End-completely-regular and End-inverse joins of graphs *

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Abstract

A graph X is said to be End-completely-regular (End-inverse) if its endomorphism monoid $End(X)$ is completely regular (inverse). In this paper, we will show that if $X+Y$ is End-completely-regular, then both X and Y are End-completely-regular. We give several approaches to construct new End-completely-regular graphs by means of the join of two graphs with certain conditions. In particular, determine the End-completely-regular joins of bipartite graphs. We also prove that $X+Y$ is End-inverse if and only if $X+Y$ is End-regular and both X and Y are End-inverse. We also determine the End-inverse joins of bipartite graphs.

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1 Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is to develop further relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The bipartite graphs are a class of famous graphs. Its endomorphism monoids are studied by several authors. In [15], the connected

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bipartite graphs whose endomorphism monoids are regular were explicitly found. In [3], Fan gave a characterization of connected bipartite graphs with an orthodox monoid. The bipartite graphs with completely regular endomorphism monoids were characterized in [2]. The joins of bipartite graphs with regular endomorphism monoids were characterized in [8]. The endomorphism monoids and endomorphism-regularity of graphs were considered by several authors (see [6],[7], [10] and [14]). In this paper, we will show that if $X+Y$ is End-completely-regular, then both X and Y are End-completely-regular. We give several approaches to construct new End-completely-regular graphs by means of the join of two graphs with certain conditions. In particular, determine the End-completely-regular joins of bipartite graphs. We also prove that $X+Y$ is End-inverse if and only if $X+Y$ is End-regular and both X and Y are End-inverse. We also determine the End-inverse bipartite graphs and the End-inverse joins of bipartite graphs.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in the graph X, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. For a vertex v of X, denote by $N_X(v)$ (or briefly by $N(v)$) the set $\{x \in V(X) | \{x, v\} \in E(X)\}$ and call it the *neighborhood* of v in X. A subgraph H is called an *induced subgraph* of X if for any $a, b \in H$, $\{a, b\} \in H$ if and only if $\{a, b\} \in V(X)$. A graph X is called *bipartite* if X has no odd cycle. It is known that, if a graph X is a bipartite graph, then its vertex set can be partitioned into two disjoint non-empty subsets, such that no edge joins two vertices in the same set. A graph X is called *complete* if for any $a, b \in V(X)$, $\{a, b\} \in E(X)$. We denote by K_n a complete graph with n vertices. A *clique* of a graph X is a maximal complete subgraph of X. A subset $K \subseteq V(X)$ is said to be *complete* if $\{a, b\} \in E(X)$ for any two vertices $a, b \in K$. A subset $S \subseteq V(X)$ is said to be *independent* if $\{a, b\} \notin E(X)$ for any two vertices $a, b \in S$. A graph X is called a *split graph* if its vertex set $V(X)$ can be partitioned into disjoint (non-empty) sets K and S such that K is a complete set and S is an independent set. In this paper, we always assume that a split graph X has a fixed partition $V(X) = K \cup S$, where $K = \{x_1, \cdots, x_n\}$ is a maximum complete set and $S = \{y_1, \cdots, y_m\}$ is an independent set. Since K is a maximum complete set of X, it is easy to see that for any $y \in S$, $0 \leq d_X(y) \leq n-1$. Let X and Y be two graphs. The *join* of X and Y, denoted by $X+Y$, is a graph such that $V(X+Y) = V(X) \cup V(Y)$ and $E(X+Y) = E(X) \cup E(Y) \cup \{\{x_1, x_2\} | x_1 \in V(X), x_2 \in V(Y)\}$.

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is said to be a *half-strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. A homomorphism f from X to itself is called an *endomorphism* of X. The sets of all endomorphisms, half-strong endomorphisms of X are denoted by $End(X)$ and $hEnd(X)$ respectively.

A *proper coloring* of a graph X is a map from $V(X)$ into some finite set of colors such that no two adjacent vertices are assigned the same colors. If X can be properly colored with a set of k colors, then we say that X can be properly k-colored. The least value of k for which X can be properly k-colored is the *chromatic number* of X, and is denoted by $\chi(X)$. We know that if there is a homomorphism from X to Y, then $\chi(X) \leq \chi(Y)$. A *retraction* of a graph X is a homomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ of f to $V(Y)$ is the identity mapping on $V(Y)$. It is known that the idempotents of $End(X)$ are retractions of X. Denote by $Idpt(X)$ the set of all idempotents of $End(X)$. A graph X is *unretractive*, if $End(X) = Aut(X)$. A subgraph Y of X is a *core* of X if Y is unretractive and there is a homomorphism from X to Y. Let X and Y be two graphs. We say X and Y are *homomorphically equivalent* if there is a homomorphism from X to Y, and there is a homomorphism from Y to X. It is known that two graphs X and Y are homomorphically equivalent if and only if their cores are isomorphic.

