On the generalized competition index of a regular or almost regular tournament

MIN SOO SIM *
School of Integrated Technology,
Yonsei University,
Incheon 406-840, Korea.

HWA KYUNG KIM †
Department of Mathematics Education,
Sangmyung University,
Seoul 110-743, Korea.

Abstract

For a primitive digraph D of order n and a positive integer m such that $1 \leq m \leq n$, we define the m-competition index of D, denoted by $k_m(D)$, as the smallest positive integer k such that distinct vertices $v_1, v_2, \cdots, v_m$ exist for each pair of vertices x and y and that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leq i \leq m$ in D. In this paper, we study the m-competition index of a regular or almost regular tournament.

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* *E-mail address:* digivice123@naver.com

† corresponding author. This author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A2001154). *E-mail address:* indices@smu.ac.kr

1 Preliminaries and notations

In this paper, we follow the terminology and notation used in [1, 3, 4, 7]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$, and order n. Loops are permitted but multiple arcs are not. A *walk* from x to y in D is a sequence of vertices $x, v_1, \cdots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \cdots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a walk from x to y where $x = y$. A *cycle* is a closed walk from x to y with distinct vertices except for $x = y$. The *length of a walk* W is determined as the number of arcs in W. The notation $x \xrightarrow{k} y$ is used to indicate that there exists a walk from x to y of length k. The notation $x \not\xrightarrow{k} y$ is used to indicate that there is no walk from x to y of length k. The notation $x \to y$ represents arc (x, y).

A digraph D is said to be *strongly connected* if for each pair of vertices x and y in $V(D)$, there exists a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D, denoted by $l(D)$, is the greatest common divisor of the lengths of the cycles in D. If D is a strongly connected digraph of order 1, then D has a loop and $l(D) = 1$. For a strongly connected digraph D, D is *primitive* if $l(D) = 1$.

If D is a primitive digraph of order n, there exists some positive integer k such that a walk exactly of length k exists from each vertex x to each vertex y. The smallest value of k is termed the *exponent* of D and is denoted by exp(D). For a positive integer m where $1 \leq m \leq n$, we define the *m-competition index* of a primitive digraph D, denoted by $k_m(D)$, as the smallest positive integer k such that distinct vertices $v_1, v_2, \cdots, v_m$ exist for each pair of vertices x and y and that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leq i \leq m$ in D.

Kim [7] introduced the m-competition index as a generalization of the competition index presented in [6]. Akelbek and Kirkland [1, 2] introduced the scrambling index of a primitive digraph D, denoted by $k(D)$. In the case of primitive digraphs, the definitions of the scrambling index and competition index are identical. Furthermore, we have $k(D) = k_1(D)$. Recently, Huang and Liu [5] introduced the generalized scrambling index of a primitive digraph D, denoted by $k(D, \lambda, \mu)$. In the case of primitive digraphs, we have $k_m(D) = k(D, 2, m)$.

For a positive integer k and a primitive digraph D, we define the *k-step outneighborhood* of a vertex x as

$$N^+(D^k : x) = \left\{ v \in V(D) | x \xrightarrow{k} v \right\}.$$

We define the *k-step outneighborhood* of a vertex set X as $N^+(D^k : X) = \cup_{x \in X} N^+(D^k : x)$. Further, we define the *k-step common outneighborhood* of vertices x and y as $N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y)$.

Next, we define the *local m-competition index* of vertices x and y as $k_m(D : x, y) = \min\{k : M(D^t : x, y) \geq m \text{ where } t \geq k\}$ and the *local m-competition index* of x as $k_m(D : x) = \max_{y \in V(D)} \{k_m(D : x, y)\}$. Hence, we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x, y \in V(D)} k_m(D : x, y).$$

From the definitions of $k_m(D)$, $k_m(D : x)$, and $k_m(D : x, y)$, we have $k_m(D : x, y) \leq k_m(D : x) \leq k_m(D)$. On the basis of the definitions of the m-competition index and the exponent of D of order n, we can write $k_m(D) \leq \exp(D)$, where m is a positive integer such that $1 \leq m \leq n$. Furthermore, we have $k_n(D) = \exp(\mathrm{D})$ and

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D). \qquad (1)$$

This is a generalization of the scrambling index and exponent.

An *n-tournament* T_n is a digraph with n vertices in which every pair of vertices is joined by exactly one arc. Assigning an orientation to each edge of a complete graph results in a tournament.

