Strict lower bounds on the multiplicative Zagreb indices of graph operations

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Abstract: The first and second multiplicative Zagreb indices of a simple graph G are defined as:

$$\Pi_1(G) = \prod_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad \Pi_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(u)$ denotes the degree of the vertex u of G. In this paper, we present some strict lower bounds on the first and second multiplicative Zagreb indices of several graph operations in terms of the first and second multiplicative Zagreb indices and multiplicative sum Zagreb index of their components.

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1. Introduction

Throughout the paper, we consider connected finite graphs without any loops or multiple edges. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$. A *topological index* $Top(G)$ of G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. In organic Chemistry, topological indices have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships (SPR), structure-activity relationships (SAR) and pharmaceutical drug design [1-5].

The *Zagreb indices* belong among the oldest topological indices, and were introduced as early as in 1972 [6]. For details on their theory and applications see [7-10], and especially the recent papers [11-16]. The first and second Zagreb indices of G are denoted by $M_1(G)$ and $M_2(G)$, respectively, and are defined as:

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$$M_1(G)=\sum_{u\in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G)=\sum_{uv\in E(G)} d_G(u)d_G(v),$$

where $d_G(u)$ denotes the degree of the vertex u of G which is the number of edges incident to u. The first Zagreb index can also be expressed as a sum over edges of G:

$$M_1(G)=\sum_{uv\in E(G)}[d_G(u)+d_G(v)].$$

The multiplicative versions of Zagreb indices were introduced by Todeschini et al. in 2010 [17]. The first and second *multiplicative Zagreb indices* of G are denoted by $\Pi_1(G)$ and $\Pi_2(G)$, respectively, and are defined as:

$$\Pi_1(G)=\prod_{u\in V(G)} d_G(u)^2 \quad \text{and} \quad \Pi_2(G)=\prod_{uv\in E(G)} d_G(u)d_G(v).$$

The second multiplicative Zagreb index can also be expressed as a product over vertices of G [18]:

$$\Pi_2(G)=\prod_{u\in V(G)} d_G(u)^{d_G(u)}.$$

The *multiplicative sum Zagreb index* of G which can be considered as another multiplicative version of the first Zagreb index is defined as [19]:

$$\Pi_1^*(G)=\prod_{uv\in E(G)}[d_G(u)+d_G(v)].$$

We refer the reader to [20-23], for mathematical properties and applications of multiplicative Zagreb indices.

Many interesting graphs are composed of simpler graphs via several operations (also known as graph products). In [24], Khalifeh et al. presented some exact formulae for computing the Zagreb indices of some graph operations. In [25], Das et al. obtained some upper bounds for the multiplicative Zagreb indices of various graph operations in terms of the Zagreb indices of their components. In this paper, we proceed to obtain some strict lower bounds for the multiplicative Zagreb indices of several graph operations in terms of the first and second multiplicative Zagreb indices and multiplicative sum Zagreb index of their components. For more information on computing topological indices of graph operations see [26-28].

2. Main results

In this section, we compare the first and second multiplicative Zagreb indices of some graph operations and their components. We refer the reader to monograph [29] for more information on graph operations. All considered operations are binary. Hence, we will usually deal with two simple connected finite graphs G_1 and G_2 which are considered to be not singleton. For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, and its order and size by n_i and m_i, respectively, where $i\in\{1,2\}$.

At first, we recall the Arithmetic- Geometric Mean inequality.

Lemma 1 (AM- GM inequality) Let $x_1,\ldots,x_n$ be nonnegative numbers. Then

$$\frac{x_1+\cdots+x_n}{n}\geq\sqrt[n]{x_1\ldots x_n},$$

with equality if and only if $x_1=x_2=\cdots=x_n$. □

The *join* $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is a graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. The degree of a vertex u of $G_1 + G_2$ is given by:

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & u \in V(G_1) \\ d_{G_2}(u) + n_1 & u \in V(G_2) \end{cases}.$$

Theorem 1 The first and second multiplicative Zagreb indices of $G_1 + G_2$ satisfy the following inequalities:

(i) $\Pi_1(G_1 + G_2) > (3\sqrt[3]{2})^{n_1+n_2} n_1^{n_2} n_2^{n_1} \sqrt{\Pi_1(G_1)\Pi_1(G_2)}$,

(ii) $\Pi_2(G_1 + G_2) > 3^{m_1+m_2} n_1^{m_2} n_2^{m_1} (4\sqrt{n_1 n_2})^{n_1 n_2} \sqrt[4]{\Pi_1(G_1)^{n_2} \Pi_1(G_2)^{n_1}}$
$\sqrt[3]{\Pi_1^*(G_1)\Pi_2(G_1)\Pi_1^*(G_2)\Pi_2(G_2)}$.

