Rooted General Maps on All Surfaces

Wenzhong Liu [1]
Department of Mathematics, Nanjing University of Aeronautics and Astronautics,
Nanjing 210016, P. R.China
E-mail: wzhliu7502@gmail.com
Yanpei Liu
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, P. R. China
E-mail: ypliu@bjtu.edu.cn

Abstract In this paper, we concentrate on rooted general maps on all surfaces(orientable and nonorientable) without regard to genus and present the enumerating equation with respect to vertices and edges, which is a Riccati's equation. To solve it, a new solution in continued fraction form is given. As two especial cases, the corresponding results of rooted general maps and rooted monopole maps on all surfaces with respect to edges regardless of genus are obtained.

Keywords Rooted map, Riccati's equation, Continued fraction

MR(2000) Subject Classification 05C45

1. Introduction

Starting with Tutte's seminal works [12–15] on the number of rooted planar maps , the enumeration of rooted maps has been studied intensively. Then, Brown [3] first counted non-planar maps in 1966. Walsh and Lehman [16–18] had laid a groundwork in the 1970s in the field. Later, some scholars, such as Arquès [1–2], Bender et al. [4], Gao [5], Liu [7–9] , Ren [10–11]and so on, further extended it and did some important works. On the object, the three main approaches are often used as follows: the bijective, algebraic and topological approaches. Arquès [2] first employed the continued fraction to give the expression of counting rooted general maps on the orientable surfaces without regard to genus. In the paper, we research rooted general maps on all surfaces regardless of genus and present the enumerating equation with respect to vertices and edges, which is a Riccati's equation. For the different equation, a new solution in continued fraction form is given because the analytic solution cannot nearly be gotten for some Riccati's equation. As two especial cases, the corresponding re-

[1] Corresponding author.

sults of rooted general maps and rooted monopole maps on all surfaces with respect to edges regardless of genus are derived. Furthermore, the counterparts of rooted general maps on the nonorientable surfaces can easily be derived. An equation of continued fraction is additionally obtained.

The article begins with some definitions. Terminologies without description can be seen in [7–9].

A *map* is a connected topological graph cellularly embedded on a surface. A map is *rooted* if a vertex and an edge with a direction along one side of it are distinguished. In this paper, maps are always rooted. For a map M, the root, the root-vertex, the root-edge and the root-face of it are denoted by $r(M)$, $v_r(M)$, $e_r(M)$ and $f_r(M)$, respectively.

A *surface* here is a compact close 2– manifolds. An *orientable* (*nonorientable*) surface of genus g is homeomorphic to a sphere with g handles (crosscaps) (*i.e.* $g = 1 - \frac{1}{2}\chi$ or $\tilde{g} = 2 - \chi$, where χ is Euler characteristic).

An edge is called *double* if it belongs to only one face in a map, or is called *single.*

Let $\mathcal{M}$ be the set of all general maps on all surfaces without regard to genus and its enumerating function $f(x, y)$ with respect to vertices and edges is defined as follows:

$$f(x,y) = \sum_{M \in \mathcal{M}} x^{m(M)} y^{n(M)}$$

where $m(M)$ and $n(M)$ are the numbers of vertices and edges of M, respectively.

Let $F(y) = f(1, y)$, *i.e.* the enumerating function of $\mathcal{M}$ with respect to edges:

$$F(y) = \sum_{M \in \mathcal{M}} y^{n(M)}$$

Let $\mathcal{T}$ be the set of all monopole maps(*i.e.* maps with only one vertex) on all surfaces regardless of genus and its enumerating function $T(y)$ with respect to edges is described as:

$$T(y) = \sum_{M \in \mathcal{T}} y^{n(M)}$$

where $n(M)$ is the number of edges of M.

2. Enumerating Equations

Let $\mathcal{M}$ be the set of all general maps on all surfaces without regard to genus and $\mathcal{M}$ is partitioned to three parts :

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 \tag{2.1}$$

where $\mathcal{M}_1$ is consisted of only one vertex-map(*i.e.* a map without edges) and

$$\mathcal{M}_2 = \{M \mid M \in \mathcal{M},\ e_r(M) \text{ is a cut-edge } \}\ ;$$

$$\mathcal{M}_3 = \{M \mid M \in \mathcal{M},\ e_r(M) \text{ is not a cut -edge } \}.$$

According to the definition of $\mathcal{M}_1$, the contribution $f_1(x,y)$ of $\mathcal{M}_1$ to $f(x,y)$ is:

$$f_1(x,y) = x^1y^0 = x \tag{2.2}$$

Lemma 1 *Let* $\mathcal{M}_{<2>} = \{M - e_r(M) \mid M \in \mathcal{M}_2\}$. *Then* $\mathcal{M}_{<2>} = \mathcal{M} \times \mathcal{M}$.