Proposition 1. (Moon and Pullman [9]) *An n-tournament T_n is primitive if and only if T_n is irreducible (strongly connected) and $n > 3$.*

For an n-tournament T_n, we define the *score* of vertex x as $s^+(x) = |N^+(T_n : x)|$ and denote $s^-(x) = |\{v|v \to x\}|$. Thus, we have $s^+(x) + s^-(x) = n - 1$. If n is odd and $s^+(x) = \frac{n-1}{2}$ for each vertex x, then T_n is said to be *regular.* If n is even and $s^+(x) = \frac{n}{2}$ or $s^+(x) = \frac{n-2}{2}$ for each vertex x, then T_n is said to be *almost regular.* If $T_n (n > 3)$ is a regular or almost regular tournament, then T_n is termed primitive. In a tournament T_n, we define a *king* as a vertex x such that

$$V(T_n)\backslash\{x\} = N^+(T_n : x) \cup N^+(T_n^2 : x).$$

Proposition 2. (Landau [8]) *Let T_n be a tournament. Every vertex in the maximum score in T_n is a king.*

Definition 3. *Suppose $A, B \subset V(D)$ and $x \in V(D)$. We define the following notations:*
$A \Rightarrow B$: *$a \to b$ for any $a \in A$ and for any $b \in B$.*
$x \Rightarrow B$: *$x \to b$ for any $b \in B$.*
$A \Rightarrow x$: *$a \to x$ for any $a \in A$.*

Lemma 4. (Sim and Kim [10]) *Let T_n be a primitive n-tournament and A be a nonempty subset of $V(T_n)$. Then, $|N^+(T_n : A)| \geq |A|$ and $|N^+(T_n : A) \cap A| \geq |A| - 1$.*

Proof. If $A = V(T_n)$, then $N^+(T_n : A) = V(T_n)$. We have the result. Suppose $A \neq V(T_n)$. Since T_n is strongly connected, we have $|N^+(T_n : A)\backslash A| \geq 1$ and $|A\backslash N^+(T_n : A)| \leq 1$. Therefore, $|N^+(T_n : A)| \geq |A|$ and $|N^+(T_n : A) \cap A| \geq |A| - 1$. This establishes the result. □

Proposition 5. (Sim and Kim [10]) *Let T_n be a primitive n-tourname where $n \geq 5$. For a positive integer m such that $1 \leq m \leq n$, we have*

$$k_m(T_n) \leq m + 2,$$

and the bound is sharp for all $n \geq 5$.

In this paper, we study $k_m(T_n)$, where T_n is regular or almost regular.

2 Main results

Proposition 6. (Tan et al. [11]) *Let T_n be a regular or almost regular tournament of order $n \geq 7$, then* $\exp(T_n) = 3$.

Example 7. Let T_8 be an almost regular 8-tournament whose adjacency matrix is given as

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The adjacency matrix of T_8^3 is thus given as

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we have $\exp(T_8) = 4$.

Example 7 demonstrates that the result of Proposition 6 is not true for $n = 8$. If $n \neq 8$, then we have Theorem 8.

Theorem 8. *Let T_n be a regular or almost regular tournament of order $n \geq 7$ and $n \neq 8$, then* $\exp(T_n) = 3$.

Proof. Since $x \overset{2}{\not\rightarrow} x$ for each vertex x, we have $\exp(T_n) \geq 3$. Suppose that there exists a primitive tournament T_n such that $\exp(T_n) \neq 3$. Then, there exist a pair of vertices u and v such that $u \overset{3}{\not\rightarrow} v$. The subsets are denoted as

$$\begin{aligned} A &= \{x|u \to x \text{ and } v \to x\}, \\ B &= \{x|u \to x \text{ and } x \to v\}, \\ C &= \{x|x \to u \text{ and } x \to v\}. \end{aligned}$$

Hence, we have $A \cup B = N^+(T_n : u)\backslash\{v\}$ and $B \cup C = \{x|x \to v\}\backslash\{u\}$. If $|B| \geq 2$, there exist two vertices p and q in B such that $u \to p \to q \to v$. In other words, $u \overset{3}{\to} v$. Therefore, $|B| \leq 1$. We have

$$|A| \geq s^+(u) - |B \cup \{v\}| \geq \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \geq 1 \quad (2)$$

$$|C| \geq s^-(v) - |B \cup \{u\}| \geq \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \geq 1. \quad (3)$$