Proof. (i) Let $G = G_1 + G_2$. By definition of the first multiplicative Zagreb index, we have:

$$\begin{aligned}
\Pi_1(G) &= \prod_{u\in V(G)} d_G(u)^2 \\
&= \prod_{u\in V(G_1)} \left(d_{G_1}(u) + n_2\right)^2 \prod_{u\in V(G_2)} \left(d_{G_2}(u) + n_1\right)^2 \\
&= \prod_{u\in V(G_1)} \left(d_{G_1}(u)^2 + 2n_2 d_{G_1}(u) + n_2^2\right) \\
&\quad \times \prod_{u\in V(G_2)} \left(d_{G_2}(u)^2 + 2n_1 d_{G_2}(u) + n_1^2\right).
\end{aligned}$$

Now by Lemma 1,

$$\begin{aligned}
\Pi_1(G) &> \prod_{u\in V(G_1)} 3\sqrt[3]{d_{G_1}(u)^2 \times 2n_2 d_{G_1}(u) \times n_2^2} \\
&\quad \times \prod_{u\in V(G_2)} 3\sqrt[3]{d_{G_2}(u)^2 \times 2n_1 d_{G_2}(u) \times n_1^2} \\
&= (3\sqrt[3]{2}n_2)^{n_1} \prod_{u\in V(G_1)} d_{G_1}(u) \times (3\sqrt[3]{2}n_1)^{n_2} \prod_{u\in V(G_2)} d_{G_2}(u) \\
&= (3\sqrt[3]{2})^{n_1+n_2} n_1^{n_2} n_2^{n_1} \sqrt{\Pi_1(G_1)\Pi_1(G_2)}.
\end{aligned}$$

(ii) By definition of the second multiplicative Zagreb index, we have:

$$\begin{aligned}
\Pi_2(G) &= \prod_{uv\in E(G)} d_G(u) d_G(v) \\
&= \prod_{uv\in E(G_1)} \left(d_{G_1}(u) + n_2\right)\left(d_{G_1}(v) + n_2\right) \\
&\quad \times \prod_{uv\in E(G_2)} \left(d_{G_2}(u) + n_1\right)\left(d_{G_2}(v) + n_1\right) \\
&\quad \times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} \left(d_{G_1}(u) + n_2\right)\left(d_{G_2}(v) + n_1\right) \\
&= \prod_{uv\in E(G_1)} \left[n_2^2 + \left(d_{G_1}(u) + d_{G_1}(v)\right) n_2 + d_{G_1}(u) d_{G_1}(v)\right] \\
&\quad \times \prod_{uv\in E(G_2)} \left[n_1^2 + \left(d_{G_2}(u) + d_{G_2}(v)\right) n_1 + d_{G_2}(u) d_{G_2}(v)\right] \\
&\quad \times \left[\prod_{u\in V(G_1)} \left(d_{G_1}(u) + n_2\right)\right]^{n_2} \left[\prod_{v\in V(G_2)} \left(d_{G_2}(v) + n_1\right)\right]^{n_1}.
\end{aligned}$$

Now by Lemma 1,

$$\begin{aligned}
\Pi_2(G) &> \prod_{uv\in E(G_1)} 3\sqrt[3]{n_2^2 \times \left(d_{G_1}(u) + d_{G_1}(v)\right) n_2 \times d_{G_1}(u) d_{G_1}(v)} \\
&\quad \times \prod_{uv\in E(G_2)} 3\sqrt[3]{n_1^2 \times \left(d_{G_2}(u) + d_{G_2}(v)\right) n_1 \times d_{G_2}(u) d_{G_2}(v)} \\
&\quad \times \left[\prod_{u\in V(G_1)} 2\sqrt{n_2 d_{G_1}(u)}\right]^{n_2} \left[\prod_{v\in V(G_2)} 2\sqrt{n_1 d_{G_2}(v)}\right]^{n_1}
\end{aligned}$$

$$= (3n_2)^{m_1} \sqrt[3]{\prod_{uv\in E(G_1)} \left(d_{G_1}(u)+d_{G_1}(v)\right) \prod_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v)}$$
$$\times (3n_1)^{m_2} \sqrt[3]{\prod_{uv\in E(G_2)} \left(d_{G_2}(u)+d_{G_2}(v)\right) \prod_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v)}$$
$$\times (2\sqrt{n_1})^{n_1 n_2} \sqrt[4]{\left[\prod_{v\in V(G_2)} d_{G_2}(v)^2\right]^{n_1}}$$
$$\times (2\sqrt{n_2})^{n_1 n_2} \sqrt[4]{\left[\prod_{u\in V(G_1)} d_{G_1}(u)^2\right]^{n_2}}$$
$$= 3^{m_1+m_2} n_1^{\ m_2} n_2^{\ m_1} (4\sqrt{n_1 n_2})^{n_1 n_2} \sqrt[4]{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}$$
$$\times \sqrt[3]{\Pi_1^{\ *}(G_1)\Pi_2(G_1)\Pi_1^{\ *}(G_2)\Pi_2(G_2)}. \square$$