Proof For $M \in \mathcal{M}_2$, according to the definition of $\mathcal{M}_2$, the root-edge $e_r(M)$ of M is a cut-edge and its deletion disconnects $\mathcal{M}_2$ into two maps in $\mathcal{M}$. Thus $M - e_r(M) \in \mathcal{M} \times \mathcal{M}$.

conversely, for any $(M_1, M_2) \in \mathcal{M} \times \mathcal{M}$,the map M can be constructed by adding a new edge as the root-edge $e_r(M)$ of M from the root-vertex of M_1 to that of M_2, and choosing the root-vertex of M_1 as the new root-vertex of M. Then $M \in \mathcal{M}$ and $M - e_r(M) = (M_1, M_2)$. So $(M_1, M_2) \in \mathcal{M}_{<2>}$.

From Lemma 1, the contribution $f_2(x,y)$ of $\mathcal{M}_2$ to $f(x,y)$ is as follows:

$$f_2(x,y) = yf^2(x,y) \tag{2.3}$$

Lemma 2 *Let* $\mathcal{M}_{<3>} = \{M - e_r(M) \mid M \in \mathcal{M}_3\}$. *Then* $\mathcal{M}_{<3>} = \mathcal{M}$.

Proof For any $M \in \mathcal{M}_3$, $M - e_r(M)$ is also a map in $\mathcal{M}$ from the definition $\mathcal{M}_3$.

On the other hand, for any map $M \in \mathcal{M}$, one can construct the map $M' \in \mathcal{M}$ from M by adding a trivial edge or twisted edge as the new root-edge $e_r(M')$ such that $M' \in \mathcal{M}_3$ described as follows. The new edge may be added from the root-vertex of M to each of vertices on all face boundaries of M. Further, considering that each double edge on the same boundary results two cases on its two different sides, and two loops accident to the root-vertex of M occur, there are $2\Big(2n(M)+1\Big)$ places where the other end of the new root-edge $e_r(M')$ can be attached altogether.

From Lemma 2, the contribution $f_3(x,y)$ of $\mathcal{M}_3$ to $f(x,y)$ can be derived:

$$f_3(x,y) = y \sum_{M\in\mathcal{M}} 2\Big(2n(M)+1\Big)x^{m(M)}y^{n(M)} = 2yf(x,y) + 4y^2\frac{\partial f(x,y)}{\partial y} \tag{2.4}$$

Theorem 1 *The enumerating function $f(x,y)$ of $\mathcal{M}$ with respect to vertices and edges satisfies the following equation:*

$$f(x,y) = x + yf^2(x,y) + 2yf(x,y) + 4y^2\frac{\partial f(x,y)}{\partial y} \tag{2.5}$$

Proof From (2.1), (2.2), (2.3) and (2.4), (2.5) can be deduced.

Corollary 1 *The enumerating function $F(y)$ of $\mathcal{M}$ with respect to edges satisfies the following equation:*

$$F(y) = 1 + yF^2(y) + 2yF(y) + 4y^2\frac{dF(y)}{dy} \tag{2.6}$$

Proof Let $x = 1$ in (2.5), then (2.6) can be gotten.

Corollary 2 *The enumerating function $T(y)$ of $\mathcal{T}$ with respect to edges satisfies the following equation:*

$$T(y) = 1 + 2yT(y) + 4y^2\frac{dT(y)}{dy} \tag{2.7}$$

Proof Considering the enumerating function $T(y)$ is the coefficient of x^1 in $f(x,y)$, $T(y) = \left[\frac{f(x,y)}{x}\right]_{x=0}$ and (2.7) is obtained from (2.5).

3. Solutions of Enumerating Equations

3.1 *Solutions in continued fractions*

In the subsection, we give new form solutions of (2.5), (2.6)and (2.7), which are continued fractions.

