, y)(u_2, x)(u_3, y) \cdots (u_k, x). \tag{2}$$

If it is not a cycle, then there exist i and j with $0 < j - i < k$ such that either (u_i, x) and (u_j, x) are the same vertex in $G \times K_2$ or (u_i, y) and (u_j, y) are the same vertex in $G \times K_2$. Both cases imply that $j - i$ is even and $u_i = u_j$, contradicting the assumption that $u_0 u_1 \cdots u_k \in \mathcal{W}_1(G)$. □

For any walk $W : u_0 u_1 \cdots u_k$ in G, let

$$n_i(W, G) = \begin{cases} |\{0 \leq j \leq k-1 : u_j \in V_i(G)\}|, & \text{if } u_0 = u_k; \\ |\{0 \leq j \leq k : u_j \in V_i(G)\}|, & \text{otherwise.} \end{cases} \tag{3}$$

That is, $n_i(W, G)$ is the number of times that the vertices of $V_i(G)$ appear in W. If W is a path or a cycle, then $n_i(W, G) = |V_i(G) \cap V(W)|$; otherwise, this equality may not be true as some vertices of $V_i(G)$ may appear in W more than once.

Lemma 2 *Let G be any graph and x, y the two vertices in K_2. Let $W : u_0 u_1 u_2 \cdots u_k$ be a walk in G and W' denote the following walk in $G \times K_2$:*

$$\begin{cases} (u_0, x)(u_1, y)(u_2, x) \cdots (u_k, x), & \text{if } k \text{ is even;} \\ (u_0, x)(u_1, y)(u_2, x) \cdots (u_k, y), & \text{otherwise.} \end{cases}$$

Then

(i) *$n_i(W, G) = n_i(W', G \times K_2)$ for any $i \geq 1$;*

(ii) *if $k \geq 4$, k is even and W is a closed walk contained in $\mathcal{W}_1$, then $n_i(W, G) = |V_i(G \times K_2) \cap V(W')|$.*

Proof. (i) The result follows from the fact that $d_G(u) = d_{G \times K_2}((u, x)) = d_{G \times K_2}((u, y))$ holds for all $u \in V(G)$.

(ii) If $k \geq 4$, k is even and W is a closed walk contained in $\mathcal{W}_1$, then W' is a cycle by Lemma 1, implying that $n_i(W', G \times K_2) = |V_i(G \times K_2) \cap V(W')|$. The result now follows from (i). □

We will apply the following result due to Tutte [2] to obtain a necessary and sufficient condition for $G \times K_2$ to admit a nowhere-zero 3-flow.

Theorem 2 ([2]) *A cubic graph admits nowhere-zero 3-flows if and only if it is bipartite.*

Notice that Theorem 2 also holds if G contains multiedges. Thus Theorem 2 can be extended to graphs G with $\Delta(G) \leq 3$ which admit nowhere-zero 3-flows.

Corollary 1 *Let G be any graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$. Then G admits a nowhere-zero 3-flow if and only if $n_3(C, G)$ is even for every cycle C in G.* □

By applying Corollary 1, we can now provide a necessary and sufficient condition for $G \times K_2$ to admit a nowhere-zero 3-flow.

Theorem 3 *Let G be any connected graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$. Then $G \times K_2$ admits a nowhere-zero 3-flow if and only if $n_2(W, G)$ is even for every even closed walk $W \in \mathcal{W}_1(G)$ of length at least 4.*

Proof. By Lemma 1, every cycle C in $G \times K_2$ corresponds to an even closed walk W of length at least 4 contained in $\mathcal{W}_1(G)$. As $2 \leq \delta(G) \leq \Delta(G) \leq 3$, we have

$$\begin{aligned} |V(C)| &= |V(C) \cap V_2(G \times K_2)| + |V(C) \cap V_3(G \times K_2)| \\ &= n_2(W, G) + n_3(C, G \times K_2), \end{aligned}$$

where the second equality follows from Lemma 2. Since C is an even cycle, $n_2(W, G)$ is even if and only if $n_3(C, G \times K_2)$ is even. Thus $n_2(W, G)$ is even for every even closed walk W belonged to $\mathcal{W}_1(G)$ if and only if $n_3(C, G \times K_2)$ is even for every cycle C in $G \times K_2$. Hence the result holds by Corollary 1. □