The *corona product* $G_1 \circ G_2$ of graphs G_1 and G_2 is a graph obtained by taking one copy of G_1 and n_1 copies of G_2 and joining the i–th vertex of G_1 to every vertex in i–th copy of G_2 for $1 \le i \le n_1$. The i–th copy of G_2 will be denoted by $G_{2,i}$, $1 \le i \le n$. The degree of a vertex $u \in V(G_1 \circ G_2)$ is equal to:

$$d_{G_1\circ G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & u \in V(G_1) \\ d_{G_2}(u) + 1 & u \in V(G_{2,i}) \end{cases}.$$

Theorem 2 The first and second multiplicative Zagreb indices of $G_1 \circ G_2$ satisfy the following inequalities:

(i) $\Pi_1(G_1 \circ G_2) > (3\sqrt[3]{2})^{n_1(n_2+1)}\ n_2^{n_1} \sqrt{\Pi_1(G_1)}\ \sqrt{\Pi_1(G_2)^{n_1}}$,

(ii) $\Pi_2(G_1 \circ G_2) > 3^{m_1+n_1 m_2}\ 4^{n_1 n_2}\ (\sqrt{n_2}\,)^{n_1 n_2+2m_1} \sqrt[4]{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}$
$$\times \sqrt[3]{\Pi_1^{\ *}(G_1)\Pi_2(G_1)\left(\Pi_1^{\ *}(G_2)\Pi_2(G_2)\right)^{n_1}}\ .$$

Proof. (i) Let $G = G_1 \circ G_2$. By definition of the first multiplicative Zagreb index, we have:

$$\Pi_1(G) = \prod_{u\in V(G)} d_G(u)^2$$
$$= \prod_{u\in V(G_1)}\left(d_{G_1}(u) + n_2\right)^2 \left[\prod_{u\in V(G_2)}\left(d_{G_2}(u) + 1\right)^2\right]^{n_1}$$
$$= \prod_{u\in V(G_1)}\left(d_{G_1}(u)^2 + 2n_2 d_{G_1}(u) + n_2^2\right)$$
$$\times \left[\prod_{u\in V(G_2)}\left(d_{G_2}(u)^2 + 2d_{G_2}(u) + 1\right)\right]^{n_1}.$$

Now by Lemma 1,

$$\Pi_1(G) > \prod_{u\in V(G_1)} 3\sqrt[3]{d_{G_1}(u)^2 \times 2n_2 d_{G_1}(u) \times n_2^2}$$
$$\times \left[\prod_{u\in V(G_2)} 3\sqrt[3]{d_{G_2}(u)^2 \times 2d_{G_2}(u) \times 1}\right]^{n_1}$$
$$= (3\sqrt[3]{2}\, n_2)^{n_1} \prod_{u\in V(G_1)} d_{G_1}(u) \times (3\sqrt[3]{2})^{n_1 n_2}\left[\prod_{u\in V(G_2)} d_{G_2}(u)\right]^{n_1}$$
$$= (3\sqrt[3]{2})^{n_1(n_2+1)}\ n_2^{n_1} \sqrt{\Pi_1(G_1)}\ \sqrt{\Pi_1(G_2)^{n_1}}.$$

(ii) By definition of the second multiplicative Zagreb index, we have:

$$\Pi_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v)$$
$$= \prod_{uv\in E(G_1)}\left(d_{G_1}(u) + n_2\right)\left(d_{G_1}(v) + n_2\right)$$
$$\times \left[\prod_{uv\in E(G_2)}\left(d_{G_2}(u) + 1\right)\left(d_{G_2}(v) + 1\right)\right]^{n_1}$$

$$\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} \left(d_{G_1}(u) + n_2\right)\left(d_{G_2}(v) + 1\right)$$
$$= \prod_{uv\in E(G_1)} \left[n_2^2 + \left(d_{G_1}(u) + d_{G_1}(v)\right) n_2 + d_{G_1}(u) d_{G_1}(v)\right]$$
$$\times \left[\prod_{uv\in E(G_2)} [1 + \left(d_{G_2}(u) + d_{G_2}(v)\right) + d_{G_2}(u) d_{G_2}(v)]\right]^{n_1}$$
$$\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} \left[d_{G_1}(u) d_{G_2}(v) + d_{G_1}(u) + n_2 d_{G_2}(v) + n_2\right]$$