Let f be an endomorphism of a graph X. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f.

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. An element a of a semigroup S is called *completely regular* if $a = axa$ and $xa = ax$ for some $x \in S$. A semigroup S is called regular (completely regular) if all its elements are regular (completely regular). An *inverse semigroup* is a regular semigroup in which the idempotents commute. A graph X is said to be End-regular (resp., End-completely-regular, End-inverse) if its endomorphism monoid $End(X)$ is regular (resp., completely regular, inverse). Clearly, End-completely-regular and End-inverse graphs are End-regular.

For undefined notations and terminology in this paper the reader is referred to [1,4,5,9]. We list some known results which will be used in the sequel.

Lemma 1.1 ([13]) Let G be a graph and let $f \in End(G)$. Then f is completely regular if and only if $f|_{I_f} \in Aut(I_f)$.

Lemma 1.2 ([2]) Let X be a bipartite graph. Then X is End-completely-regular if and only if X is one of K_1, K_2, P_2, $2K_1$, $2K_2$ and $K_1 \bigcup K_2$.

Lemma 1.3 ([11]) Let X and Y be two graphs. If $X + Y$ is End-regular, then both X and Y are End-regular.

Lemma 1.4 ([8]) Let X and Y be two End-regular graphs. If for any $f \in End(X+Y)$, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, then $X + Y$ is End-regular.

Lemma 1.5 ([12]) Let X be a split graph with $V(X) = K \bigcup S$. Then $End(X)$ is completely regular if and only if $|S| = 1$.

Lemma 1.6 ([8]) Let X and Y be two K_3-free graphs. If both of them are non-bipartite, then for any endomorphism f of $X + Y$, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(X) \subseteq Y$ and $f(Y) \subseteq X$.

2 Main results

Recall that End-regular bipartite graphs are characterized in [15] and End-regular joins of bipartite graphs are determined in [8]. In this section, we shall characterize the End-completely-regular and End-inverse joins of bipartite graphs.

Theorem 2.1 Let X and Y be two graphs. If $X+Y$ is End-completely-regular, then both X and Y are End-completely-regular.

Proof Since $X+Y$ is End-completely-regular, $X+Y$ is End-regular. By Lemma 1.3, both X and Y are End-regular. To show X is End-completely-regular, let $f \in End(X)$. By Lemma 1.1, we only need to prove that $f|_{I_f}$ is an automorphism of I_f.

Now we define a mapping F from $X + Y$ to itself by

$$F(x) = \begin{cases} f(x), & if\ x \in V(X), \\ x, & if\ x \in V(Y). \end{cases}$$

Then it is easy to check that $F \in End(X + Y)$. Since $X + Y$ is End-completely-regular, by Lemma 1.1, $F|_{I_F} \in Aut(I_F)$. Note that $F(x) = f(x) \in V(X)$ for any $x \in V(x)$. Then $F|_{I_f} \in Aut(I_f)$. It follows from $F|_{I_f} = f|_{I_f}$ that $f|_{I_f} \in Aut(I_f)$. By Lemma 1.1, f is completely regular. Hence X is End-completely-regular. A similar argument will show that Y is End-completely-regular.

The following example shows that X and Y being End-completely-regular may not yield that $X+Y$ is End-completely-regular.

Example 2.2 Let X and Y be two graphs with $V(X) = \{x_1, x_2\}$, $V(Y) = \{y_1, y_2\}$ and $E(X) = E(Y) = \phi$. By Lemma 1.2, X and Y are End-completely-regular. It is easy to see that $X + Y \cong C_4$. By Lemma 1.2, it is not End-completely-regular.

In the following, we give some sufficient conditions for $X+Y$ to be End-completely-regular.

Lemma 2.3 Let X and Y be two End-completely-regular graphs. If for any $f \in End(X+Y)$, $f(X) \subseteq X$ and $f(Y) \subseteq Y$, then $X+Y$ is End-completely-regular.

