Spanning trees in subcubic graphs*

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Abstract

We prove that every connected subcubic graph G has two spanning trees T_1, T_2 such that every component of $G - E(T_1)$ is a path of length at most 3, and every component of $G - E(T_2)$ is either a path of length at most 2 or a cycle.

Key words: Spanning tree; subcubic graphs.

1 Introduction

We only consider finite graphs without loops or multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For any $S \subseteq V(G) \cup E(G)$, define $G - S$ to be the subgraph of G with vertex set $V(G) - (S \cap V(G))$ and edge set $\{e \in E(G) : e \notin S$ or e is not incident with any vertex in $S\}$. For any edge subset R of $E(G)$, let $G[R]$ be the subgraph of G induced by R. We write $A := B$ to

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rename B as A. A graph is said to be subcubic if it has maximum degree at most three.

The problem of finding spanning trees that satisfying a specified property in given graphs has been studied extensively. For example, finding spanning trees with maximum number of leaves. Griggs, Kleitman and Shastri [4] proved that every connected cubic graph on n vertices contains a spanning tree with at least $\frac{n}{4}+2$ leaves, and this is best possible for all (even) n. In the same paper, they also proved that every connected n-vertex cubic graph containing no K_4^- has a spanning tree with at least $\frac{n}{3}+\frac{4}{3}$ leaves. Kleitman and West [6] showed that every connected n-vertex graph with minimum degree at least 3 (resp., 4) always admits a spanning tree with at least $\frac{n}{4}+2$ (resp., $\frac{2n}{5}+\frac{8}{5}$) leaves. For the case that G has minimum degree at least 5, the lower bound $\frac{n}{2}+2$ was given by Griggs and Wu in [5]. There are several other results concerning spanning trees with maximum number of leaves in certain graphs, see [3, 1, 2].

In this paper, we consider a different direction. A well-known result of Thomassen [7] implies that every connected subcubic graph contains a spanning tree whose deletion results in paths of length at most 5. (This will be explained in Section 2.) By using the classic Depth-First Search algorithm, we prove that every connected subcubic graph G has a spanning tree T such that every component of $G-E(T)$ is a path of length at most 3. The complete graph K_4 shows that this bound is best possible. If we allow cycle exists, then the above bound can be further improved to 2, that is, every connected subcubic graph G contains a spanning tree T such that $G-E(T)$ consists of paths of length at most 2 and cycles. Moreover, the number of cycles is at most one.

This work is related to a conjecture of Hoffman and Ostenhof that every connected cubic graph G has a spanning tree T such that every component of $G-E(T)$ is either a path of length at most 1 or a cycle. (See [9] for more information.) Such trees exist for some special class of cubic graphs, such as Peterson graph and prisms over cycles. As far as we know, this conjecture is still open.

This work is inspired by the following conjecture proposed by Hoffman and Ostenhof [9].

Conjecture 1.1. *Every connected cubic graph G contains a spanning tree T such that every component of $G - E(T)$ is either a path of length at most 1 or a cycle.*

It would be nice to know whether Conjecture 1.1 holds for all connected subcubic graphs.

2 Main results

In this section, we prove the main results of this paper.

In [7], Thomassen showed that every subcubic graph has an edge-coloring in two colors such that each monochromatic component is a path of length at most 5. The complete bipartite graph $K_{3,3}$ shows that the number 5 can not be replaced by 4. In fact, Thomassen's result means that every connected subcubic graph G has a spanning tree T such that $G - E(T)$ consists of paths of length at most 5. This can be seen as follows. Let G be a connected subcubic graph. Then G can be edge-colored in two colors, say red and blue, such that each monochromatic component is a path of length at most 5. We use E_R (resp., E_B) to denote the set of red (resp., blue) edges of G, then both $G[E_R]$ and $G[E_B]$ are forests of G. It is easy to see that we can always choose a subset of E_B, say E'_B, such that $T := G[E_R \cup E'_B]$ is a spanning tree of G as desired. (Since every component of $G - E(T)$ is a subpath of a blue component of $G[E_B]$, and hence has length at most 5.)

By applying the well-known Depth-First Search algorithm, our first result shows that we can improve the number 5 to 3. The complete graph K_4 shows that this bound is sharp.

Theorem 2.1. *Let G be a connected subcubic graph. Then G contains a spanning tree T such that every component of $G - E(T)$ is a path of length at most 3. Moreover, the number of paths of length 3 is at most one.*

Proof. Let u^* be an arbitrary vertex of G. Then we apply the Depth-First Search algorithm to find a spanning tree T of G such that u^* is the first vertex traversed. For convenience, let v^* be the last vertex of G traversed by the algorithm. Since G is subcubic, all the components of $G - E(T)$ are paths and cycles. We will show that T is as required.

Let C be any component of $G - E(T)$. We claim that C must be a path of length at most 3. For otherwise, C is either a path of length at least 4 or a cycle. Then either C is a triangle $uvwu$ or C contains the path $suvwt$ as its subgraph. (It is possible that $s = t$ when C is a cycle of length 4.) In both cases, all the vertices of $\{u, v, w\}$ have degree 2 in C. Without loss of generality, we may assume that $v \notin \{u^*, v^*\}$. By the Depth-First Search algorithm, when we traverse the vertex v, the next candidate must be u or w, that is, $vu \in E(T)$ or $vw \in E(T)$, a contradiction.

We next show that at most one component of $G-E(T)$ has length 3. Let $P := suvt$ be a path of length 3 in $G - E(T)$. If $\{u, v\} \neq \{u^*, v^*\}$, then by the same argument as above, we can also deduce a contradiction. Therefore we have $\{u, v\} = \{u^*, v^*\}$. This completes the proof of the theorem. ∎

It follows from the proof of Theorem 2.1 that if $G - E(T)$ has a path P of length 3, then P must contain the first vertex u^* and the last vertex v^* traversed by the algorithm, and both u^* and v^* have degree 3 in G. This leads to the following result.

Corollary 2.2. *Let G be a connected subcubic graph. If some vertex of G, say v, has degree at most 2, then G contains a spanning tree T such that every component of $G - E(T)$ is a path of length at most 2.*

Proof. We need only use the Depth-First Search algorithm to find a spanning tree T of G such that v is the first vertex traversed. Then the resulting tree T is as desired. ∎

Our next result shows that if we allow some components of $G-E(T)$ are cycles, then the bound in Theorem 2.1 can be further improved to 2. For this purpose, we need the following lemma due to Thomassen and Toft [8].

Lemma 2.3. *Every connected graph G with minimum degree at least 3 has an induced cycle C such that $G - V(C)$ is connected.*

Theorem 2.4. *Let G be a connected subcubic graph. Then G contains a spanning tree T such that every component of $G - E(T)$ is either a path of length at most 2 or a cycle. Moreover, the number of cycles is at most one.*

Proof. If some vertex of G has degree at most 2, then the assertion of the theorem follows directly from Corollary 2.2.

So we may assume that G is a cubic graph. Then by Lemma 2.3, G contains an induced cycle C such that $G - V(C)$ is connected. It is easy to check that $G' := G - E(C)$ is still a connected subcubic graph (since C is an induced cycle). By Corollary 2.2, there is a spanning tree T in G' such that every component of $G' - E(T)$ is a path of length at most 2 (by letting some vertex of C be the first vertex traversed by the algorithm). Then T is also a spanning tree in G as desired. In this case, C is the unique cycle in $G - E(T)$. ∎

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