2-semiarcs in PG$(2,q)$, $q \leq 13$

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Abstract

A 2-semiarc is a pointset $\mathcal{S}_2$ with the property that the number of tangent lines to $\mathcal{S}_2$ at each of its points is two. Using some theoretical results and computer aided search, the complete classification of 2-semiarcs in PG$(2,q)$ is given for $q \leq 7$, the spectrum of their sizes is determined for $q \leq 9$, and some results about the existence are proven for $q = 11$ and $q = 13$. For several sizes of 2-semiarcs in PG$(2,q)$, $q \leq 7$, classification results have been obtained by theoretical proofs.

1 Introduction

Ovals, k-arcs, and semiovals of finite projective planes are not only interesting geometric structures, but they have important applications to coding theory and cryptography, as well. For details about these objects we refer the reader to [11, 28, 29, 31].

Semiarcs are a natural generalization of arcs. Let Π_q be a projective plane of order q. A non-empty pointset $\mathcal{S}_t \subset \Pi_q$ is called a *t-semiarc* if for every point $P \in \mathcal{S}_t$ there exist exactly t lines $\ell_1, \ell_2, \ldots \ell_t$ such that $\mathcal{S}_t \cap \ell_i = \{P\}$ for $i = 1, 2, \ldots, t$. These lines are called the *tangents* to $\mathcal{S}_t$ at P. If a line ℓ meets $\mathcal{S}_t$ in $2, 3$ or k points (where $k > 3$), then ℓ is called a bisecant, trisecant or k-secant of $\mathcal{S}_t$, respectively. The classical examples of semiarcs are the semiovals ($t = 1$) and the subplanes ($t = q - m$, where m is the order of the subplane).

Semiarcs are closely connected to other combinatorial structures, too. Without the pursuit of wholeness we mention $(r, 1)$-designs and configurations.

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Definition 1.1. *A finite point-line incidence structure is called* linear space *if each line contains at least two points and any two distinct points are on exactly one line. If there are exactly r lines through each point, then the linear space is called* $(r, 1)$-design.

A (v_r, b_k)-configuration *is a finite point-line incidence structure with the following properties:*

- *There are v points and b lines.*
- *There are r lines through each point and there are k points on each line.*
- *Two distinct lines intersect each other at most once and two distinct points are connected by at most one line.*

If $v = b$ and $r = k$, then the configuration is called symmetric (v_k)-configuration.

The following proposition gives a natural correspondence between embeddable $(r, 1)$-designs and semiarcs in finite planes. Its proof is straightforward.

Proposition 1.2. *If $\mathcal{S}_t$ is a t-semiarc in Π_q, then the points of $\mathcal{S}_t$ and the secants of $\mathcal{S}_t$ form a $(q+1-t, 1)$-design. If an $(r, 1)$-design is embeddable into Π_q, then its points form a $(q+1-r)$-semiarc.*

Gropp investigated $(r, 1)$-designs with small r [26, 27]. He constructed all $(r, 1)$-designs with at most 12 points, his list contains 974 elements, most of them are configurations. His proof is computer assisted and he has not considered the embeddability of these designs.

In the last years the interest and research on the fundamental problem of determining the spectrum of the values for which there exists a given subconfiguration of points in $PG(n, q)$ have increased considerably (see for example [2,4–10,18–20,22,23,28,30,39,40,45]). In particular semiovals were investigated by several authors. Among others Lisonek [34] determined the spectrum of sizes of semiovals by exhaustive computer search for $q \leq 9$, q odd, Bartoli [3], Ranson and Dover [21,41], Kiss, Marcugini and Pambianco [32,33], and Nakagawa and Suetake [38,44] gave characterization theorems for semiovals in planes of small order.

Because of the huge diversity of semiarcs, their complete classification seems out of reach. The aim of this paper is to investigate and characterize 2-semiarcs in projective planes of order $q \leq 13$. Throughout the paper Π_q denotes an arbitrary projective plane of order q, while $PG(2, q)$ denotes the desarguesian projective plane over the field of q elements. It is well-known, that if $q = 2, 3, 4, 5, 7$ or 8, then each projective plane of order q is isomorphic to $PG(2, q)$.

The paper is organized as follows. In Section 2 we give lower and upper bounds and prove some number theoretical conditions on the sizes of 2-semiarcs in Π_q. Using these propositions and the results of Gropp, in Section 3 the complete characterization is provided for $q \leq 5$. In Section 4 we consider 2-semiarcs in PG(2,7). A computer-free description is given for semiarcs having sizes at most 12, and a computer-assisted proof shows that there are no 2-semiarcs in the plane with $|\mathcal{S}_2| \geq 13$. In Section 5 the description of the algorithm used to obtain the classification of 2-semiarcs is given. Finally in Section 6 results about the existence of 2-semiarcs in PG$(2,q)$ for $q \in \{8,9,11,13\}$ are given. The computer search is supported by the structural constraints proven in Section 2.

