Spanning trees whose stems have at most k leaves

Mikio Kano *
Zheng Yan
Department of Computer and Information Sciences
Ibaraki University, Hitachi, Ibaraki, Japan
kano@mx.ibaraki.ac.jp yanzhenghubei@163.com
http://gorogoro.cis.ibaraki.ac.jp

Abstract

For a tree T, the set of leaves of T is denoted by $Leaf(T)$, and the subtree $T - Leaf(T)$ is called the stem of T. We prove that if a connected graph G either satisfies $\sigma_{k+1}(G) \geq |G| - k - 1$ or has no vertex set of size $k+1$ such that the distance between any two their vertices is at least 4, then G has a spanning tree whose stem has at most k leaves, where $\sigma_{k+1}(G)$ denotes the minimum degree sum of $k+1$ independent vertices of G. Moreover, we show that the condition on $\sigma_{k+1}(G)$ is sharp. Also we give another similar sufficient degree condition for a claw-free graph to have such a spanning tree.

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. Let $N_G(v)$ denote the neighborhood of v in G. Thus $\deg_G(v) = |N_G(v)|$. A graph G is said to be *claw-free* if G has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$.

Let T be a tree. A vertex of T with degree one is often called a *leaf*, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of

*Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) No.25400187

T is called the *stem* of T and denoted by $Stem(T)$. A spanning tree with specified stem was first considered in [4].

Let $k \geq 2$ be an integer. A tree whose maximum degree at most k is called a *k-tree*. Similarly, a stem whose maximum degree at most k is called a *k-stem*, and a tree whose stem is a k-tree is called a *tree with k-stem* (see Figure 1).

For two vertices x and y of a graph G, the distance between x and y in G, which is the length of a shortest path connecting x and y in G, is denoted by $d_G(x,y)$. For an integer $k \geq 2$, $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G. Furthermore for an integer $s \geq 2$, let $\sigma_k^s(G)$ denote the minimum degree sum of k vertices $v_1, v_2, \ldots, v_k$ of G such that $d_G(v_i, v_j) \geq s$ for any two distinct vertices v_i and v_j. Then

$$\sigma_k(G) = \sigma_k^2(G) \quad \text{and} \quad \sigma_k^m(G) \geq \sigma_k^\ell(G) \quad \text{for every integers} \quad 2 \leq \ell \leq m. \quad (1)$$

The following theorem gives a sufficient condition using $\sigma_k(G)$ for a graph G to have a spanning tree with k-stem.

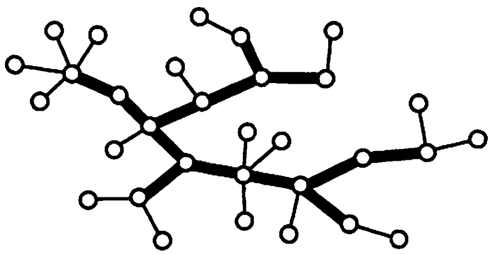

Figure 1: A tree with 3-stem, which is also a tree with 6-ended stem, where the bold lines form the stem.

Theorem 1 (Kano, Tsugaki and Yan [4]) *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree with k-stem.*

A tree having at most k leaves is called a *k-ended tree*, and a stem having at most k leaves is called a *k-ended stem*. A tree whose stem has at most k leaves is called a *tree with k-ended stem* (see Figure 1). In [5], Tsugaki and Zhang gave a sufficient condition using $\sigma_3(G)$ for a graph to have a spanning tree with k-ended stem as follows.

Theorem 2 (Tsugaki and Zhang [5]) *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree with k-ended stem.*

For an integer $s \geq 2$, we call a vertex set X of G an *s-stable set* if the distance between each pair of distinct vertices of S is at least s. Note that if G has no s-stable of size k, then we define $\sigma_k^s(G) = \infty$. In this paper, we prove the following two theorems.