Let $\mathcal{W}_2(G)$ be the family of closed walks $u_0 u_1 u_2 \cdots u_k$ in G, where $k \geq 3$ and $u_0 = u_k$, such that $u_i \neq u_{i+2}$ for $i = 0, 1, 2, \cdots, k-2$ and $u_{k-1} \neq u_1$. In other words, a closed walk W belongs to $\mathcal{W}_2(G)$ if and only if for every $u_i \in V(W)$, its two neighbours along W are distinct. Thus every even closed walk in $\mathcal{W}_1(G)$ belongs to $\mathcal{W}_2(G)$.

Lemma 3 *Let G be a connected graph. Assume that $n_2(C, G)$ is even for every cycle C in G. Then $n_2(W, G)$ is even for every closed walk $W \in \mathcal{W}_2(G)$.*

Proof. Let $W \in \mathcal{W}_2(G)$. It is clear that $n_2(W, G)$ is even if W is of length 3.

Assume that $n_2(W,G)$ is even if W is of length less than k, where $k \geq 4$. Now let W be of length k.

We need only consider the case that W is not a cycle by the given condition. Write W as $u_0u_1u_2\cdots u_k$, where $k \geq 4$ and $u_0 = u_k$. For convenience, we assume that $u_t = u_{t+k}$ if $-k \leq t \leq -1$ and $u_t = u_{t-k}$ if $k+1 \leq t \leq 2k$.

As W is not a cycle, there exist i, j with $0 \leq i < j < k$ such that $u_i = u_j$ and $u_i, u_{i+1}, \cdots, u_{j-1}$ are distinct. Thus $u_iu_{i+1}\cdots u_j$ is a cycle and we denote it by C. By the definition of $\mathcal{W}_2(G)$, $j \geq i+3$.

Let $s \geq 0$ be the largest integer such that $u_{i-r} = u_{j+r}$ for all $0 \leq r \leq s$. As $W \in \mathcal{W}_2(G)$, we have $u_{i-1} \neq u_{i+1}$ and $u_{j-1} \neq u_{j+1}$. Thus, if $s \geq 1$, then $u_{j+1} \notin \{u_{i-1}, u_{j-1}\}$ and so $d(u_i) = 3$.

Now notice that $k \geq j - i + 2s$. As W is not a cycle, we have $k > j - i$. If $k = j - i + 2s$, then $s > 0$ and u_{i-s+1} and u_{i-s-1} are the same vertex in G, implying that $W \notin \mathcal{W}_2(G)$, a contradiction. Hence $k > j - i + 2s$. It further implies that $d(u_{i-s}) = 3$.

Partition W into two closed walks W_1 and W_2, where

$$W_1 : u_{i-s}u_{i-s+1}\cdots u_{j+s};$$
$$W_2 : u_{i-s}u_{j+s+1}u_{j+s+2}\cdots u_{i-s+k-1}u_{i-s+k}.$$

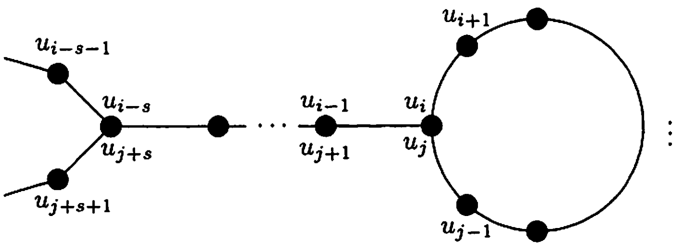

Figure 1

Since $u_{i-s-1} \neq u_{j+s+1}$ and $W \in \mathcal{W}_2(G)$, $W_2 \in \mathcal{W}_2(G)$. By induction, $n_2(W_2,G)$ is even. As $d(u_{i-s}) = d(u_i) = 3$ when $s \geq 1$,

$$n_2(W_1,G) = 2|\{r : d_G(u_r) = 2, i-s+1 \leq r \leq i-1\}| + n_2(C,G) \equiv 0 (\text{mod } 2).$$