2 Some conditions on the sizes of 2-semiarcs

It follows from the definition that each t-semiarc in Π_q satisfies $t \leq q+1$. If t is close to this upper bound, then we can easily classify the t-semiarcs. The following proposition was proved by Csajbók and Kiss [17].

Proposition 2.1. *Let $\mathcal{S}_t$ be a t-semiarc in Π_q. The following properties hold:*

- *if $t = q+1$, then $\mathcal{S}_t$ is a single point,*
- *if $t = q$, then $\mathcal{S}_t$ is a subset of a line, and vice versa any subset of a line containing at least two points is a q-semiarc,*
- *if $t = q-1$, then $\mathcal{S}_t$ is a set of three non-collinear points.*

□

A semiarc cannot contain large collinear subsets. If $\mathcal{S}_t$ is a t-semiarc in Π_q, $\mathcal{S}_t$ is not contained in a line and it has a k-secant, then $k \leq q+1-t$ obviously holds. Semiarcs with long secants were investigated by Csajbók. He proved the following results; see [15, Theorems 2.4 and 4.6].

Theorem 2.2. *Let $\mathcal{S}_t$ be a t-semiarc in* PG$(2,q)$. *Then the following properties hold.*

- *If $t < (q-1)/2$, then $\mathcal{S}_t$ has no $(q+1-t)$-secants.*
- *If $\mathcal{S}_t$ has two $(q-t)$-secants such that the common point of these secants is not contained in $\mathcal{S}_t$ and* $\gcd(q,t) = \gcd(q-1,t-1) = 1$, *then $\mathcal{S}_t$ is the union of these two $(q-t)$-secants.*

Bounds on the sizes of t-semiarcs were also given by Csajbók and Kiss [17]. In the case $t = 2$ their result is the following.

Theorem 2.3. *Let $\mathcal{S}_2$ be a 2-semiarc in a projective plane of order q. Then*

$$q \leq |\mathcal{S}_2| \leq 1 + \left\lfloor \frac{q(1+\sqrt{8q-7})}{4} \right\rfloor.$$

The simplest example of a 2-semiarc of size q is a q-arc, a set of q points such that no three of them are collinear. As the following proposition shows, there are no more examples of 2-semiarc of size q.

Proposition 2.4. *Let $\mathcal{S}_2$ be a 2-semiarc of size q in a projective plane of order q. Then $\mathcal{S}_2$ is an arc.*

Proof. We have to prove that no three points of $\mathcal{S}_2$ are collinear. Suppose that the line ℓ is a trisecant of $\mathcal{S}_2$. If P is a point in $\ell \cap \mathcal{S}_2$, then $|\mathcal{S}_2| = q$ implies that there are at least $(q+1) - (q-2) = 3$ tangents to $\mathcal{S}_2$ at P, contradiction. □

Theorem 2.5. *In $\mathrm{PG}(2, p^h)$, $p \neq 2$, there exists, up to collineations, a unique 2-semiarc $\mathcal{S}_2$ of size $q = p^h$. Its stabilizer group has size $hq(q-1)$.*

Proof. $\mathcal{S}_2$ is an arc of size $q = p^h$. It is known, that in $\mathrm{PG}(2, q)$ each q-arc is contained in a $(q+1)$-arc, and if q is odd, then by the Theorem of Segre, it is contained in an irreducible conic [43]. The stabilizer of a conic is transitive on its points, hence all the q-point subsets of the conic are projectively equivalent. Since the number of conics is $q^2(q^2+q+1)(q-1)$ and each has $q+1$ subsets of size q, there exist exactly $q^2(q^2+q+1)(q-1)(q+1)$ different 2-semiarcs of size q. Thus the stabilizer group has size $\frac{|\mathrm{P\Gamma L}(3,q)|}{q^2(q^2+q+1)(q-1)(q+1)} = hq(q-1)$. □

If Π_q contains a 2-semiarc whose size is close to the lower bound q, then the order of the plane must satisfy some number theoretical conditions.