Theorem 3 *Let G be a connected graph and $k \geq 2$ be an integer. If G has no 4-stable set of order $k+1$, then G has a spanning tree with k-ended stem.*

Theorem 4 *Let G be a connected graph and $k \geq 2$ be an integer. If*

$$\sigma_{k+1}(G) \geq |G| - k - 1, \tag{2}$$

then G has a spanning tree with k-ended stem.

For a claw-free graph, we obtain the following theorem.

Theorem 5 *Let G be a connected claw-free graph and $k \geq 2$ be an integer. If*

$$\sigma_{k+1}^4(G) \geq |G| - 2k - 1, \tag{3}$$

then G has a spanning tree with k-ended stem.

It is clear that our Theorem 4 implies Theorem 1 since a k-ended stem is a k-stem. Notice that the condition of Theorem 1 is also best possible. Moreover, Theorem 4 implies Theorem 2. Namely, if $k = 2$, then (2) is equivalent to the condition of Theorem 2. Assume that $k \geq 3$ and $\sigma_3(G) \geq |G| - 2k + 1$. Let $\{v_1, v_2, \cdots, v_{k+1}\}$ be an independent set of size $k+1$ such that $\sigma_{k+1}(G) = \sum_{i=1}^{k+1} \deg_G(v_i)$. Then

$$\begin{aligned}\sigma_{k+1}(G) = \sum_{i=1}^{k+1} \deg_G(v_i) &\geq \sigma_3(G) + \sum_{i=4}^{k+1} \deg_G(v_i) \\ &\geq |G| - 2k + 1 + k - 2 = |G| - k - 1.\end{aligned}$$

Hence the condition of Theorem 2 implies (2).

Sufficient conditions for a graph to have a spanning k-ended tree were obtained as follows.

Theorem 6 (Broersma and Tuinstra [2]) *Let $k \geq 2$ be an integer and G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Theorem 7 (Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita [3]) *Let G be a connected claw-free graph and $k \geq 2$ be an integer. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning k-ended tree.*

Some other results on spanning trees can be found in a survey paper [6] and Chapter 8 of book [1]. We conclude this section by showing that the two conditions in Theorems 4 and 5 are sharp.

Let $k \geq 2$ and $m \geq 1$ be integers, and let $D_1, D_2, \ldots, D_{k+1}$ be $k+1$ disjoint copies of the complete graph K_m of order m. Let $w, v_1, \ldots, v_{k+1}$ be $k+2$ vertices not contained in $D_1 \cup D_2 \cup \cdots \cup D_{k+1}$. Join w to all the vertices of $D_1 \cup D_2 \cup \cdots \cup D_{k+1}$ by edges, and join v_i to all the vertices of D_i by edges for every $1 \leq i \leq k+1$. Let G_1 denote the resulting graph (see Figure 2). Then $|G_1| = (k+1)m + k + 2$ and

$$\sigma_{k+1}(G_1) = \sum_{i=1}^{k+1} \deg_{G_1}(v_i) = (k+1)m = |G_1| - k - 2,$$

but G_1 has no spanning tree with k-ended stem. Hence the condition on $\sigma_{k+1}(G)$ in Theorem 4 is sharp.

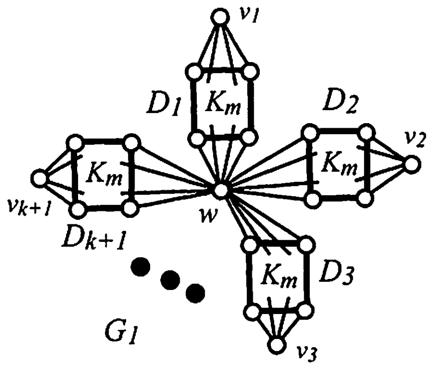

Figure 2: G_1 is a graph that has no spanning tree with k-ended stem and satisfies $\sigma_{k+1}(G_1) = |G_1| - k - 2$,