Now by Lemma 1,

$$\Pi_2(G) > \prod_{uv\in E(G_1)} 3\sqrt[3]{n_2{}^2 \times \left(d_{G_1}(u) + d_{G_1}(v)\right) n_2 \times d_{G_1}(u) d_{G_1}(v)}$$
$$\times \left[\prod_{uv\in E(G_2)} 3\sqrt[3]{1 \times \left(d_{G_2}(u) + d_{G_2}(v)\right) \times d_{G_2}(u) d_{G_2}(v)}\right]^{n_1}$$
$$\times \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} 4\sqrt[4]{d_{G_1}(u) d_{G_2}(v) \times d_{G_1}(u) \times n_2 d_{G_2}(v) \times n_2}$$
$$= (3n_2)^{m_1} \sqrt[3]{\prod_{uv\in E(G_1)} \left(d_{G_1}(u) + d_{G_1}(v)\right) \prod_{uv\in E(G_1)} d_{G_1}(u) d_{G_1}(v)}$$
$$\times 3^{n_1 m_2} \left[\sqrt[3]{\prod_{uv\in E(G_2)} \left(d_{G_2}(u) + d_{G_2}(v)\right) \prod_{uv\in E(G_2)} d_{G_2}(u) d_{G_2}(v)}\right]^{n_1}$$
$$\times (4\sqrt{n_2})^{n_1 n_2} \left[\sqrt[4]{\prod_{u\in V(G_1)} d_{G_1}(u)^2}\right]^{n_2} \left[\sqrt[4]{\prod_{v\in V(G_2)} d_{G_2}(v)^2}\right]^{n_1}$$
$$= 3^{m_1 + n_1 m_2}\, 4^{n_1 n_2}\, (\sqrt{n_2})^{n_1 n_2 + 2m_1} \sqrt[4]{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}$$
$$\times \sqrt[3]{\Pi_1{}^*(G_1)\Pi_2(G_1)\left(\Pi_1{}^*(G_2)\Pi_2(G_2)\right)^{n_1}}\ . \square$$

The *composition* $G_1[G_2]$ of graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \times V(G_2)$ and vertex (u_1, u_2) is adjacent with vertex (v_1, v_2) whenever [u_1 is adjacent with v_1 in G_1] or [$u_1 = v_1$ and u_2 is adjacent with v_2 in G_2]. The degree of a vertex $u = (u_1, u_2)$ of $G_1[G_2]$ is given by:

$$d_{G_1[G_2]}(u) = n_2 d_{G_1}(u_1) + d_{G_2}(u_2).$$

Theorem 3 The first and second multiplicative Zagreb indices of $G_1[G_2]$ satisfy the following inequalities:

(i) $\Pi_1(G_1[G_2]) > \left(3\sqrt[3]{2}\, n_2\right)^{n_1 n_2} \sqrt{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}$,

(ii) $\Pi_2(G_1[G_2]) > 3^{n_1 m_2}\, 4^{n_2^2 m_1}\, n_2{}^{n_1 m_2 + n_2^2 m_1} \sqrt{\Pi_1(G_1)^{m_2}\, \Pi_1(G_2)^{m_1 n_2}}$

$$\times \sqrt{\Pi_2(G_1)^{n_2^2}}\ \sqrt[3]{\left(\Pi_1{}^*(G_2)\Pi_2(G_2)\right)^{n_1}}\ .$$

Proof. (i) Let $G = G_1[G_2]$. By definition of the first multiplicative Zagreb index, we have:

$$\Pi_1(G) = \prod_{(u_1,u_2)\in V(G)} d_G((u_1, u_2))^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} \left(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\right)^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} [n_2^2 d_{G_1}(u_1)^2 + 2n_2 d_{G_1}(u_1) d_{G_2}(u_2) + d_{G_2}(u_2)^2].$$

Now by Lemma 1,

$$\Pi_1(G) >$$

$$\prod_{u_1 \in V(G_1)} \prod_{u_2 \in V(G_2)} 3\sqrt[3]{n_2^2 d_{G_1}(u_1)^2 \times 2n_2 d_{G_1}(u_1) d_{G_2}(u_2) \times d_{G_2}(u_2)^2}$$

$$= \left(3\sqrt[3]{2}\, n_2\right)^{n_1 n_2} \left[\prod_{u_1 \in V(G_1)} d_{G_1}(u_1)\right]^{n_2} \left[\prod_{u_2 \in V(G_2)} d_{G_2}(u_2)\right]^{n_1}$$

$$= \left(3\sqrt[3]{2}\, n_2\right)^{n_1 n_2} \sqrt{\Pi_1(G_1)^{n_2} \Pi_1(G_2)^{n_1}}\ .$$