Proposition 2.6. *Let $\mathcal{S}_2$ be a 2-semiarc of size $q+1$ in a projective plane of order q. Then $q+1$ is divisible by 3.*

Proof. Let P be any point of $\mathcal{S}_2$. The total number of lines through P is $q+1$, and two of them are tangents to $\mathcal{S}_2$. The remaining q points of $\mathcal{S}_2$ are distributed among the $q-1$ secants through P. Hence there are $q-2$ bisecants and one trisecant through P. Thus each point of $\mathcal{S}_2$ lies on exactly one trisecant, hence $|\mathcal{S}_2|$ is divisible by 3. □

Proposition 2.7. *Let $\mathcal{S}_2$ be a 2-semiarc of size $q+2$ in a projective plane of order q. Then there exist integers $0 \leq \alpha$ and $0 \leq \beta \neq 1$ such that $q+2 = 4\alpha + 3\beta$.*

Proof. Let P be any point of $\mathcal{S}_2$. The total number of lines through P is $q+1$, two of them are tangents to $\mathcal{S}_2$. The remaining $q+1$ points of $\mathcal{S}_2$ are distributed among the $q-1$ secants through P. Hence there are either two trisecants and $q-3$ bisecants, or one 4-secant and $q-2$ bisecants through P. Thus each point lies on either two trisecants or one 4-secant. Let $\mathcal{T}_3$ be the set of points lying on two trisecants. Then it is a configuration (v_2, k_3), where $v = |\mathcal{T}_3|$ and, by [25, Theorem 3.1], $|\mathcal{T}_3| = 3\beta$, with $\beta \neq 1$. Let $\mathcal{T}_4$ be the set of points lying on one 4-secant, then $|\mathcal{T}_4| = 4\alpha$. Then $q+2 = 4\alpha + 3\beta$, with $\alpha, \beta \geq 0$ and $\beta \neq 1$. □

3 2-semiarcs in small planes

The classification of 2-semiarcs in the cases $q = 2$ and $q = 3$ follows from Proposition 2.1.

Theorem 3.1.

- *In* PG(2, 2) *each* 2*-semiarc* $\mathcal{S}_2$ *consists of two or three collinear points.*
- *In* PG(2, 3) *each* 2*-semiarc* $\mathcal{S}_2$ *is a set of three non-collinear points.*

If $q = 4$, then 2-semiarcs correspond to (3, 1)-designs by Proposition 1.2. Gropp [26, Table 1] proved that there are three such designs, they consist of 4, 6 and 7 points, respectively. He also gave a detailed combinatorial description of these objets. We show that each of these designs is embeddable into PG(2, 4).

Theorem 3.2. *In* PG(2, 4) *there exist three projectively non-equivalent* 2*-semiarcs.*

- $|\mathcal{S}_2| = 4$, *four points in general position.*
- $|\mathcal{S}_2| = 6$, *the vertices of a complete quadrilateral.*
- $|\mathcal{S}_2| = 7$, *the points of a subplane of order 2.*

Proof. It is easy to verify (without applying Gropp's results), that there are only three possible sizes of a 2-semiarc. Theorem 2.3 gives

$$4 \leq |\mathcal{S}_2| \leq 7.$$

From Proposition 2.6 we get $|\mathcal{S}_2| \neq 5$, because $q+1 = 5$ is not divisible by 3. Hence $|\mathcal{S}_2| \in \{4, 6, 7\}$.

The case $|\mathcal{S}_2| = 4$ follows from Proposition 2.4. The combinatorial description of Gropp gives that if $|\mathcal{S}_2| = 6$, then there are two trisecants and one bisecant through each point, hence the design corresponds to the six vertices of a complete quadrilateral and it is obviously embeddable into PG(2,4).

If $|\mathcal{S}_2| = 7$, then according to Gropp, the design is a (7_3)-configuration. In other words this is the Fano plane PG(2,2), which is embeddable into PG(2,4). □

Table 1: Description of the three 2-semiarcs in PG(2,4)

$\|\mathcal{S}_2\|$	x_0	x_1	x_2	x_3	G
4	7	8	6	0	$\mathbb{Z}_2 \times S_4$
6	2	12	3	4	$\mathbb{Z}_2 \times S_4$
7	0	14	0	7	$\mathrm{PSL}(3,2) \times \mathbb{Z}_2$

Table 1 contains the non-equivalent 2-semiarcs in PG(2,4), the number of their i-secants, x_i, and the description of the stabilizer groups in PΓL(3,4).

If $q = 5$, then 2-semiarcs correspond to $(4,1)$-designs by Proposition 1.2. Gropp [26, Table 1] proved that there are eight such designs with at most 12 points. We show that only three of them are embeddable into PG(2,5).

Theorem 3.3. *In* PG(2,5) *there exist three projectively non-equivalent 2-semiarcs.*

- $|\mathcal{S}_2| = 5$, *five points of a conic.*
- $|\mathcal{S}_2| = 6$, *the union of two trisecants.*
- $|\mathcal{S}_2| = 9$, *the projective triangle.*

Proof. It is easy to see that there are only four possible sizes of a 2-semiarc. Theorem 2.3 gives $5 \leq |\mathcal{S}_2| \leq 9$. From Proposition 2.7 we get $|\mathcal{S}_2| \neq 7$, because $q + 2 = 7$ cannot be written as $4\alpha + 3\beta$ with $\beta \neq 1$.