Hence $n_2(W,G) = n_2(W_1,G) + n_2(W_2,G) \equiv 0(\text{mod } 2)$. □

Corollary 2 *Let G be a connected graph. Assume that $n_2(C,G)$ is even for every cycle C in G. Then $n_2(W,G)$ is even for every even closed walk $W \in \mathcal{W}_1(G)$.* □

To prove the sufficiency of Theorem 1, we shall also apply the following result due to Zhang, Zheng and Mamut [4].

Theorem 4 ([4]) *If a graph G admits a nowhere-zero k-flow, then $G \times H$ also admits a nowhere-zero k-flow for any graph H.* □

Proof of the sufficiency of Theorem 1: If $|V(C) \cap V_2(G)|$ is even for every cycle C in G, then, by Corollary 2, $n_2(W, G)$ is even for every even closed walk $W \in \mathcal{W}_1(G)$ and thus $G \times K_2$ admits a nowhere-zero 3-flow by Theorem 3.

If $|V(C) \cap V_3(G)|$ is even for every cycle C in G, then, by Corollary 1, G admits a nowhere-zero 3-flow, and so $G \times K_2$ admits a nowhere-zero 3-flow by Theorem 4. □

3 Necessity of Theorem 1

For any graph $G = (V, E)$, two subgraphs of G are said to be *edge-disjoint* if they do not have any edge in common. A *cycle-partition* of G is a family of pairwise edge-disjoint cycles $C_1, C_2, \cdots, C_k$ in G such that

$$\bigcup_{i=1}^{k} E(C_i) = E. \tag{4}$$

A graph is called *an even graph* if every vertex of this graph is of even degree. The following is a well-known characterization for even graphs (see, for example, [1]).

Lemma 4 *A graph $G = (V, E)$ possesses a cycle-partition if and only if G is an even graph.* □

We will strengthen Lemma 4 to the result that every even graph G has a cycle-partition $\{C_i : 1 \le i \le k\}$ such that $|V(C_i) \cap V(C_j)| \le 2$ for every pair $i, j : 1 \le i < j \le k$. Let us first prove the following result.

Lemma 5 *If $G = (V, E)$ is an even graph and $|\{x \in V : d(x) \ge 4\}| \ge 3$, then there exist three pairwise edge-disjoint cycles in G.*

Proof. By Lemma 4, G has a cycle-partition. Since $|\{x \in V : d(x) \geq 4\}| \geq 3$, G contains at least two edge-disjoint cycles, say C_1 and C_2.

If $|V(C_1) \cap V(C_2)| \leq 2$, then $E(C_1) \cup E(C_2) \neq E(G)$, as $|\{x \in V : d(x) \geq 4\}| \geq 3$. It follows that G contains at least one more cycle, and the result holds.

Now assume that $|V(C_1) \cap V(C_2)| \geq 3$. Let u, v be two vertices in $V(C_1) \cap V(C_2)$ such that one $u-v$ path P on C_2 satisfies that $V(P) \cap V(C_1) = \{u, v\}$. Let Q_1 and Q_2 be the two $u-v$ paths on C_1. Since $|V(C_1) \cap V(C_2)| \geq 3$, there exists $w \in (V(C_1) \cap V(C_2)) \setminus \{u, v\}$. Let $w \in V(Q_2)$ and C the cycle formed by P and Q_1.