Let $k \geq 2$ and $m \geq 1$ be integers. Let H be a copy of the complete graph K_{k+1} with vertex set $V(H) = \{u_1, u_2, \ldots, u_{k+1}\}$, and let $D_1, D_2, \ldots, D_{k+1}$ be $k+1$ disjoint copies of the complete graph K_m. We construct a graph G_2 as follows: $V(G_2) = V(H) \cup V(D_1) \cup \cdots \cup V(D_{k+1}) \cup \{v_1, \ldots, v_{k+1}\}$ (disjoint union). For every $1 \leq i \leq k+1$, join u_i and v_i to all the vertices of D_i. Denote the resulting graph by G_2 (see Figure 3). It is immediate that $|G_2| = k + 1 + (k+1)(m+1)$ and G_2 is claw-free. Moreover,

$$\sigma^4_{k+1}(G_2) = \sum_{i=1}^{k+1} \deg_{G_2}(v_i) = (k+1)m = |G_2| - 2k - 2.$$

On the other hand, it is easy to see that G_2 has no spanning tree with k-ended stem. Therefore the condition on $\sigma^4_{k+1}(G)$ in Theorem 5 is sharp.

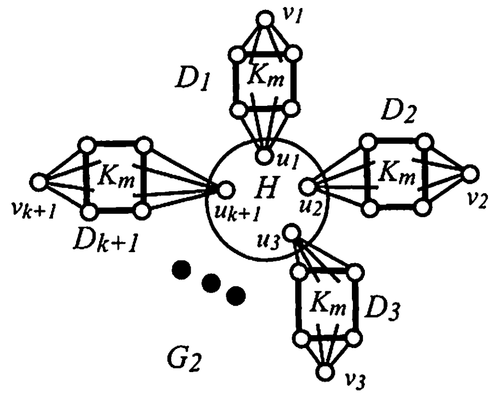

Figure 3: G_2 is a claw-free graph that has no spanning tree with k-ended stem and satisfies $\sigma^4_{k+1}(G_2) = |G_2| - 2k - 2$.

2 Proofs of Theorems 3, 4 and 5

In order to prove Theorems 4 and 5, we needs the following proposition.

Proposition 8 *Let G be a connected graph and $k \geq 2$ be an integer. Assume that for every spanning tree T of G such that $|Leaf(Stem(T))|$ is minimum, it follows that either $Leaf(T)$ has no 4-stable set of G with size $k+1$ or $\sum_{x \in S} \deg_G(x) \geq |Leaf(T)|+1$ for every 4-stable set $S \subseteq Leaf(T)$ of G with size $k+1$. Then G has a spanning tree whose stem has at most k leaves.*

Proof. For convenience, we often write $Stem(T)$ for $V(Stem(T))$ when no confusion arises. Suppose that G satisfies the assumption of Proposition 8. Choose a spanning tree T of G so that

(T1) $|Leaf(Stem(T))|$ is minimum,

(T2) $|Stem(T)|$ is as small as possible subject to (T1),

We may assume that $|Leaf(Stem(T))| \geq k+1$ since otherwise T is the desired spanning tree of G. Let $x_1, x_2, \cdots, x_{k+1}$ be $k+1$ distinct leaves of $Stem(T)$. We begin with the following claim.

Claim 1. For every x_i, $1 \le i \le k+1$, there exists a leaf y_i of T such that y_i is adjacent to x_i in T and satisfies $N_G(y_i) \subseteq Leaf(T) \cup \{x_i\}$.