First we prove that $|\mathcal{S}_2| \neq 8$. Suppose to the contrary that $\mathcal{S}_2$ is a 2-semiarc with 8 points. Gropp proved that there is only one $(4,1)$-design with eight points, the symmetric (8_3)-configuration (also called Möbius-Kantor configuration). But it was proven by Abdul-Elah, Al-Dhahir and Jungnickel [1] that this configuration cannot be embedded into PG(2,5). Hence $|\mathcal{S}_2| \in \{5,6,9\}$.

The case $|\mathcal{S}_2| = 5$ follows from Proposition 2.4.

In the case $|\mathcal{S}_2| = 6$, if $P \in \mathcal{S}_2$ is a point, then there are $6 - 2 = 4$ non-tangents through P, hence $\mathcal{S}_2$ has no 4-secants. Let a be the number of trisecants, and b be the number of bisecants through P. Then we get $a+b=4$ and $2a+b=5$, hence $a=1$ and $b=3$. So $\mathcal{S}_2$ is the union of two trisecants, ℓ_1 and ℓ_2. This is the second case of Theorem 2.2.

Finally consider the case $|\mathcal{S}_2| = 9$. Gropp proved that there are two $(4,1)$-designs with nine points. One of them is the affine plane of order 3. But AG$(2,3)$ cannot be embedded into PG$(2,q)$ if $q \equiv 2 \pmod 3$ (see e.g. [12]).

The points of the other $(4,1)$-design are of two types: (i) the vertices of a triangle $\mathcal{T}$, (ii) the points on exactly one side of $\mathcal{T}$, two points on each side. If a point is of type (i), then it is on two 4-secants and on two bisecants; if a point is of type (ii), then it is on one 4-secant and hence on two trisecants and on one bisecant. Hence $\mathcal{S}_2$ has three 4-secants, $6\cdot 2/3 = 4$ trisecants and $(3\cdot 2+6\cdot 1)/2 = 6$ bisecants. $\mathcal{S}_2$ also has $9\cdot 2 = 18$ tangents, so $\mathcal{S}_2$ is a blocking set because $3+4+6+18 = 31$ equals to the total number of lines in PG$(2,5)$. This blocking set has cardinality $3(q+1)/2$, hence by a theorem of Lovász and Schrijver [35] it is a projective triangle.

A possible embedding into PG$(2,5)$ is the following. The vertices of $\mathcal{T}$: $\{(1:0:0),(0:1:0),(0:0:1)\}$, the points on the sides of $\mathcal{T}$: $\{(1:1:0),(4:1:0),(1:0:1),(4:0:1),(0:1:1),(0:4:1)\}$. □

Table 2: Description of the three 2-semiarcs in PG$(2,5)$

$\|\mathcal{S}_2\|$	x_0	x_1	x_2	x_3	x_4	G
5	11	10	10	0	0	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
6	8	12	9	2	0	D_4
9	0	18	6	4	3	S_4

Table 2 contains the non-equivalent 2-semiarcs in PG$(2,5)$, the number of their i-secants, x_i, and the description of the stabilizer groups in PGL$(3,5)$.

4 2-semiarcs in PG$(2,7)$

The number of $(6,1)$-designs with at most 12 points is 47.

Instead of considering the list of Gropp [26], we give a geometric characterization of the embeddable designs and we prove that there exist 25 non-equivalent 2-semiarcs in PG$(2,7)$. First consider the long secants of the semiarcs. If $q=7$ and $t=2$, then Theorem 2.2 gives the following corollary.

Corollary 4.1. *Let $\mathcal{S}_2$ be a 2-semiarc in* PG(2, 7). *Then $\mathcal{S}_2$ has no 6-secants. If $\mathcal{S}_2$ has two 5-secants such that the common point of these secants is not contained in $\mathcal{S}_2$, then $\mathcal{S}_2$ is the union of these two 5-secants.*

If the common point of the long secants belongs to $\mathcal{S}_2$, then the size of the semiarc cannot be small.

Proposition 4.2. *Let $\mathcal{S}_2$ be a 2-semiarc in* PG(2, 7). *If $\mathcal{S}_2$ has two 5-secants such that the common point of these secants is contained in $\mathcal{S}_2$, then* $|\mathcal{S}_2| > 12$.