(ii) By definition of the second multiplicative Zagreb index, we have:

$$\Pi_2(G) = \prod_{(u_1,u_2)(v_1,v_2) \in E(G)} d_G((u_1,u_2)) d_G((v_1,v_2))$$

$$= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \left(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\right)\left(n_2 d_{G_1}(u_1) + d_{G_2}(v_2)\right)$$

$$\times \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 \in V(G_2)} \prod_{v_2 \in V(G_2)} \left(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)\right) \times \left(n_2 d_{G_1}(v_1) + d_{G_2}(v_2)\right)$$

$$= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \left[n_2^2 d_{G_1}(u_1)^2 + \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right) n_2 d_{G_1}(u_1) + d_{G_2}(u_2) d_{G_2}(v_2)\right]$$

$$\times \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 \in V(G_2)} \prod_{v_2 \in V(G_2)} \left[n_2^2 d_{G_1}(u_1) d_{G_1}(v_1) + n_2 d_{G_1}(u_1) d_{G_2}(v_2) + n_2 d_{G_1}(v_1) d_{G_2}(u_2) + d_{G_2}(u_2) d_{G_2}(v_2)\right].$$

Now by Lemma 1,

$$\Pi_2(G) >$$

$$\prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} 3\sqrt[3]{n_2^3 d_{G_1}(u_1)^3 \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right) d_{G_2}(u_2) d_{G_2}(v_2)}$$

$$\times \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 \in V(G_2)} \prod_{v_2 \in V(G_2)} 4\sqrt[4]{n_2^4 \left(d_{G_1}(u_1) d_{G_1}(v_1)\right)^2 d_{G_2}(u_2)^2 d_{G_2}(v_2)^2}$$

$$= (3n_2)^{n_1 m_2} \left[\prod_{u_1 \in V(G_1)} d_{G_1}(u_1)\right]^{m_2} \sqrt[3]{\left[\prod_{u_2 v_2 \in E(G_2)} \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right)\right]^{n_1}}$$

$$\times \sqrt[3]{\left[\prod_{u_2 v_2 \in E(G_2)} d_{G_2}(u_2) d_{G_2}(v_2)\right]^{n_1}}$$

$$\times (4n_2)^{n_2^2 m_1} \sqrt{\left[\prod_{u_1 v_1 \in E(G_1)} d_{G_1}(u_1) d_{G_1}(v_1)\right]^{n_2^2}} \sqrt[4]{\left[\prod_{u_2 \in V(G_2)} d_{G_2}(u_2)^2\right]^{m_1 n_2}}$$

$$\times \sqrt[4]{\left[\prod_{v_2 \in V(G_2)} d_{G_2}(v_2)^2\right]^{m_1 n_2}}$$

$$= 3^{n_1 m_2} 4^{n_2^2 m_1} n_2^{n_1 m_2 + n_2^2 m_1} \sqrt{\Pi_1(G_1)^{m_2} \Pi_1(G_2)^{m_1 n_2}} \sqrt{\Pi_2(G_1)^{n_2^2}}$$

$$\times \sqrt[3]{\left(\Pi_1^*(G_2) \Pi_2(G_2)\right)^{n_1}}\ . \square$$

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1 \ and \ u_2 v_2 \in E(G_2)]$ or $[u_2 = v_2 \ and \ u_1 v_1 \in E(G_1)]$. The degree of a vertex $u = (u_1, u_2)$ of $G_1 \times G_2$ is given by:

$$d_{G_1 \times G_2}(u) = d_{G_1}(u_1) + d_{G_2}(u_2)\ .$$

Theorem 4 The first and second multiplicative Zagreb indices of $G_1 \times G_2$ satisfy the following inequalities:

(i) $\Pi_1(G_1 \times G_2) > (3\sqrt[3]{2})^{n_1 n_2} \sqrt{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}$,

(ii) $\Pi_2(G_1 \times G_2) > 3^{n_1 m_2 + n_2 m_1} \sqrt{\Pi_1(G_1)^{m_2}\, \Pi_1(G_2)^{m_1}}$

$$\times \sqrt[3]{\left(\Pi_1{}^*(G_1)\Pi_2(G_1)\right)^{n_2} \left(\Pi_1{}^*(G_2)\Pi_2(G_2)\right)^{n_1}}.$$