Let x_a be a leaf of $Stem(T)$, where $1 \le a \le k+1$. It is obvious that there exists a leaf of T which is adjacent to x_a in T. Assume that every leaf y of T adjacent to x_a in T satisfies $N_G(y) \cap (Stem(T) - \{x_a\}) \ne \emptyset$. Then for every leaf y of T adjacent to x_a in T, remove the edge yx_a from T and add an edge yz of G, where z is a vertex of $N_G(y) \cap (Stem(T) - \{x_a\})$. Denote the resulting tree of G by T_1. Then T_1 is a spanning tree of G, $|Leaf(Stem(T_1))| \le |Leaf(Stem(T))|$ and $Stem(T_1) = Stem(T) - \{x_a\}$, which contradicts the condition (T2). Therefore there exists a leaf y_a adjacent to x_a in T such that $N_G(y_a) \cap (Stem(T) - \{x_a\}) = \emptyset$. Since $V(G) = Stem(T) \cup Leaf(T)$, the claim holds.

Claim 2. $d_G(y_i, y_j) \ge 4$ for every $1 \le i, j \le k+1$ with $i \ne j$.

Let $P(y_a, y_b)$ be a shortest path connecting y_a and y_b in G, where $1 \le a, b \le k+1$ and $a \ne b$. Assume first that all the vertices of $P(y_a, y_b)$ are contained in $Leaf(T)$. Then add $P(y_a, y_b)$ to T and remove the edges of T joining $P(y_a, y_b)$ to $Stem(T)$ except the edges $y_a x_a$ and $y_b x_b$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_1 of $Stem(T)$ incident with a vertex of degree at least three in $Stem(T)$. By removing the edge e_1, we obtain a spanning tree whose stem has a smaller number of leaves than $|Leaf(Stem(T))|$. This contradicts the choice (T1). Hence $P(y_a, y_b)$ passes through a vertex s of $Stem(T)$.

If $s \notin Stem(T) - \{x_a, x_b\}$, then $d_G(y_a, s) \ge 2$ and $d_G(s, y_b) \ge 2$ by Claim 1, and thus $d_G(y_a, y_b) = d_G(y_a, s) + d_G(s, y_b) \ge 4$. So we may assume that $s = x_a$ by symmetry. Namely, $P(y_a, y_b) = y_a x_a + P(x_a, y_b)$, where $P(x_a, y_b)$ is the subpath of $P(y_a, y_b)$ connecting x_a and y_b. If $P(x_a, y_b)$ passes through a vertex, say t, of $Stem(T) - \{x_b\}$, then $d_G(y_a, y_b) = d_G(y_a, x_a) + d_G(x_a, t) + d_G(t, y_b) \ge 4$ by Claim 1. Thus $P(x_a, y_b)$ does not pass through $Stem(T) - \{x_b\}$.

Add $P(x_a, y_b)$ to T and remove the edges of T joining $P(x_a, y_b) \cap Leaf(T)$ to $Stem(T)$ except $y_b x_b$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_2 of $Stem(T)$ incident with a vertex degree at least three in $Stem(T)$. By removing the edge e_2, we obtain a spanning tree whose stem has a smaller number of leaves than $|Leaf(Stem(T))|$. This contradicts the choice (T1). Hence, Claim 2 holds.

By Claim 2, we may assume that $Leaf(T)$ satisfies the latter condition on $Leaf(T)$ in Proposition 8. By Claims 1 and 2, it follows that $N_G(y_i) \cap N_G(y_j) = \emptyset$ for every $1 \le i, j \le k+1$ with $i \ne j$ and

$$\bigcup_{1 \le i \le k+1} N_G(y_i) \subseteq (Leaf(T) - \{y_1, \cdots, y_{k+1}\}) \cup \{x_1, \cdots, x_{k+1}\}.$$

Hence,

$$\sum_{1\leq i\leq k+1} \deg_G(y_i) \leq |Leaf(T)|.$$

This contradicts the latter condition on $Leaf(T)$ in Proposition 8. Consequently, the proposition is proved. □