Proof. Let ℓ_1 and ℓ_2 be the 5-secants and let $P \in \ell_1 \cap \ell_2$. Then $P \in \mathcal{S}_2$ implies that there are six secants of $\mathcal{S}_2$ through P. Hence $\mathcal{S}_2 \setminus (\ell_1 \cup \ell_2)$ must contain at least four points. So $|\mathcal{S}_t| \geq 9 + 4 = 13$ holds. □

Theorem 4.3. *In* PG(2, 7) *there are nine combinatorially non-equivalent 2-semiarcs (there are projectively non-equivalent subclasses in some combinatorial classes).*

- $|\mathcal{S}_2| = 7$, *seven points of a conic.*
- $|\mathcal{S}_2| = 9$, *there are two types,*
  1. *nine vertices of a* 3×3 *grid,*
  2. *the six vertices of two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$, and the three points of intersections of the corresponding sides of $\mathcal{T}_1$ and $\mathcal{T}_2$.*
- $|\mathcal{S}_2| = 10$, *there are two types,*
  1. *the union of two 5-secants,*
  2. *the points of a* 10_3 *configuration.*
- $|\mathcal{S}_2| = 11$, *then the semiarc has no 5-secant. There are two types,*
  1. *four 4-secants and four trisecants,*
  2. *one 4-secant and ten trisecants.*
- $|\mathcal{S}_2| = 12$, *then it has three 4-secants and these lines form a triangle $\mathcal{T}$. There are two types,*
  1. *two vertices of $\mathcal{T}$ belong to $\mathcal{S}_2$,*
  2. *three vertices of $\mathcal{T}$ belong to $\mathcal{S}_2$.*

Proof. Theorem 2.3 gives $7 \leq |\mathcal{S}_2| \leq 15$. Let s be the number of points of $\mathcal{S}_2$, let $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_{57}\}$ be the set of lines of PG(2,7) and let $c_i = |\mathcal{S}_2 \cap \ell_i|$ for $i = 1, 2, \ldots, 57$. If we count in two different ways the number of incident point-line pairs (P, ℓ_j) where $\ell_j \in \mathcal{L}$ and $P \in \mathcal{S}_2$, and the ordered triples (P_1, P_2, ℓ_j) where $\ell_j \in \mathcal{L}$ and the distinct points P_1 and P_2 are in $\mathcal{S}_2 \cap \ell_j$, then we get

$$\sum_{i=1}^{57} c_i = 8s \quad \text{and} \quad \sum_{i=1}^{57} c_i(c_i - 1) = s(s-1).$$

Hence

$$\sum_{i=1}^{57} c_i^2 = s^2 + 7s.$$

We may assume without loss of generality that the lines $\ell_{58-2s}, \ell_{59-2s}, \ldots, \ell_{57}$ are the tangents to $\mathcal{S}_2$, for these lines $c_i = 1$. If we subtract these values, then we get

$$\sum_{i=1}^{57-2s} c_i = 6s \quad \text{and} \quad \sum_{i=1}^{57-2s} c_i^2 = s^2 + 5s. \tag{1}$$

It follows from Corollary 4.1 that if $k \geq 6$, then $\mathcal{S}_2$ has no k-secant. Let x_i be the number of i-secants of $\mathcal{S}_2$ for $i = 0, 1, \ldots, 5$. Then

$$\sum_{i=1}^{57-2s} (c_i - 2)(c_i - 3) = 6x_0 + 2x_4 + 6x_5$$

and

$$\sum_{i=1}^{57-2s} (c_i - 3)(c_i - 4) = 12x_0 + 2x_2 + 2x_5.$$

On the other hand, Equations (1) give

$$\sum_{i=1}^{57-2s} (c_i - 2)(c_i - 3) = \sum_{i=1}^{57-2s} (c_i^2 - 5c_i + 6) =$$

$$= s^2 + 5s - 5 \cdot 6s + 6(57 - 2s) = s^2 - 37s + 342$$

and

$$\sum_{i=1}^{57-2s} (c_i - 3)(c_i - 4) = \sum_{i=1}^{57-2s} (c_i^2 - 7c_i + 12) =$$

$$= s^2 + 5s - 7 \cdot 6s + 12(57 - 2s) = s^2 - 61s + 684.$$

Hence

$$6x_0+2x_4+6x_5 = s^2-37s+342 \quad \text{and} \quad 12x_0+2x_2+2x_5 = s^2-61s+684. \quad (2)$$

First we prove the non-existence parts of the theorem. From Proposition 2.6 we get $|S_2| \neq 8$, because $q+1=8$ is not divisible by 3.

Suppose, that $s = 15$. Then Equations (1) give

$$\sum_{i=1}^{27} c_i = 90 \quad \text{and} \quad \sum_{i=1}^{27} c_i^2 = 300.$$

Applying the inequality between the arithmetic and quadratic means we get

$$\frac{90}{27} = \frac{\sum_{i=1}^{27} c_i}{27} \leq \sqrt{\frac{\sum_{i=1}^{27} c_i^2}{27}} = \sqrt{\frac{300}{27}} = \frac{10}{3}.$$

Thus equality holds, hence $c_1 = c_2 = \ldots = c_{27}$. But $90/27$ is not an integer, contradiction.