Proof Since X and Y are End-completely-regular, X and Y are End-regular. By Lemma 1.4, $X+Y$ is End-regular.

Let $f \in End(X+Y)$. Denote $f_1 = f|_X$ and $f_2 = f|_Y$. Since $f(X) \subseteq X$ and $f(Y) \subseteq Y$, $f_1 \in End(X)$ and $f_2 \in End(Y)$. Note that X and Y are End-completely-regular. Then $f_1|_{I_{f_1}}$ is an automorphism of I_{f_1} and $f_2|_{I_{f_2}}$ is an automorphism of I_{f_2}. Now $I_f = I_{f_1} + I_{f_2}$. Hence $f|_{I_f}$ is an automorphism of I_f. Consequently, $X+Y$ is End-completely-regular.

Theorem 2.4 Let X and Y be two End-completely-regular graphs. Then

(1) If X and Y are two K_3-free non-bipartite graphs and the cores of X and Y are not isomorphic, then $X+Y$ is End-completely-regular.

(2) If X is a bipartite graph and Y is a K_3-free non-bipartite graph, then $X+Y$ is End-completely-regular.

(3) If X is a K_3-free non-bipartite graph and Y has at least one triangle with $\chi(Y) < \chi(X)+1$, then $X+Y$ is End-completely-regular.

Proof (1) Let $f \in End(X+Y)$. By Lemma 1.6, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(Y) \subseteq X$ and $f(X) \subseteq Y$. In the second case, $f|_X$ is a homomorphism from X to Y and $f|_Y$ is a homomorphism from Y to X. Thus X and Y are homomorphically equivalent and so the cores of X and Y are isomorphic. This is a contradiction. Hence $f(X) \subseteq X$ and $f(Y) \subseteq Y$. By Lemma 2.3, $X+Y$ is End-completely-regular.

(2) Let $f \in End(X+Y)$. We will prove that $f(X) \subseteq X$ and $f(Y) \subseteq Y$. There are two cases.

Case 1. Assume that $E(X) = \phi$. First we show that $f(X) \subseteq X$. Otherwise, there exists a vertex $x_1 \in V(X)$ such that $f(x_1) \in Y$. Since Y is non-bipartite, Y contains an odd cycle. Thus $f(Y)$ also has an odd cycle. As $E(X) = \phi$, then $f(Y)$ has an edge in Y, say $\{f(y_1), f(y_2)\}$. Then $f(y_1), f(y_2), f(x)$ form a triangle in Y. This is a contradiction.

Next we prove $f(Y) \subseteq Y$. Otherwise, there exists $y_1 \in V(X)$ such that $f(y_1) \in X$. Since $\{y_1, x\} \in E(X+Y)$ for any $x \in V(X)$, $\{f(y_1), f(x)\} \in E(X+Y)$. Note that $f(y_1), f(x) \in V(X)$. Then $\{f(y_1), f(x)\} \in E(X)$. This is a contradiction.

Case 2. Assume that $E(X) \neq \phi$. First we show that $f(X) \subseteq X$. Assume that $f(X) \nsubseteq X$. Then either $f(X) \subseteq Y$, or there exist two vertices x_1 and x_2 in $V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$. In the first case, since X contains at least one edge, $f(X)$ contains at least one edge, say $\{a, b\}$. Now we have that $f(Y) \subseteq X$. Otherwise, there exists $y_0 \in V(Y)$ such that $f(y_0) \in V(Y)$, then $a, b, f(y_0)$ form a triangle. This is a contradiction. Hence $f|_Y$ is a homomorphism from Y to X, and we have $\chi(Y) \leq \chi(X)$. Note that $\chi(X) = 2$ and $\chi(Y) \geq 3$. This is a contradiction. In the second case, since Y contains an odd cycle, $f(Y)$ also contains an odd cycle. Thus $f(Y)$ either has an edge in X or has an edge in Y. Without loss of generality, suppose $\{f(y_1), f(y_2)\} \in E(Y)$ for some $y_1, y_2 \in V(Y)$. Note that $\{f(y_1), f(x_2)\} \in E(Y)$ and $\{f(y_2), f(x_2)\} \in E(Y)$. Then $f(x_2), f(y_1), f(y_2)$ form a triangle in Y. This is a contradiction. Hence $f(X) \subseteq X$.