Therefore, we have $A \neq \phi$ and $C \neq \phi$.
If there exist a vertex a in A and a vertex x in $B \cup C$ such that $a \to x$, then we have $u \to a \to x \to v$. That is, $u \overset{3}{\to} v$. Therefore, $B \cup C \Rightarrow A$. If there exist a vertex c in C and a vertex y in B such that $y \to c$, then we have $u \to y \to c \to v$. That is, $u \overset{3}{\to} v$. Therefore, we have

$$\begin{aligned} B &\Rightarrow A, \\ C &\Rightarrow A, \\ C &\Rightarrow B. \end{aligned}$$

Case 1. $n = 7$.
T_7 is regular and $s^+(x) = s^-(x) = 3$ for each vertex x. Since $|B \cup C| \geq s^-(v) - 1 = 2$, we have $s^-(a) \geq |\{u, v\} \cup B \cup C| \geq 4$ for $a \in A$. Since this is a contradiction, we have $\exp(T_7) = 3$.

Case 2. $n \geq 9$.
We have $|A| \geq s^+(u) - |B \cup \{v\}| \geq 2$ and $|C| \geq s^-(v) - |B \cup \{u\}| \geq 2$ by (2) and (3). Thus, there exist two vertices c_1 and c_2 in C such that $c_1 \to c_2$. Further, we have $s^+(c_1) \geq |\{u, v, c_2\} \cup A \cup B| = |A \cup B| + 3$ and $|A \cup B| \geq s^+(u) - 1 \geq \frac{n}{2} - 2$. Therefore, $s^+(c_1) \geq \frac{n}{2} + 1$. Since this is a contradiction, we have $\exp(T_n) = 3$.
This establishes the result. □

Lemma 9. *If T_n is a regular or almost regular tournament where $n \geq 7$ and $n \neq 8$, then we have $k_{n-1}(T_n) = 3$.*

Proof. According to Theorem 8, $k_{n-1}(T_n) \leq k_n(T_n) = 3$ For each pair of vertices u and v, we have $u \overset{2}{\not\to} u$ and $v \overset{2}{\not\to} v$. Further, we have $N^+(T_n^2 : u, v) \subset V \backslash \{u, v\}$. Therefore, $|N^+(T_n^2 : u, v)| \leq n - 2$, which implies that $k_{n-1}(T_n) \geq 3$. This establishes the result. □

Lemma 10. *Let T_n be a regular or almost regular tournament where $n \geq 7$. For a positive integer m such that $m \geq \lceil \frac{n-2}{4} \rceil$, we have*

$$k_m(T_n) \geq 2.$$

Proof. Suppose there exists an n-tournament T_n such that $k_m(T_n) = 1$ for a positive integer m $(\geq \lceil \frac{n-2}{4} \rceil)$. Consider the set

$$\Theta = \{(u, v, w) | u \to w \text{ and } v \to w \text{ for } u \neq v\}.$$

Since $k_m(T_n) = 1$, we have

$$|\Theta| \geq 2\binom{n}{2} \cdot m. \tag{4}$$

Case 1. n is odd.
Let $n = 2n' + 1$. Since $s^-(w) = n'$, we have

$$|\Theta| = \binom{n}{1} \cdot 2\binom{n'}{2}.$$

By (4), we have

$$\binom{n}{1} \cdot 2\binom{n'}{2} = \binom{2n'+1}{1} \cdot 2\binom{n'}{2} = |\Theta| \geq 2\binom{2n'+1}{2} \cdot m.$$

Therefore,

$$m \leq \frac{n'-1}{2} = \frac{n-3}{4}.$$

Case 2. n is even.
Let $n = 2n'$. Since $s^-(w) = n'$ or $n' - 1$, we have

$$|\Theta| = \binom{n'}{1} \cdot 2\binom{n'}{2} + \binom{n'}{1} \cdot 2\binom{n'-1}{2}.$$

By (4), we have

$$\binom{n'}{1}\cdot 2\binom{n'}{2}+\binom{n'}{1}\cdot 2\binom{n'-1}{2}=|\Theta|\geq 2\binom{2n'}{2}\cdot m.$$

Therefore,

$$m\leq\frac{(n'-1)^2}{2n'-1}=\frac{(n-2)^2}{4(n-1)}<\frac{n-2}{4}.$$

Since $m\geq\lceil\frac{n-2}{4}\rceil$, a contradiction is observed in all the cases. This establishes the result. □