Now suppose, that $s = 14$. Then Equations (2) give

$$3x_0 + x_4 + 3x_5 = 10 \quad \text{and} \quad 6x_0 + x_2 + x_5 = 13.$$

Elementary counting shows that there are only nine possibilities for the numbers $x_0, x_1, \ldots, x_5$. These are the following.

x_0	x_1	x_2	x_3	x_4	x_5
2	28	1	22	4	0
1	28	7	14	7	0
0	28	13	6	10	0
2	28	0	25	1	1
1	28	6	17	4	1
0	28	12	9	7	1
1	28	5	20	1	2
0	28	11	12	4	2
0	28	10	15	1	3

Now suppose, that $s = 13$. Then Equations (2) give

$$3x_0 + x_4 + 3x_5 = 15 \quad \text{and} \quad 6x_0 + x_2 + x_5 = 30.$$

Elementary counting shows that there are only twelve possibilities for the numbers $x_0, x_1, \ldots, x_5$. These are as follows.

x_0	x_1	x_2	x_3	x_4	x_5
5	26	0	26	0	0
4	26	6	18	3	0
3	26	12	10	6	0
2	26	18	2	9	0
4	26	5	21	0	1
3	26	11	13	3	1
2	26	17	5	6	1
3	26	10	16	0	2
2	26	16	8	3	2
1	26	22	0	6	2
2	26	15	11	0	3
1	26	21	3	3	3

In these cases an exhaustive computer search shows that there are no 2-semiarcs of sizes 14 and 13 in PG(2,7).

Now consider the existence parts. The case $|\mathcal{S}_2| = 7$ follows from Proposition 2.4.

If $|\mathcal{S}_2| = 9$, then we can apply Proposition 2.7. As $9 = 4\alpha + 3\beta$ implies $\alpha = 0$ and $\beta = 3$, we get that there is no 4-secant of $\mathcal{S}_2$ and there are two trisecants through each point of $\mathcal{S}_2$. Hence the total number of trisecants is $9 \times 2/3 = 6$. There are two possibilities.

(i) There do not exist three trisecants such that they form a triangle whose three vertices are in $\mathcal{S}_2$. Then the points of $\mathcal{S}_2$ are the nine vertices of a 3×3 grid, whose six lines are the trisecants of $\mathcal{S}_2$. An example for this case is the following. The points of $\mathcal{S}_2$ are the points of intersections of three horizontal and three vertical lines. Their cartesian coordinates are the following: $(0,0)$, $(1,0)$, $(3,0)$, $(0,1)$, $(1,1)$, $(3,1)$, $(0,4)$, $(1,4)$ and $(3,4)$.

The grid has two triples of lines. There are two possibilities in each triples: the lines either form a triangle or they belong to a pencil. Hence there are projectively non-isomorphic examples of this combinatorial type (see Table 3).

(ii) There exist three trisecants such that they form a triangle $\mathcal{T}_1$ whose three vertices, say P_1, P_2 and P_3 are in $\mathcal{S}_2$. In this case $\mathcal{S}_2$ contains three points, say Q_1, Q_2 and Q_3 from the sides of $\mathcal{T}_1$, and three more points, say R_1, R_2 and R_3. Consider the three other trisecants of $\mathcal{S}_2$. If Q_iQ_j were a trisecant, then it ought to contain exactly one point from the set $\{R_1, R_2, R_3\}$, hence both of the remaining two trisecants would pass on the other two R_i, contradiction. So each of the remaining three trisecants contains one point of the set $\{Q_1, Q_2, Q_3\}$, hence two points from the set $\{R_1, R_2, R_3\}$. So the points R_1, R_2 and R_3 form a triangle $\mathcal{T}_2$. An example for this case is the following. The homogeneous coordinates of the vertices of $\mathcal{T}_1$ are $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$, the coordinates of the vertices of $\mathcal{T}_2$ are $(2:3:1)$, $(3:4:1)$ and $(5:5:1)$. The points of intersections of the corresponding sides are $(1:4:0)$, $(0:1:1)$ and $(1:0:1)$.

There are projectively non-isomorphic examples of this combinatorial type, too (see Table 3).

If $|\mathcal{S}_2| = 10$, then first we consider the largest collinear subset of $\mathcal{S}_2$. Because of Theorem 2.2 its cardinality is at most $q-2=5$. If $\mathcal{S}_2$ has a 5-secant, then Csajbók, Héger and Kiss [16, Proposition 2.3] proved that $\mathcal{S}_2$ is the union of two 5-secants.