Next we prove that $f(Y) \subseteq Y$. Otherwise, there exists $y_1 \in V(Y)$ such that $f(y_1) \in V(X)$ and $f(y_1)$ is adjacent to every vertex of $f(X)$. Since $f(X)$ contains at least one edge, X contains a triangle. This is a contradiction. Now the assertion follows from Lemma 2.3.

(3) We show that $f(Y) \nsubseteq X$. Otherwise, $f|_Y$ is a homomorphism from Y to X. Note that any homomorphism f maps a triangle to a triangle and Y has at least one triangle. Then X also has at least one triangle. A contradiction. Hence either $f(Y) \subseteq Y$, or there exist two vertices y_1 and y_2 in Y such that $f(y_1) \in Y$ and $f(y_2) \in X$.

In the second case, if $f(X) \subseteq X$, then $f|_X$ is a homomorphism from X to itself, so $\chi(X) = \chi(I_{f|_X})$. Note that $f(y_2)$ is adjacent to every vertex of $I_{f|_X}$, then $\chi(X) \geq \chi(I_{f|_X}) + 1$. A contradiction. If $f(X) \subseteq Y$, then $f|_X$ is a homomorphism from X to Y and $f(y_1)$ is adjacent to every vertex of $I_{f|_X}$, thus $\chi(Y) \geq \chi(I_{f|_X}) + 1 \geq \chi(X) + 1$. A contradiction. If there exist $x_1, x_2 \in V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$, then both $f(X)$ and $f(Y)$ have no edge in X, otherwise, there exists a triangle in X. This is impossible, because X is K_3-free.

Now $f(Y) \subseteq Y$. If $f(X) \nsubseteq X$, then there exists $x \in V(X)$ such that $f(x) \in Y$ and $f(x)$ is adjacent to every vertex in $V(I_{f|_Y})$. Thus we have $\chi(Y) \leq \chi(I_{f|_Y}) + 1 = \chi(Y) + 1$. A contradiction. Hence $f(X) \subseteq X$. By Lemma 2.3, $X + Y$ is End-completely-regular.

The next theorem characterize the End-completely-regular joins of bipartite graphs.

Theorem 2.5 Let X and Y be two bipartite graphs. Then $X+Y$ is End-completely-regular if and only if one of them is End-completely-regular and the other is K_1 or K_2.

Proof Sufficiency. It is easy to see that K_1+K_1, K_1+K_2 and K_2+K_2 are unretractive. Clearly, they are End-completely-regular. K_1+P_2, K_1+2K_1, $K_1+(K_1\bigcup K_2)$, K_2+P_2, K_2+2K_1 and $K_2+(K_1\bigcup K_2)$ are split graphs. By Lemma 1.5, they are End-completely-regular. In the following, we prove that K_1+2K_2 and K_2+2K_2 are End-completely-regular.

Fig. 1: Graphs K_1+2K_2 and K_2+2K_2

Let $f\in End(K_1+2K_2)$. If $I_f=K_1+2K_2$, then $f\in Aut(K_1+2K_2)$ and so it is completely regular; If $I_f\not\cong K_1+2K_2$, then $f(2)=f(4)$ or $f(2)=f(5)$. Without loss of generality, we may suppose $f(2)=f(4)$. Then $f(3)=f(5)$. Otherwise, we have $[3]_{\rho_f}=\{3\}$. Let $A=\{1,4,5\}$. Then the subgraph of K_1+2K_2 induced by A is isomorphic to K_3. Since $\{1,3\}\in E$, $\{f(1),f(3)\}\in E$. Now $\{f(3),f(4)\}=\{f(3),f(2)\}\in E$ implies that $f(3)$ is adjacent to two vertices of $f(A)$. Note that there is no vertex in K_1+2K_2 adjacent to two vertices of a clique. This is a contradiction. Hence in this case $I_f\cong K_3$. Since K_3 is unretractive, $f(I_f)=I_f$, by Lemma 1.1, f is completely regular. Hence K_1+2K_2 is End-completely-regular.