Let $H = G - E(C)$. Since $w \notin V(C)$, $d_H(w) = d_G(w) \geq 4$. As every vertex of H is of even degree, H contains two edge-disjoint cycles. Hence G contains at least three pairwise edge-disjoint cycles. □

Lemma 6 *Let $G = (V, E)$ be an even graph. If $\{C_i : 1 \leq i \leq k\}$ is a cycle-partition of G with maximum value of k, then $|V(C_i) \cap V(C_j)| \leq 2$ for every pair i, j with $1 \leq i < j \leq k$.*

Proof. Suppose that $|V(C_1) \cap V(C_2)| \geq 3$, without loss of generality. Let H be the subgraph of G induced by $E(C_1) \cup E(C_2)$. Then H is an even graph and H has at least 3 vertices of degree at least 4. By Lemma 5, H contains (at least) three pairwise edge-disjoint cycles. Thus G has a cycle-partition with more than k cycles, a contradiction. □

We now apply Lemma 6 to get a result which will be used in proving the necessity of Theorem 1.

Lemma 7 *Let $G = (V, E)$ be any graph and $U \subseteq V_2(G)$. Assume that $|V(C) \cap U| \equiv 0 (mod\ 2)$ for every even cycle C in G. If there exist two odd cycles C_1 and C_2 in G such that*

$$|V(C_1) \cap U| + |V(C_2) \cap U| \equiv 1 (mod\ 2), \tag{5}$$

then there must exist odd cycles C_1' and C_2' in G such that

$$|V(C_1') \cap V(C_2')| \leq 1 \quad \textit{and} \quad |V(C_1') \cap U| + |V(C_2') \cap U| \equiv 1 (mod\ 2). \tag{6}$$

Proof. Let Φ be the family of $\{C_1, C_2\}$, where C_1 and C_2 are odd cycles in G such that

$$|V(C_1) \cap U| + |V(C_2) \cap U| \equiv 1 (\text{mod } 2). \tag{7}$$

Assume that $\Phi \neq \emptyset$ and let

$$\tau = \min_{\{C_1,C_2\}\in\Phi} |V(C_1) \cap V(C_2)|. \quad (8)$$

Choose $\{C_1, C_2\} \in \Phi$ such that $|V(C_1) \cap V(C_2)| = \tau$.

Let H be the subgraph of G induced by $(E(C_1) \cup E(C_2)) \setminus (E(C_1) \cap E(C_2))$. Notice that H is an even graph as every vertex in H is of degree 2 or 4, and by Lemma 6, H has a cycle-partition $\{C'_i : 1 \leq i \leq k\}$ such that $|V(C'_i) \cap V(C'_j)| \leq 2$ for every pair i, j: $1 \leq i < j \leq k$. As

$$\sum_{i=1}^{k} |E(C'_i)| \equiv |E(C_1)| + |E(C_2)| \equiv 0 (\text{mod } 2), \quad (9)$$

the number of odd cycles in $\{C'_i : 1 \leq i \leq k\}$ is even. Observe that

$$\begin{aligned} & |V(C_1) \cap U| + |V(C_2) \cap U| \\ = \; & |(V(C_1) \setminus V(C_2)) \cap U| + |(V(C_2) \setminus V(C_1)) \cap U| \\ & +2|V(C_1) \cap V(C_2) \cap U| \\ = \; & \sum_{i=1}^{k} |V(C'_i) \cap U| + 2|V(C_1) \cap V(C_2) \cap U|. \end{aligned}$$

Thus

$$\sum_{i=1}^{k} |V(C'_i) \cap U| \equiv |V(C_1) \cap U| + |V(C_2) \cap U| \equiv 1 (\text{mod } 2). \quad (10)$$

Since $|V(C) \cap U| \equiv 0(\text{mod } 2)$ for every even cycle C of G, by (9) and (10), there must exist i, j with $1 \leq i < j \leq k$ such that $\{C'_i, C'_j\} \in \Phi$. This implies that $\tau \leq 2$.

If $\tau = 2$ and $E(C_1) \cap E(C_2) \neq \emptyset$, then H itself is an even cycle, and by (10), $|V(H) \cap U|$ is odd, contradicting the condition that $|V(C) \cap U| \equiv 0(\text{mod } 2)$ for every even cycle C in G.