If $\mathcal{S}_2$ has no 5-secants, then the points of $\mathcal{S}_2$ can be partitioned into two subsets. Let $\mathcal{A} \subset \mathcal{S}_2$ be the set of points belonging to three trisecants of $\mathcal{S}_2$ and let $\mathcal{B} \subset \mathcal{S}_2$ be the set of points belonging to one trisecant and one 4-secant of $\mathcal{S}_2$. If $|\mathcal{A}| = a$ and $|\mathcal{B}| = b$, then the total number of trisecants of $\mathcal{S}_2$ is $(3a+b)/3$, hence $3|b$. Thus if $b>0$, then $b \geq 3$, and no point of $\mathcal{S}_2$ lies on more than one 4-secant. Hence $b>0$ implies $|\mathcal{S}_2| \geq 3 \times 4 = 12$, contradiction. So $\mathcal{S}_2$ has no 4-secant, hence it is a (10_3)-configuration. An example for this case is the Desargues configuration.

It is known that there are ten projectively non-isomorphic (10_3)-configuration [25]. The embeddability of these configurations were investigated by Glynn [24], who proved that one of them is not embeddable into any pappian plane. It is also known, that the other nine can be embedded into the classical euclidean plane [13]. Our exhaustive computer search shows that these nine can also be embedded into PG(2, 7).

If $|\mathcal{S}_2| = 11$, then it is a 2-semiarc with $q+4$ points. For each point $P \in \mathcal{S}_2$ there are $q-1$ secants through P, thus $q+3$ points of $\mathcal{S}_2$ are distributed among the secants through P. It follows from Corollary 4.1 that $\mathcal{S}_2$ has no 6-secant. Thus the points of $\mathcal{S}_2$ can be partitioned into four subsets. Let $\mathcal{A} \subset \mathcal{S}_2$ be the set of points belonging to four trisecants of $\mathcal{S}_2$, let $\mathcal{B} \subset \mathcal{S}_2$ be the set of points belonging to two trisecants and one 4-secant of $\mathcal{S}_2$, let $\mathcal{C} \subset \mathcal{S}_2$ be the set of points belonging to two 4-secants of $\mathcal{S}_2$ and finally let $\mathcal{D} \subset \mathcal{S}_2$ be the set of points belonging to one trisecant and one 5-secant of $\mathcal{S}_2$.

First we prove that $\mathcal{D} = \emptyset$. Let $|\mathcal{A}| = a$, $|\mathcal{B}| = b$, $|\mathcal{C}| = c$ and $|\mathcal{D}| = d$. Let s be the number of 5-secants. Then Corollary 4.1 and Proposition 4.2 imply that $s \leq 1$. Suppose that $s=1$. Then we show that $c=0$ also holds. The 4-secants cannot meet the 5-secant in a point of $\mathcal{S}_2$ and the union of two intersecting 4-secants contains 7 points, so if $c \neq 0$, then $\mathcal{S}_2$ contains at least $5+7>11$ points, contradiction. So $s=1$ implies $a+b=6$. The number of the 4-secants of $\mathcal{S}_2$ is $b/4$, hence $4|b$. There are only two possibilities, either $b=0$ or $b=4$. In the first case $a=6$, in the second $a=2$. The number of the trisecants of $\mathcal{S}_2$ is $(5+4a+2b)/3$. If $b=0$, then this number is $5 + 4 \cdot 6 = 29$ and it is not divisible by 3, contradiction. If $b = 4$ then $a = 2$, and $\mathcal{S}_2$ has one 5-secant, ℓ_5, one 4-secant, ℓ_4 and seven trisecants. Let $\mathcal{S}_2 \setminus (\ell_5 \cup \ell_4) = \{P, R\}$. Then there are four trisecants through both P and R, hence the line PR is a trisecant. Each of the other $2 \times 3 = 6$ trisecants through P or R must contain one point of ℓ_5, but there exists a unique trisecant at each point of ℓ_5. This contradiction proves $d = 0$.

If $d = 0$, then $a + b + c = 11$. The number of the trisecants of $\mathcal{S}_2$ is $(4a + 2b)/3$, hence

$$b \equiv a \pmod 3.$$

The number of the 4-secants of $\mathcal{S}_2$ is $(b + 2c)/4 = (22 - 2a - b)/4$, hence

$$b \equiv 2a + 2 \pmod 4.$$

Thus the Chinese Remainder Theorem gives

$$b \equiv 10a + 6 \pmod{12}.$$

We know that $0 \leq a, b \leq 11$, hence if a is given, then this congruence uniquely determines b, and also $0 \leq c = 11 - a - b$. We have the following possibilities.

a	0	1	2	3	4	5	6	7	8	9	10	11
b	6	4	2	0	10	8	6	4	4	0	10	8
c	5	6	7	8	–	–	–	0	–	2	–	–
Case	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}

An example for Case A_1 is the following. Let

$$\mathcal{C} = \{(1:0:0), (0:1:0), (0:0:1), (1:1:0), (1:0:1)\}.$$

The two 4–secants through $(1:0:0)$ contain the points

$$(1:0:0), (0:1:0), (1:1:0), (1:5:0)$$

and

$$(1:0:0), (0:0:1), (1:0:1), (1:0:4),$$

respectively. The 4-secant through $(0:0:1)$ and $(0:1:0)$ contains the points $(0:1:1), (0:1:5) \in \mathcal{B}$. The 4-secant through $(1:1:0)$ and $(1:0:1)$ contains the points $(1:3:5), (1:2:6) \in \mathcal{B}$.