Lemma 11. *Let T_n be a regular tournament where $n\geq 7$. For a positive integer m such that $m\leq n-4$, we have*

$$k_m(T_n)\leq 2.$$

Proof. Every vertex in T_n has the same score. By Proposition 2, every vertex is a king. Let $n'=\frac{n-1}{2}$ and v be a vertex in T_n. Since $|N^+(T_n:v)|=n'$ and $v\notin N^+(T_n^2:v)$, we have $|N^+(T_n^2:v)\backslash N^+(T_n:v)|=n'$. By Lemma 4,

$$\begin{aligned}&|N^+(T_n^2:v)|\\ =\;&|T^+(T_n^2:v)\backslash N^+(T_n:v)|+|T^+(T_n^2:v)\cap N^+(T_n:v)|\\ \geq\;&n'+(n'-1)=n-2.\end{aligned}$$

For each pair of vertices x and y,

$$\begin{aligned}&|N^+(T_n^2:x,y)|\\ =\;&|N^+(T_n^2:x)|+|N^+(T_n^2:y)|-|N^+(T_n^2:x)\cup N^+(T_n^2:y)|\\ \geq\;&(n-2)+(n-2)-n\\ =\;&n-4\\ \geq\;&m.\end{aligned}$$

This establishes the result. □

Lemma 12. *Let T_n be an almost regular tournament, where $n\geq 8$. For a positive integer m such that $m\leq n-6$, we have*

$$k_m(T_n)\leq 2.$$

Proof. Let $n' = \frac{n}{2}$ and v be a vertex in T_n.

Case 1. $s^+(v) = n'$.
Since v is a king, we have

$$|N^+(T_n^2 : v)\backslash N^+(T_n : v)| = n' - 1.$$

Case 2. $s^+(v) = n' - 1$.
The subsets are denoted as $A = N^+(T_n : v)$, $B = N^+(T_n^2 : v)$ and $C = V(T_n)\backslash A\backslash B\backslash\{v\}$. Note $|A| = s^+(v) = n' - 1$. If there exist two vertices c_1 and c_2 in C, then $c_1 \Rightarrow A \cup \{v\}$ and $c_2 \Rightarrow A \cup \{v\}$. Without loss of generality, we may assume that $c_1 \rightarrow c_2$. Hence, we have $s^+(c_1) > n'$. This is a contradiction. In addition, we have $|C| \leq 1$. Then

$$|N^+(T_n^2 : v)\backslash N^+(T_n : v)| \geq n' - 1.$$

In all cases, by Lemma 4, we have

$$\begin{aligned}
& |N^+(T_n^2 : v)| \\
= \; & |T^+(T_n^2 : v)\backslash N^+(T_n : v)| + |T^+(T_n^2 : v) \cap N^+(T_n : v)| \\
\geq \; & (n' - 1) + (n' - 2) = n - 3.
\end{aligned}$$

For each pair of vertices x and y,

$$\begin{aligned}
& |N^+(T_n^2 : x, y)| \\
= \; & |N^+(T_n^2 : x)| + |N^+(T_n^2 : y)| - | N^+(T_n^2 : x) \cup N^+(T_n^2 : y)| \\
\geq \; & (n - 3) + (n - 3) - n \\
= \; & n - 6 \\
\geq \; & m.
\end{aligned}$$

This establishes the result. □

Let Z_n be a cyclic group of order n and A be a subset of Z_n. The *Caley digraph* is the digraph $\text{Cay}(Z_n, A) = (V, E)$, where $V = Z_n$ and $E = \{(x, y) | y - x \in A\}$.

Example 13. Let F_1 and F_2 be regular $(2n+1)$-tournaments, where $n \geq 3$. These tournaments are defined as follows.

$$\begin{aligned} F_1 &= \mathrm{Cay}\left(Z_{2n+1}, \{1, 2, \cdots, n\}\right), \\ F_2 &= \mathrm{Cay}(Z_{2n+1}, \{2, 4, \cdots, 2n-2\} \cup \{1\}). \end{aligned}$$