Theorem 2 *The enumerating function $f(x,y)$ of $\mathcal{M}$ with respect to vertices and edges can be expressed in continued fraction form:*

$$f(x,y) = \cfrac{x}{1-\cfrac{(x+2)y}{1-\cfrac{(x+4)y}{1-\cfrac{(x+6)y}{1-\cdots}}}} \tag{3.1}$$

Proof First, we defined a function sequence $\Big(f_k(x,y)\Big)_{k\geq 1}$:

(1) $f_1(x,y) = f(x,y)$, *i.e.* the enumerating function of $\mathcal{M}$;

(2) For any positive integer k, we construct so recurrence relation between $f_k(x,y)$ and $f_{k+1}(x,y)$:

$$f_k(x,y) = \frac{x+2(k-1)}{1-yf_{k+1}(x,y)} \qquad (a_k)$$

Claim: For every positive integer k, $f_k(x,y)$ is a solution of the following equation:

$$f_k(x,y) = x+2(k-1)+yf_k^2(x,y)+2yf_k(x,y)+4y^2\frac{\partial f_k(x,y)}{\partial y} \qquad (b_k)$$

which can be proved by an induce means as follows:

(i) For $k=1$, this is (2.5);

(ii) Let k be a positive integer and suppose that $f_k(x,y)$ is a solution of (b_k), *i.e.*

$$f_k(x,y) = x+2(k-1)+yf_k^2(x,y)+2yf_k(x,y)+4y^2\frac{\partial f_k(x,y)}{\partial y}$$

For $k+1$, one can substitute $f_k(x,y)$ by its expression with respect to $f_{k+1}(x,y)$ from (a_k), then

$$\begin{aligned}\frac{x+2(k-1)}{1-yf_{k+1}(x,y)} &= x+2(k-1)+y\Big[\frac{x+2(k-1)}{1-yf_{k+1}(x,y)}\Big]^2+2y\frac{x+2(k-1)}{1-yf_{k+1}(x,y)}\\ &\quad +4y^2\frac{\partial}{\partial y}\Big[\frac{x+2(k-1)}{1-yf_{k+1}(x,y)}\Big]\end{aligned}$$

Thus, $f_{k+1}(x,y) = x+2k+yf_{k+1}^2(x,y)+2yf_{k+1}(x,y)+4y^2\frac{\partial f_{k+1}(x,y)}{\partial y}$

This concludes the proof of the claim.

Now, (3.1) can be deduced by an iterating process as follows:

$$\begin{aligned}f(x,y) &= f_1(x,y) = \frac{x}{1-yf_2(x,y)} = \frac{x}{1-\dfrac{(x+2)y}{1-f_3(x,y)}} = \cdots\\ &= \cfrac{x}{1-\cfrac{(x+2)y}{1-\cfrac{(x+4)y}{1-\cfrac{(x+6)y}{1-\cdots}}}}\end{aligned}$$

Corollary 3 *The enumerating function $F(y)$ of $\mathcal{M}$ with respect to edges can be expressed in continued fraction form:*

$$F(y) = \cfrac{1}{1 - \cfrac{3y}{1 - \cfrac{5y}{1 - \cfrac{7y}{1 - \cdots}}}} \tag{3.2}$$

Proof Let $x = 1$ in (3.1), then (3.2) can be obtained.

Corollary 4 *The enumerating function $T(y)$ of $\mathcal{T}$ with respect to edges can be expressed in continued fraction form:*

$$T(y) = \cfrac{1}{1 - \cfrac{2y}{1 - \cfrac{4y}{1 - \cfrac{6y}{1 - \cdots}}}} \tag{3.3}$$

Proof Since $T(y) = \left[\frac{f(x,y)}{x}\right]_{x=0}$, (3.3) is easily derived from (3.1).

3.2 *Explicit enumeration formula*

In the subsection, we obtain an explicit enumeration formula for Equation (2.7).

Theorem 3 *The enumerating function $T(y)$ of $\mathcal{T}$ with respect to edges has the explicit formula as follows:*

$$T(y) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} y^n. \tag{3.4}$$

Proof According to the definition of $T(y)$, let

$$T(y) = T_0 + T_1 y + T_2 y^2 + \cdots + T_n y^n + \cdots \qquad (*)$$

where $T_i (i \geq 0)$ is an integer.

Taking $(*)$ into Equation (2.7), one can deduce the following recursive relation:

$$\begin{cases} T_0 = 1 & \text{(i);} \\ T_1 = 2T_0 & \text{(ii);} \\ T_n = (4n-2)T_{n-1} \quad (n \geq 1) & \text{(iii).} \end{cases}$$

Then we have

$$T_n = \prod_{i=1}^{n} (4i - 2) = \frac{2(2n-1)!}{(n-1)!} \quad (n \geq 1).$$

Together with (i), we can obtain

$$T(y) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} y^n.$$

According to Theorem 3 and Corollary 4, we additionally obtain an equation of continued fraction as follows:

$$T(y) = \cfrac{1}{1 - \cfrac{2y}{1 - \cfrac{4y}{1 - \cfrac{6y}{1 - \cdots}}}} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} y^n.$$

Remark : For three numbers of rooted general maps on all surfaces, the orientable surfaces and the nonorientable surfaces, the third one is clear as long as any two of three numbers are known. From Theorem 2 and [2, Theorem 3], the corresponding results of rooted general maps on the nonorientable surfaces can easily be deduced.