Proof. (i) Let $G = G_1 \times G_2$. By definition of the first multiplicative Zagreb index, we have:

$$\Pi_1(G) = \prod_{(u_1,u_2)\in V(G)} d_G((u_1,u_2))^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} \left(d_{G_1}(u_1) + d_{G_2}(u_2)\right)^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} \left(d_{G_1}(u_1)^2 + 2d_{G_1}(u_1)\, d_{G_2}(u_2) + d_{G_2}(u_2)^2\right).$$

Now by Lemma 1,

$$\Pi_1(G) > \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} 3\sqrt[3]{d_{G_1}(u_1)^2 \times 2\, d_{G_1}(u_1)\, d_{G_2}(u_2) \times d_{G_2}(u_2)^2}$$
$$= (3\sqrt[3]{2})^{n_1 n_2} \left[\prod_{u_1\in V(G_1)} d_{G_1}(u_1)\right]^{n_2} \left[\prod_{u_2\in V(G_2)} d_{G_2}(u_2)\right]^{n_1}$$
$$= (3\sqrt[3]{2})^{n_1 n_2} \sqrt{\Pi_1(G_1)^{n_2}\, \Pi_1(G_2)^{n_1}}.$$

(ii) By definition of the second multiplicative Zagreb index, we have:

$$\Pi_2(G) = \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} d_G((u_1,u_2))d_G((v_1,v_2))$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2 v_2\in E(G_2)} \left(d_{G_1}(u_1) + d_{G_2}(u_2)\right)\left(d_{G_1}(u_1) + d_{G_2}(v_2)\right)$$
$$\times \prod_{u_2\in V(G_2)} \prod_{u_1 v_1\in E(G_1)} \left(d_{G_1}(u_1) + d_{G_2}(u_2)\right)\left(d_{G_1}(v_1) + d_{G_2}(u_2)\right)$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2 v_2\in E(G_2)} \left[d_{G_1}(u_1)^2 + \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right) d_{G_1}(u_1) + d_{G_2}(u_2)d_{G_2}(v_2)\right]$$
$$\times \prod_{u_2\in V(G_2)} \prod_{u_1 v_1\in E(G_1)} \left[d_{G_2}(u_2)^2 + \left(d_{G_1}(u_1) + d_{G_1}(v_1)\right) d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_1}(v_1)\right].$$

Now by Lemma 1,

$$\Pi_2(G) >$$
$$\prod_{u_1\in V(G_1)} \prod_{u_2 v_2\in E(G_2)} 3\sqrt[3]{d_{G_1}(u_1)^3 \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right) d_{G_2}(u_2)d_{G_2}(v_2)}$$
$$\times \prod_{u_2\in V(G_2)} \prod_{u_1 v_1\in E(G_1)} 3\sqrt[3]{d_{G_2}(u_2)^3 \left(d_{G_1}(u_1) + d_{G_1}(v_1)\right) d_{G_1}(u_1)d_{G_1}(v_1)}$$
$$= 3^{n_1 m_2} \left[\prod_{u_1\in V(G_1)} d_{G_1}(u_1)\right]^{m_2}$$
$$\times \sqrt[3]{\left[\prod_{u_2 v_2\in E(G_2)} \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right) \prod_{u_2 v_2\in E(G_2)} d_{G_2}(u_2)d_{G_2}(v_2)\right]^{n_1}}$$
$$\times 3^{n_2 m_1} \left[\prod_{u_2\in V(G_2)} d_{G_2}(u_2)\right]^{m_1}$$
$$\times \sqrt[3]{\left[\prod_{u_1 v_1\in E(G_1)} \left(d_{G_1}(u_1) + d_{G_1}(v_1)\right) \prod_{u_1 v_1\in E(G_1)} d_{G_1}(u_1)d_{G_1}(v_1)\right]^{n_2}}$$
$$= 3^{n_1 m_2 + n_2 m_1} \sqrt{\Pi_1(G_1)^{m_2}\, \Pi_1(G_2)^{m_1}}$$

$$\times \sqrt[3]{\left(\Pi_1{}^*(G_1)\Pi_2(G_1)\right)^{n_2}\left(\Pi_1{}^*(G_2)\Pi_2(G_2)\right)^{n_1}}\ .\ \square$$

The *strong product* $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $[u_1 = v_1 \ and \ u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \ and \ u_1v_1 \in E(G_1)]$ or $[u_1v_1 \in E(G_1) \ and \ u_2v_2 \in E(G_2)]$. The degree of a vertex $u = (u_1, u_2)$ of $G_1 \boxtimes G_2$ is given by:

$$d_{G_1\boxtimes G_2}(u) = d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)\, d_{G_2}(u_2)\ .$$