Let $f\in End(K_2+2K_2)$. If $I_f=K_2+2K_2$, then it is completely regular; If $I_f\not\cong K_2+2K_2$, then $f(3)=f(5)$ or $f(3)=f(6)$. Without loss of generality, we may suppose $f(3)=f(5)$. Then $f(4)=f(6)$. Otherwise, we have $[4]_{\rho_f}=\{4\}$. Let $B=\{1,2,5,6\}$. Then the subgraph of K_2+2K_2 induced by B is isomorphic to K_4. Since $\{1,4\}\in E$ and $\{2,4\}\in E$, $\{f(1),f(4)\}\in E$ and $\{f(2),f(4)\}\in E$. Now $\{f(4),f(5)\}=\{f(4),f(3)\}\in E$ implies that $f(4)$ is adjacent to three vertices of $f(A)$. Note that there is no vertex in K_2+2K_2 adjacent to three vertices of a clique of order 4. This is a contradiction. Hence in this case $I_f\cong K_4$. Since K_4 is unretractive, $f(I_f)=I_f$, by Lemma 1.1, f is completely regular. Hence K_2+2K_2 is End-completely-regular.

Necessity. We only need to show that $X+Y$ is not End-completely-regular for the following 10 cases. The main idea of the proof is that, for each cases, we will find an endomorphism $f\in End(X+Y)$ which is not completely regular.

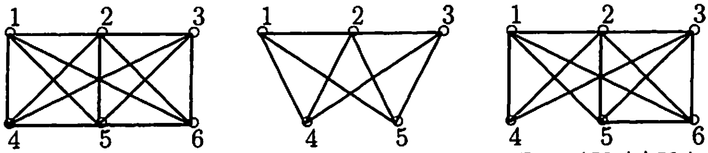

Fig. 2: Graphs $P_2 + P_2$, $P_2 + 2K_1$ and $P_2 + (K_1 \bigcup K_2)$

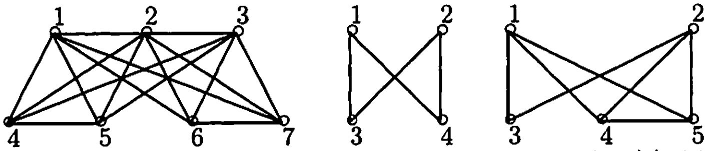

Fig. 3: Graphs $P_2 + 2K_2$, $2K_1 + 2K_1$ and $2K_1 + (K_1 \bigcup K_2)$

Fig. 4: Graphs $2K_1 + 2K_2$ and $2K_2 + (K_1 \bigcup K_2)$

Fig. 5: Graphs $2K_2 + 2K_2$ and $(K_1 \bigcup K_2) + (K_1 \bigcup K_2)$

Case (1) $P_2 + P_2$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

Case (2) $P_2 + 2K_1$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 2 & 1 & 3 \end{pmatrix}$$

Case (3) $P_2 + (K_1 \bigcup K_2)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

Case (4) $P_2 + 2K_2$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 4 & 1 & 2 & 2 & 3 \end{pmatrix}$$

Case (5) $2K_1 + 2K_1$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 2 \end{pmatrix}$$

case (6) $2K_1 + (K_1 \bigcup K_2)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 1 & 4 & 2 \end{pmatrix}$$

Case (7) $2K_1 + 2K_2)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 6 & 5 & 1 & 5 & 2 \end{pmatrix}$$

Case (8) $2K_2 + 2K_2$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Case (9) $2K_2 + (K_1 \bigcup K_2)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 6 & 7 & 1 & 3 & 4 \end{pmatrix}$$

Case (10) $(K_1 \bigcup K_2) + (K_1 \bigcup K_2)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

The proof is completed.

Next we seek the conditions for a join of bipartite graphs $X + Y$ under which $X + Y$ is End-inverse.

Lemma 2.6 Let X and Y be two graphs. If $X + Y$ is End-inverse, then both X and Y are End-inverse.

Proof Since $X+Y$ is End-inverse, $X+Y$ is End-regular. By Lemma 1.3, both X and Y are End-regular. To show X is End-inverse, we only need to prove that the idempotents of $End(X)$ commute.

Let f_1 and f_2 be two idempotents in $End(X)$. Define two mappings g_1 and g_2 from $V(X+Y)$ to itself by

$$g_1(x)=\begin{cases} f_1(x), & x\in V(X), \\ x, & x\in V(Y), \end{cases} \quad \text{and} \quad g_2(x)=\begin{cases} f_2(x), & x\in V(X), \\ x, & x\in V(Y). \end{cases}$$

Then g_1 and g_2 are two idempotents of $End(X+Y)$ and so $g_1g_2=g_2g_1$ since $X+Y$ is End-inverse. For any $x\in V(X)$, we have

$$f_1f_2(x)=f_1(f_2(x))=f_1(g_2(x))=g_1(g_2(x))=g_1g_2(x)=g_2g_1(x)$$

$$=g_2(g_1(x))=g_2(f_1(x))=f_2(f_1(x))=f_2f_1(x).$$

Clearly, $f_1f_2=f_2f_1$. Hence the idempotents of $End(X)$ commute and so X is End-inverse. A similar argument will show that Y is End-inverse.