For $0 \leq x \leq 2n$, we have

$$\begin{aligned} N^+(F_1^2 : x) &= V(F_1) \backslash \{x, x+1\}, \\ N^+(F_2 : x) &\supset \{x-3, x+1, x+2, x+4\}, \\ N^+(F_2 : x+1) \cup N^+(F_2 : x+2) &= V(F_2) \backslash \{x, x+1\}, \end{aligned}$$

and $x \to (x-3) \to (x+1)$. In addition, for $0 \leq x < y \leq 2n$, we have

$$\begin{aligned} N^+(F_1^2 : 0, 2) &= V(F_1) \backslash \{0, 1, 2, 3\}, \\ N^+(F_2^2 : x) &\supset N^+(F_2 : x+1) \cup N^+(F_2 : x+2) \cup \{(x+1)\} \\ &= V(F_2) \backslash \{x\}, \\ N^+(F_2^2 : x, y) &= V(F_2) \backslash \{x, y\}. \end{aligned}$$

By (1) and Theorem 8, we have

$$3 \leq k_{2n-2}(F_1) \leq k_{2n-1}(F_1) \leq \exp(\mathrm{F}_1) = 3.$$

And by (1) and Lemma 10, we have

$$2 \leq k_{2n-2}(F_2) \leq k_{2n-1}(F_2) = 2.$$

Example 14. Let F_3 be an almost regular $(2n)$-tournament, where $n \geq 5$. This tournament is defined as follows.

$$\begin{aligned} & V(F_3) = Z_{2n}, \\ & E(F_3) \\ = \; & E(\mathrm{Cay}(Z_{2n}, \{1, 2, \cdots, n-1\})) \\ & \cup \{(0, n), (2, n+2), \cdots, (2\left\lfloor \frac{n-1}{2} \right\rfloor, n+2\left\lfloor \frac{n-1}{2} \right\rfloor)\} \\ & \cup \{(n+1, 1), (n+3, 3), \cdots, (n+2\left\lceil \frac{n-1}{2} \right\rceil - 1, 2\left\lceil \frac{n-1}{2} \right\rceil - 1)\} \end{aligned}$$

We have

$$\begin{aligned} N^+(F_3^2:1) &= V(F_3)\backslash\{0,1,2\}, \\ N^+(F_3^2:5) &= V(F_3)\backslash\{4,5,6\}. \end{aligned}$$

Therefore,

$$N^+(F_3^2:1,5) = V(F_3)\backslash\{0,1,2,4,5,6\}.$$

By (1) and Theorem 8, we have

$3 \le k_{2n-5}(F_3) \le k_{2n-4}(F_3) \le k_{2n-3}(F_3) \le k_{2n-2}(F_3) \le \exp(\mathrm{F}_3) = 3.$

Example 15. Let D_4 be a $(2n-1)$-tournament, where $n \ge 5$. This tournament is defined as follows.

$$\begin{aligned} V(D_4) &= Z_{2n-1}, \\ E(D_4) &= E(\mathrm{Cay}(Z_{2n-1}, \{2,4,\cdots,2n-4\} \cup \{1\})) \\ &\quad \cup\{(0,2n-2),(4,0)\}\backslash\{(2n-2,0),(0,4)\}. \end{aligned}$$

Further, let F_4 be an almost regular $(2n)$-tournament, where $n \ge 5$. This tournament is defined as follows.

$$\begin{aligned} V(F_4) &= V(D_4) \cup \{a\}, \\ E(F_4) &= E(D_4) \\ &\quad \cup\{(a,0),(a,2),\cdots,(a,2n-4)\} \\ &\quad \cup\{(3,a),(5,a),\cdots,(2n-3,a)\} \\ &\quad \cup\{(a,1),(2n-2,a)\}. \end{aligned}$$

In the tournament D_4, for $0 \le x \le 2n-3$, we have

$$\begin{aligned} N^+(D_4:x) &\supset \{x+1,x+2\}, \\ N^+(D_4:2n-2) &\supset \{1,3,\cdots,2n-5\}, \end{aligned}$$

and for $0 \le x \le 2n-4$, we have

$$\begin{aligned} N^+(D_4:0) \cup N^+(D_4:1) &\supset V(D_4)\backslash\{0,4\}, \\ N^+(D_4:x+1) \cup N^+(D_4:x+2) &\supset V(D_4)\backslash\{x,x+1\}, \\ N^+(D_4:2n-2) \cup N^+(D_4:0) &\supset V(D_4)\backslash\{0,4,2n-3\}. \end{aligned}$$