Theorem 5 The first and second multiplicative Zagreb indices of $G_1 \boxtimes G_2$ satisfy the following inequalities:

(i) $\Pi_1(G_1 \boxtimes G_2) > (6\sqrt{2})^{n_1n_2} \sqrt[3]{\Pi_1(G_1)^{2n_2}\, \Pi_1(G_2)^{2n_1}}$,

(ii) $\Pi_2(G_1 \boxtimes G_2) > 7^{m_1m_2}\left(6\sqrt[6]{2}\right)^{n_1m_2+n_2m_1} \sqrt[6]{\Pi_1(G_1)^{3n_2+4m_2}\, \Pi_1(G_2)^{3n_1+4m_1}}$

$$\times \sqrt[7]{\Pi_2(G_1)^{4m_2}\, \Pi_2(G_2)^{4m_1}} \sqrt[21]{\Pi_1{}^*(G_1)^{7n_2+3m_2}\, \Pi_1{}^*(G_2)^{7n_1+3m_1}}\ .$$

Proof. (i) Let $G = G_1 \boxtimes G_2$. By definition of the first multiplicative Zagreb index, we have:

$$\Pi_1(G) = \prod_{(u_1,u_2)\in V(G)} d_G\left((u_1, u_2)\right)^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} \left(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)\, d_{G_2}(u_2)\right)^2$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} \big[d_{G_1}(u_1)^2 + d_{G_2}(u_2)^2 + d_{G_1}(u_1)^2 d_{G_2}(u_2)^2 + 2d_{G_1}(u_1)d_{G_2}(u_2) + 2\, d_{G_1}(u_1)^2\, d_{G_2}(u_2) + 2\, d_{G_2}(u_2)^2\, d_{G_1}(u_1)\big].$$

Now by Lemma 1,

$$\Pi_1(G) > \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} 6\sqrt[6]{2^3 d_{G_1}(u_1)^8 d_{G_2}(u_2)^8}$$
$$= \prod_{u_1\in V(G_1)} \prod_{u_2\in V(G_2)} 6\sqrt{2}\ \sqrt[3]{d_{G_1}(u_1)^4 d_{G_2}(u_2)^4}$$
$$= (6\sqrt{2})^{n_1n_2} \sqrt[3]{\left[\prod_{u_1\in V(G_1)} d_{G_1}(u_1)^4\right]^{n_2} \left[\prod_{u_2\in V(G_2)} d_{G_2}(u_2)^4\right]^{n_1}}$$
$$= (6\sqrt{2})^{n_1n_2} \sqrt[3]{\Pi_1(G_1)^{2n_2}\, \Pi_1(G_2)^{2n_1}}\ .$$

(ii) By definition of the second multiplicative Zagreb index, we have:

$$\Pi_2(G) = \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} d_G((u_1, u_2))d_G((v_1, v_2)).$$

By definition of the strong product, we can partition the above product into three products as follows:

The first product P_1 is taken over all edges $(u_1, u_2)(v_1, v_2) \in E(G)$ such that $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. The calculation of P_1 is as follows:

$$P_1 = \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} \left(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)\, d_{G_2}(u_2)\right)$$
$$\times \left(d_{G_1}(u_1) + d_{G_2}(v_2) + d_{G_1}(u_1)\, d_{G_2}(v_2)\right)$$

$$= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \Big[d_{G_1}(u_1)^2 + \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big) d_{G_1}(u_1)^2 + \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big) d_{G_1}(u_1) + d_{G_2}(u_2) d_{G_2}(v_2) d_{G_1}(u_1)^2 + 2 d_{G_2}(u_2) d_{G_2}(v_2) d_{G_1}(u_1) + d_{G_2}(u_2) d_{G_2}(v_2) \Big].$$

Now by Lemma 1,

$$P_1 > \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} 6 \sqrt[6]{2 d_{G_1}(u_1)^8 \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big)^2 \big(d_{G_2}(u_2) d_{G_2}(v_2)\big)^3}$$

$$= \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} 6\sqrt[6]{2}\ \sqrt[3]{d_{G_1}(u_1)^4}\ \sqrt[3]{d_{G_2}(u_2) + d_{G_2}(v_2)} \times \sqrt{d_{G_2}(u_2) d_{G_2}(v_2)}$$

$$= \left(6\sqrt[6]{2}\right)^{n_1 m_2} \sqrt[3]{\left[\prod_{u_1 \in V(G_1)} d_{G_1}(u_1)^4\right]^{m_2}} \sqrt{\left[\prod_{u_2 v_2 \in E(G_2)} d_{G_2}(u_2) d_{G_2}(v_2)\right]^{n_1}} \times \sqrt[3]{\left[\prod_{u_2 v_2 \in E(G_2)} \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big)\right]^{n_1}}$$

$$= \left(6\sqrt[6]{2}\right)^{n_1 m_2} \sqrt[3]{\Pi_1(G_1)^{2m_2}}\ \sqrt[3]{\Pi_1^{*}(G_2)^{n_1}}\ \sqrt{\Pi_1(G_2)^{n_1}}\ .$$