4. Number Tables

In the section, we give two number tables. In the first one, it is computed that the first terms of $f(x, y)$ with respect to vertices and edges according to (3.1). In the second one, we calculate the first terms of $F(y)$ and $T(y)$ with respect to edges from (3.2) and (3.3)(or (3.4)), respectively.

y/x	1	2	3	4	5
0	1	*	*	*	*
1	2	1	*	*	*
2	12	10	2	*	*
3	120	128	44	5	*
4	1680	2080	936	186	14
5	3024	41424	22000	5800	772
6	665280	981408	584000	183600	32712
7	17297280	27022848	17487232	6210176	1328880
8	518918400	849070080	586447104	227960960	55418144
9	17643225600	30001455360	21841559040	9109883776	2435456448
10	670442572800	1178093836800	896081597952	395684768000	113945118592

Table 1: The number of general maps regardless of genus, with respect to vertices(x) and edges(y)

y	$T(y)$	F(y)
0	1	1
1	2	3
2	12	24
3	120	297
4	1680	4896
5	30240	100278
6	665280	2450304
7	17297280	6953397
8	518918400	2247492096
9	17643225600	81528066378
10	670442572800	3280382613504

Table 2: The numbers of monopole maps and general maps regardless of genus, with respect to edges(y)

Acknowledgment: The authors are grateful to the anonymous referees for their constructive suggestions which improved the presentation and made it more readable.

References:

1. D. Arquès, Relation fonctionelles et dénombremant des cartes pointées sur letore, *J.Comb. Theory,Series B*43(1987),253–274.

2. D. Arquès and Jean-Francois Béraud, Rooted maps on orientable surfaces, Riccati's equation and continued fractions, *Discrete Mathematics* 215(2000),1–12.

3. W.G.Brown, On the enumeration of non-planar maps , Mem. Amer. Math. Soc. 65(1966).

4. E.A.Bender, E.R.Canfield and R.W.Robinson, The enumeration of maps on the torus and the projective plane, *Canada.Math.Bull.* 31(1988), 257–271.

5. Zhicheng Gao, The number of rooted 2–connected triangular maps on the projective plane. *J.Comb. Theory,Series B* 53(1991),130–142.

6. Wenzhong Liu and Yanpei Liu, The Number of Rooted Essential Maps on Surfaces. *Acta Mathematics Sinica, English Series*, 8(2008), 1303-1312.

7. Yanpei Liu, Enumerative Theory of Maps, Kluwer, /Boston/London 1999.

8. Yanpei Liu, Theory of Counting Combinatorial Maps [in Chinese], Science Press, Beijing, 2001.

9. Yanpei Liu, Algebraic Principles of Maps [in Chinese], Higher Education Press, Beijing, 2006.

10. Han Ren and Yanpei Liu The number of loopless 4–regular maps on the projective plane. *J.Comb. Theory,Series B* 84(2002),84–99.

11. Han Ren and Yanpei Liu 4-regular maps on the Klein bottle *J.Comb. Theory,Series B* 82(2001),118–137.

12. W.T.Tutte , A census of planar triangulations , *Canad.J.Math* 14(1962), 21–38.

13. W.T.Tutte , A census of slicings, *Canad.J.Math* 14(1962), 708–722

14. W.T.Tutte , A census of planar maps, *Canad.J.Math* 15(1963), 249–271.

15. W.T.Tutte , The enumerative theory of planar maps, in A Survey of Combinatorial Theory, J.N.Srivatava et al., eds., North Holland, 1973, 437–448.

16. T. Walsh and A. B. Lehman, Counting rooted maps by genus I , *J. Comb. Theory, Series B*13 (1972), 192-218. 43(1987),253–274.

17. T. Walsh and A. B. Lehman, Counting rooted maps by genus II , *J. Comb. Theory, Series B*13 (1972), 122-144.

18. T. Walsh and A. B. Lehman, Counting rooted maps by genus III , *J. Comb. Theory, Series B*18 (1975), 222-259.