Proof of Theorem 3. Theorem 3 follows immediately from Proposition 8. □

Proof of Theorem 4. By Theorem 3, we may assume that G has a 4-stable set with size $k+1$. Let S be a 4-stable set of G with size $k+1$ such that $\sigma^4_{k+1}(G) = \sum_{x\in S} \deg_G(x)$. Then for any two distinct vertices x and y of S, it follows that $(N_G(x) \cup \{x\}) \cap (N_G(y) \cup \{y\}) = \emptyset$, and by the existence of S, there exists at least one vertex in G that is not contained in $\bigcup_{x\in S}(N_G(x) \cup \{x\})$. Hence

$$|G| \geq \sum_{x\in S} |N_G(x) \cup \{x\}| + 1 = \sigma^4_{k+1}(G) + k + 1 + 1.$$

Thus by (1),

$$\sigma_{k+1}(G) \leq \sigma^4_{k+1}(G) \leq |G| - k - 2.$$

This contradicts the assumption of the theorem. Therefore Theorem 4 holds. □

Lemma 9 *Let G be a connected claw-free graph, and let T be a spanning tree of G such that $|Leaf(Stem(T))|$ is minimum. If $|Stem(T)| \geq 4$, then $|Stem(T)| \geq 2|Leaf(Stem(T))|$.*

Proof. Assume that two vertices x_1 and x_2 of $Leaf(Stem(T))$ are adjacent to a vertex z_1 of $Stem(Stem(T))$ in T. By the condition $|Stem(T)| \geq 4$, there exists a vertex z_2 of $Stem(T)$ that is adjacent to z_1 in T and different from x_1, x_2 and z_1. If x_1 and z_2 are adjacent in G, then $T - z_1z_2 + x_1z_2$ is a spanning tree whose stem has a smaller number of leaves than $|Leaf(Stem(T))|$, which is a contradiction. Hence, by symmetry, neither x_1 nor x_2 are adjacent to z_2 in G. Since G is claw-free, x_1 and x_2 are adjacent in G. Then $T - x_1z_1 + x_1x_2$ is a spanning tree whose stem has a smaller number of leaves than $|Leaf(Stem(T))|$. This is a contradiction. Therefore no two vertices of $Leaf(Stem(T))$ are adjacent to the same vertex of $Stem(Stem(T))$ in T. This implies that $|Stem(Stem(T)) \geq |Leaf(Stem(T))|$. Consequently, we have $|Stem(T)| \geq 2|Leaf(Stem(T))|$. □

Proof of Theorem 5. Assume that G has no spanning tree with k-ended stem. Let T be a spanning tree of G such that $|Leaf(Stem(T))|$ is minimum. Then $|Leaf(Stem(T))| \geq k+1 \geq 3$, and so $|Stem(T)| \geq 2(k+1)$ by

Lemma 9. It follows that $|Leaf(T)| = |T| - |Stem(T)| = |G| - |Stem(T)| \leq |G| - 2k - 2$. Therefore, by the condition of Theorem 5 and the above inequality, we have $\sigma^4_{k+1}(G) \geq |G| - 2k - 1 \geq |Leaf(T)| + 1$. By Proposition 8, G has a spanning tree with k-ended stem, a contradiction. Consequently Theorem 5 is proved. □

Acknowledgment The authors would like to thank anonymous referee for his/her valuable suggestions. In particular, a suggestion of introducing Proposition 8 makes the proofs of Theorems 3, 4 and 5 clearer and shorter.

References

[1] J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, Lecture Note in Mathematics (LNM **2031**), Springer (2011).

[2] H. Broersma and H. Tuinstra, Independence trees and hamilton cycles, *J. Graph Theory* **29** (1998) 227-237.

[3] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito and T. Yamashita. Spanning trees with a bounded number of leaves in a claw-free graph, *Ars Combinatoria* Vol. CIII, (2012) 137–154.

[4] M. Kano, M. Tsugaki and G. Yan, Spanning trees whose stems have bounded degrees, preprint.

[5] M. Tsugaki and Y. Zhang, Spanning trees whose stems have a few leaves, to appear in Ars Combinatoria.

[6] K. Ozeki and T. Yamashita, Spanning trees: A survey, *Graphs Combinatorics* **22** (2011) 1-26.