Theorem 2.7 Let X and Y be two graphs. Then $X+Y$ is End-inverse if and only if

(1) $X+Y$ is End-regular, and

(2) Both X and Y are End-inverse.

Proof Necessity. This follows immediately from Lemma 2.6.

Sufficiency. Since $X+Y$ is End-regular, to show $X+Y$ is End-inverse, we only need to prove that the idempotents of $End(X+Y)$ commute.

Let f be an idempotent of $End(X+Y)$. Then $f(X)\subseteq X$. Otherwise, there exists a vertex $x\in V(X)$ such that $f(X)\subseteq V(Y)$. Since $f^2=f$, then $f(f(x))=f^2(x)=f(x)$. Note that $\{x,f(x)\}\in E(X+Y)$, then $\{f(x),f(x)\}$ is a loop of $X+Y$. A contradiction. A similar argument will show that $f(Y)\subseteq V(Y)$.

If f_1 and f_2 are two idempotents of $End(X+Y)$, let $g_1=f_1|_X$, $g_2=f_1|_Y$, $h_1=f_2|_X$ and $h_2=f_2|_Y$. Then $g_1,h_1\in Idpt(X)$ and $g_2,h_2\in Idpt(Y)$. Since both of X and Y are End-inverse, $g_1h_1=h_1g_1$ and $g_2h_2=h_2g_2$. Now $f_1f_2|_X=g_1h_1$, $f_2f_1|_X=h_1g_1$, $f_1f_2|_Y=g_2h_2$ and $f_2f_1|_Y=h_2g_2$ imply that $f_1f_2=f_2f_1$. Consequently, $X+Y$ is End-inverse.

In the following, we start to characterize the End-inverse joins of bipartite graphs.

Lemma 2.8 Let X be a bipartite graph. Then X is End-inverse if and only if $X=K_1$ or $X=K_2$.

Proof Sufficiency. If $X=K_1$ or $X=K_2$, then X is unretractive and so $End(X)$ is a group. Clearly, X is End-inverse.

Necessity. Let X be a bipartite graph. Then its vertex set can be partitioned into two disjoint non-empty subsets A and B, such that no edge joins two vertices in the same set. We only need to show that $X+Y$ is not End-inverse for the following 3 cases.

Case 1. Assume X has no edge. Then $End(X) \cong T_X$, the full transformation semigroup on set $V(X)$. Hence X is not End-inverse.

Case 2. Assume X contains at least two edges. Then we may denote it by $e_1 = \{x_1, x_2\}$ and $e_2 = \{y_1, y_2\}$. Since $e_1 \neq e_2$, we can suppose $x_1 \neq y_1$. Without loss of generality, we suppose $x_1, y_1 \in A$ and $x_2, y_2 \in B$. Define two mappings f_1 and f_2 from $V(X)$ to itself by

$$f_1(x) = \begin{cases} x_1, & x \in A, \\ x_2, & x \in B, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} y_1, & x \in A, \\ y_2, & x \in B. \end{cases}$$

Then $f_1, f_2 \in Idpt(X)$. But $f_1 f_2(x_1) = x_1 \neq y_1 = f_2 f_1(x_1)$. Thus $f_1 f_2 \neq f_2 f_1$. Hence X is not End-inverse.

Case 3. Assume X contains only one edge $e = \{z_1, z_2\}$ and has at least one isolated vertex x_0. Define two mappings g_1 and g_2 from $V(X)$ to itself by

$$g_1(x) = \begin{cases} z_1, & x = x_0, \\ x, & others, \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} z_2, & x = x_0 \\ x, & others. \end{cases}$$

Then $g_1, g_2 \in Idpt(X)$. But $g_1 g_2(x_0) = z_2 \neq z_1 = g_2 g_1(x_0)$. Thus $g_1 g_2 \neq g_2 g_1$. Hence X is not End-inverse.

Theorem 2.9 Let X and Y be two bipartite graphs. Then $X+Y$ is End-inverse if and only if $X+Y$ is one of the $K_1 + K_1$, $K_1 + K_2$ and $K_2 + K_2$.

Proof This follows directly from Theorem 2.7 and Lemma 2.8.

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