Further, for $1 \leq x \leq 2n-2$, we have

$$N^+(D_4 : x) \supset \{x-3, x+4\},$$

and for $1 \leq x \leq 3$ and $4 \leq y \leq 2n-2$, we have

$$\begin{aligned} x &\to (x+4) \to (x+1), \\ y &\to (y-3) \to (y+1). \end{aligned}$$

Also we have $0 \to (2n-2) \to 1$ in D_4. Therefore, for $0 \leq x \leq 2n-4$, we have

$$\begin{aligned} N^+(D_4^2 : x) &\supset V(D_4)\backslash\{x\}, \\ N^+(D_4^2 : 2n-3) &\supset V(D_4)\backslash\{0, 4, 2n-3\}, \\ N^+(D_4^2 : 2n-2) &\supset V(D_4)\backslash\{1, 2n-2\}. \end{aligned}$$

Since D_4 is an induced subdigraph of F_4, for $0 \leq x \leq 2n-4$, we have

$$\begin{aligned} N^+(F_4^2 : x) &\supset V(F_4)\backslash\{x, a\}, \\ N^+(F_4^2 : 2n-3) &\supset V(F_4)\backslash\{0, 4, 2n-3, a\}, \\ N^+(F_4^2 : 2n-2) &\supset V(F_4)\backslash\{1, 2n-2, a\}. \end{aligned}$$

In tournament F_4, for $x = (2n-3)$ or $(2n-2)$, we have $x \to a \Rightarrow \{0, 1, 2, 4\}$, and for $x \in \{0, 2, 4, \cdots, 2n-4\} \cup \{2n-3\}$ and $y \in \{1, 3, 5, \cdots, 2n-5\}$, we have

$$\begin{aligned} x &\to (2n-2) \to a, \\ y &\to (2n-3) \to a, \\ (2n-2) &\to 3 \to a. \end{aligned}$$

Since $N^+(F_4 : a) \supset \{0, 1, 2, 4\}$, we have $N^+(F_4^2 : a) \supset V(F_4)\backslash\{a\}$. For each vertex x in $V(F_4)$, we have

$$N^+(F_4^2 : x) = V(F_4)\backslash\{x\}.$$

Therefore, we have $k_{2n-2}(F_4) = 2$. By (1) and Lemma 10, we also have

$$2 \leq k_{2n-5}(F_4) \leq k_{2n-4}(F_4) \leq k_{2n-3}(F_4) \leq k_{2n-2}(F_4) = 2.$$

Theorem 16. *Let $n \geq 7$ and $n \neq 8$.*

(i) *If T_n is regular and*

(a) $m \leq \frac{n-3}{4}$, *then* $k_m(T_n) \leq 2$.
(b) $\frac{n-1}{4} \leq m \leq n-4$, *then* $k_m(T_n) = 2$.
(c) $n-3 \leq m \leq n-2$, *then* $k_m(T_n) = 2$ *or* 3.
(d) $n-1 \leq m \leq n$, *then* $k_m(T_n) = 3$.

(ii) *If T_n is almost regular and*

(a) $m \leq \frac{n-4}{4}$, *then* $k_m(T_n) \leq 2$.
(b) $\frac{n-2}{4} \leq m \leq n-6$, *then* $k_m(T_n) = 2$.
(c) $n-5 \leq m \leq n-2$, *then* $k_m(T_n) = 2$ *or* 3.
(d) $n-1 \leq m \leq n$, *then* $k_m(T_n) = 3$.

Proof. By Theorem 8, Lemma 9, Lemma 10, Lemma 11, and Lemma 12, we have the bound. By Example 13, Example 14, and Example 15, we have the following.

(i) if $n-3 \leq m \leq n-2$, $k_m(F_1) = 3$ and $k_m(F_2) = 2$.

(ii) if $n-5 \leq m \leq n-2$, $k_m(F_3) = 3$ and $k_m(F_4) = 2$.

This establishes the result. □

3 Closing remark

Akelbek and Kirkland [1] provided the concept of the scrambling index of a primitive digraph. Kim [7] introduced a generalized competition index $k_m(D)$ as another generalization of the exponent $\exp(D)$ and scrambling index $k(D)$ for a primitive digraph D. Sim and Kim [10] studied $k_m(T_n)$ as an extension of the results presented in [6, 9]. In this paper, we study $k_m(T_n)$ where T_n is a regular or almost regular tournament.

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