If $\tau = 2$ and $E(C_1) \cap E(C_2) = \emptyset$, then H actually consists of four paths between the two vertices contained in $V(C_1) \cap V(C_2)$. Since both C_1 and C_2 are odd cycles, two of these four paths are of even length and the other two are of odd length. Thus H has a cycle-partition of two even cycles, say D'_1 and D'_2. Notice that

$$|V(D'_1) \cap U| + |V(D'_2) \cap U| = |V(C_1) \cap U| + |V(C_2) \cap U| \equiv 1 (\text{mod } 2), \quad (11)$$

contradicting the condition that $|V(C) \cap U| \equiv 0 (\mathrm{mod}\ 2)$ for every even cycle C of G.

Therefore $\tau \leq 1$ and the result holds. □

By letting $U = V_2(G)$, we obtain the following result by Lemma 7.

Corollary 3 *Let $G = (V, E)$ be any graph. Assume that $n_2(C, G) \equiv 0(mod\ 2)$ for every even cycle C in G. If there exist two odd cycles C_1 and C_2 in G such that*

$$n_2(C_1, G) + n_2(C_2, G) \equiv 1(mod\ 2), \tag{12}$$

then there must exist odd cycles C_1' and C_2' in G such that

$$|V(C_1') \cap V(C_2')| \leq 1 \quad and \quad n_2(C_1', G) + n_2(C_2', G) \equiv 1(mod\ 2). \tag{13}$$

Now we are ready to prove the necessity of Theorem 1.

Proof of Necessity of Theorem 1: Assume that G is a connected graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ and that $G \times K_2$ admits a nowhere-zero 3-flow.

Claim 1: For every even cycle C, $n_2(C, G) \equiv 0(\bmod 2)$.

Suppose that C is an even cycle in G such that $n_2(C, G)$ is odd. Then C is an even closed walk contained in $\mathcal{W}_1(G)$ of length at least 4. As $n_2(C, G)$ is odd, by Theorem 3, $G \times K_2$ does not admit a nowhere-zero 3-flow, a contradiction.

Claim 2: For every two odd cycles C_1 and C_2, $n_2(C_1, G) + n_2(C_2, G) \equiv 0(\bmod 2)$.

Suppose that G contains two odd cycles C_1 and C_2 such that $n_2(C_1, G) + n_2(C_2, G)$ is odd. By Corollary 3, we can assume that C_1 and C_2 have at most one vertex in common. Write C_1 as $x_1 x_2 \cdots x_s x_1$, where $s = |V(C_1)|$, and C_2 as $y_1 y_2 \cdots y_t y_1$, where $t = |V(C_2)|$.

Case 1: $|V(C_1) \cap V(C_2)| = 1$.

Assume that $x_1 = y_1$. Let W be the closed walk: $x_1 x_2 \cdots x_s x_1 y_2 \cdots y_t x_1$. As s and t are odd, $W \in \mathcal{W}_1(G)$. The length of W is $|V(C_1)| + |V(C_2)|$, which is even. Since $n_2(W, G) = n_2(C_1, G) + n_2(C_2, G)$ is odd, by Theorem 3, $G \times K_2$ does not admit a nowhere-zero 3-flow, a contradiction.

Case 2: $|V(C_1) \cap V(C_2)| = 0$.

Let P be a shortest path among all paths between a vertex on C_1 and a vertex in C_2. Without loss of generality, assume that P is between x_1 and y_1. Thus $d(x_1) = d(y_1) > 2$. Write P as $x_1 u_1 \cdots u_k y_1$, where $k = |E(P)|-1$. Let W denote the following closed walk in G formed by C_1, C_2 and P (edges in P are repeated):

$$x_1 x_2 \cdots x_s x_1 u_1 \cdots u_k y_1 y_2 \cdots y_t y_1 u_k \cdots u_1 x_1.$$

Note that W is a closed walk of length $s + 2k + t$. As s and t are odd, $W \in \mathcal{W}_1(G)$ and W is an even closed walk. Since $d(x_1) = d(y_1) > 2$, we have

$$\begin{aligned} n_2(W,G) &= n_2(C_1,G) + n_2(C_2,G) + 2|\{u_i : 1 \le i \le k, d(u_i) = 2\}| \\ &\equiv 1(\bmod 2). \end{aligned}$$

By Theorem 3, $G \times K_2$ does not admit a nowhere-zero 3-flow, a contradiction. Hence Claim 2 holds.