The second product P_2 is taken over all edges $(u_1, u_2)(v_1, v_2) \in E(G)$ such that $u_1 v_1 \in E(G_1)$ and $u_2 = v_2$. So,

$$P_2 = \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 \in V(G_2)} \big(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)\, d_{G_2}(u_2)\big) \times \big(d_{G_1}(v_1) + d_{G_2}(u_2) + d_{G_1}(v_1)\, d_{G_2}(u_2)\big).$$

By symmetry,

$$P_2 > \left(6\sqrt[6]{2}\right)^{n_2 m_1} \sqrt[3]{\Pi_1(G_2)^{2m_1}}\ \sqrt[3]{\Pi_1^{*}(G_1)^{n_2}}\ \sqrt{\Pi_1(G_1)^{n_2}}\ .$$

The third product P_3 is taken over all edges $(u_1, u_2)(v_1, v_2) \in E(G)$ such that $u_1 v_1 \in E(G_1)$ and $u_2 v_2 \in E(G_2)$. The calculation of P_3 is as follows:

$$P_3 = \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 v_2 \in E(G_2)} \big(d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)\, d_{G_2}(u_2)\big) \times \big(d_{G_1}(v_1) + d_{G_2}(v_2) + d_{G_1}(v_1)\, d_{G_2}(v_2)\big)$$

$$= \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 v_2 \in E(G_2)} \Big[d_{G_1}(u_1) d_{G_1}(v_1) + d_{G_1}(u_1) d_{G_2}(v_2) + d_{G_2}(u_2)\, d_{G_1}(v_1) + d_{G_2}(u_2) d_{G_2}(v_2) + d_{G_1}(u_1) d_{G_1}(v_1) \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big) + \big(d_{G_1}(u_1) + d_{G_1}(v_1)\big) d_{G_2}(u_2) d_{G_2}(v_2) + d_{G_1}(u_1) d_{G_1}(v_1) d_{G_2}(u_2) d_{G_2}(v_2) \Big].$$

Now by Lemma 1,

$$P_3 > \prod_{u_1 v_1 \in E(G_1)} \prod_{u_2 v_2 \in E(G_2)} 7 \sqrt[7]{\big(d_{G_1}(u_1) d_{G_1}(v_1)\big)^4 \big(d_{G_1}(u_1) + d_{G_1}(v_1)\big)} \times \sqrt[7]{\big(d_{G_2}(u_2) d_{G_2}(v_2)\big)^4 \big(d_{G_2}(u_2) + d_{G_2}(v_2)\big)}$$

$$= 7^{m_1 m_2} \sqrt[7]{\left[\prod_{u_1 v_1 \in E(G_1)} d_{G_1}(u_1) d_{G_1}(v_1)\right]^{4m_2}}$$

$$\times \sqrt[7]{\left[\prod_{u_1 v_1 \in E(G_1)} \left(d_{G_1}(u_1) + d_{G_1}(v_1)\right)\right]^{m_2}}$$

$$\times \sqrt[7]{\left[\prod_{u_2 v_2 \in E(G_2)} d_{G_2}(u_2) d_{G_2}(v_2)\right]^{4m_1} \left[\prod_{u_2 v_2 \in E(G_2)} \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right)\right]^{m_1}}$$

$$= 7^{m_1 m_2} \sqrt[7]{\left(\Pi_2(G_1)^4\, \Pi_1^*(G_1)\right)^{m_2} \left(\Pi_2(G_2)^4\, \Pi_1^*(G_2)\right)^{m_1}}.$$

Hence,

$$\Pi_2(G) = P_1 P_2 P_3 > 7^{m_1 m_2} \left(6\sqrt[6]{2}\right)^{n_1 m_2 + n_2 m_1} \sqrt[6]{\Pi_1(G_1)^{3n_2+4m_2}\, \Pi_1(G_2)^{3n_1+4m_1}}$$
$$\times \sqrt[7]{\Pi_2(G_1)^{4m_2}\, \Pi_2(G_2)^{4m_1}} \sqrt[21]{\Pi_1^*(G_1)^{7n_2+3m_2}\, \Pi_1^*(G_2)^{7n_1+3m_1}}. \square$$

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