By Claims 1 and 2, the necessity of Theorem 1 holds. □

4 Further result

Let $\mathcal{G}_{3e}$ be the family of connected graphs G such that $V_i(G) = \emptyset$ for all odd integer i with $i \neq 3$. It is clear that $G \in \mathcal{G}_{3e}$ if $2 \le \delta(G) \le \Delta(G) \le 3$.

For a cycle C in G, let $n_e(C,G) = \sum_{i \text{ is even}} n_i(C,G)$. Let $\mathcal{G}'_{3e}$ be the family of graphs G in $\mathcal{G}_{3e}$ such that $n_3(C,G)$ is even for every cycle C in G, and $\mathcal{G}''_{3e}$ the family of graphs G in $\mathcal{G}_{3e}$ such that $n_e(C,G)$ is even for every cycle C in G.

Clearly, for any graph $G \in \mathcal{G}'_{3e} \cup \mathcal{G}''_{3e}$, if $\Delta(G) \le 3$, then $G \times K_2$ admits a nowhere-zero 3-flow by Theorem 1. We shall prove that this result holds without the condition "$\Delta(G) \le 3$".

Theorem 5 *For any $G \in \mathcal{G}'_{3e} \cup \mathcal{G}''_{3e}$, $G \times K_2$ admits nowhere-zero 3-flows.*

Proof. For any graph G, let $w(G) = \sum_{x \in V(G), d(x) > 3} d_G(x)$. We will prove this result by induction on $w(G)$.

Let $G \in \mathcal{G}'_{3e} \cup \mathcal{G}''_{3e}$. If $w(G) = 0$, then $\Delta(G) \le 3$ and so the result holds for G by Theorem 1.

Now assume that $w(G) > 0$. Then there is a graph $H \in \mathcal{G}_{3e}$ with two non-adjacent vertices u, v such that $d_H(u) = 2$, $d_H(v)$ is a positive even number and $H \cdot uv \cong G$, where $H \cdot uv$ denotes the graph obtained from H by identifying u and v.

It is clear that if $G \in \mathcal{G}'_{3e}$ (i.e., $H \cdot uv \in \mathcal{G}'_{3e}$), then $H \in \mathcal{G}'_{3e}$.

Now assume that $G \in \mathcal{G}''_{3e}$ (i.e., $H \cdot uv \in \mathcal{G}''_{3e}$). So $n_e(C, H \cdot uv)$ is even for every cycle C in $H \cdot uv$. Let C' be any cycle in H. If $\{u, v\} \not\subseteq V(C')$, then $n_e(C', H) = V_e(C, H \cdot uv)$ is even, where C is the cycle in $H \cdot uv$ formed by the edge set $E(C')$. If $\{u, v\} \subseteq V(C')$, then the subgraph of $H \cdot uv$ induced by edge set $E(C)$ consists of two cycles, say C_1 and C_2, with one vertex in common. Then $n_e(C', H) = n_e(C_1, H \cdot uv) + n_e(C_2, H \cdot uv)$ is also even. Hence $H \in \mathcal{G}''_{3e}$.

As $w(H) \leq w(G) - 2 < w(G)$, by induction, $H \times K_2$ admits nowhere-zero 3-flows. As $H \cdot uv \times K_2$ can be obtained from $H \times K_2$ by identifying (u, x) with (v, x) and identifying (u, y) with (v, y), $(H \cdot uv) \times K_2$ (i.e., $G \times K_2$) also admits nowhere-zero 3-flows. □

Remark: We do not know whether there exists a graph $G \in \mathcal{G}_{3e} \backslash (\mathcal{G}'_{3e} \cup \mathcal{G}''_{3e})$ such that $G \times K_2$ admits nowhere-zero 3-flows.

Acknowledgement. The authors wish to thank the referees for their suggestions.

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