An example for Case A_8 is the following. Let

$$(1:0:0) \in \mathcal{A} \quad \text{and} \quad \mathcal{B} = \{(0:1:0), (0:0:1), (0:1:1), (0:1:3)\}.$$

The four 3–secants through $(1:0:0)$ contain the points

$$\{(1:0:0), (0:1:0), (1:2:2)\}, \quad \{(1:0:0), (0:1:1), (1:4:4)\},$$

$$\{(1:0:0), (1:1:5), (1:4:6)\}, \quad \{(1:0:0), (1:6:1), (1:3:4)\}.$$

We prove that the other cases do not appear. Cases $A_5, A_6, A_7, A_9, A_{11}$, and A_{12} cannot appear, because they do not satisfy the condition $a+b+c=11$. The number of the 4-secants of $\mathcal{S}_2$ is $f=(b+2c)/4$. If $b=0$ and $c=2$, then $f=1$, but then obviously no point which is on two 4-secants exists. In the Cases A_2, A_3 and A_4 we have $f=4$. But four lines have at most 6 points of intersections, hence $c=7$ and $c=8$ are impossible. If $c=6$, then the four 4-secants form a complete quadrilateral, the sides of it contain the four points of the set $\mathcal{B}$, and $\mathcal{A}$ consists of a single point, say P. Then each of the four trisecants through P must contain two points from $\mathcal{B}$. But then the pigeonhole principle implies that some of these trisecants have more than one point in common. This contradiction proves the nonexistence of this configuration.

If $|\mathcal{S}_2| = 12$, then Equations (2) give

$$6x_0 + 2x_4 + 6x_5 = 42 \quad \text{and} \quad 6x_0 + x_2 + x_5 = 48.$$

Proposition 4.2 gives that $x_5 \leq 1$, hence elementary counting shows that there are only five possibilities for the numbers $x_0, x_1, \ldots, x_5$. These are the following.

x_0	x_1	x_2	x_3	x_4	x_5	Case
7	24	6	20	0	0	B_1
6	24	12	12	3	0	B_2
5	24	18	4	6	0	B_3
6	24	11	15	0	1	B_4
5	24	17	7	3	1	B_5

We show that only Case B_2 appears. For each point $P \in \mathcal{S}_2$ there are six secants through P. We have to distribute 11 points among the secants through P. It follows from Corollary 4.1 that $\mathcal{S}_2$ has no 6-secant. Thus the points of $\mathcal{S}_2$ can be partitioned into four subsets.

If $\mathcal{S}_2$ has a 5-secant, ℓ, then let $\mathcal{R} = \mathcal{S}_2 \setminus \ell$ and let $\ell \setminus \mathcal{S}_2 = \{P, Q, R\}$.

- Case B_1. In this case $\mathcal{S}_2$ is a $(12,3)$-arc. In [14] the intersection sizes with lines of all the regular complete $(12,3)$-arcs in $\mathrm{PG}(2,7)$ are presented and there exist no regular complete $(12,3)$-arcs in $\mathrm{PG}(2,7)$ having 20 trisecants. An exhaustive computer search among incomplete $(12,3)$-arcs in $\mathrm{PG}(2,7)$ shows that all of them have less than 20 trisecants.

- Case B_3. First we prove that no three of the 4-secants have a point in common. There are at most two 4-secants through any point of $\mathcal{S}_2$, and if three 4-secants would meet in a point outside $\mathcal{S}_2$, then the union of these lines would contain $\mathcal{S}_2$, so any other line could contain at most three points of $\mathcal{S}_2$, but the total number of 4-secants is six. Hence through each point of $\mathcal{S}_2$ there are exactly two 4-secants and one trisecant of $\mathcal{S}_2$.

  The six 4-secants have $6 \cdot 5/2 = 15$ points of intersection. Three of these points are not in $\mathcal{S}_2$, let X and Y be two of them and let O and E be two points of $\mathcal{S}_2$ such that OX, OY, EX and EY are 4-secants of $\mathcal{S}_2$. There is a projectivity mapping the points of the projective frame to $\{X, Y, O, E\}$. After this projectivity the points of $\mathcal{S}_2$ are in the affine plane. If we use cartesian coordinates, we get $O = (0,0)$, $E = (1,1)$, and the points $P = OX \cap EY = (1,0)$ and $R = OY \cap EX = (0,1)$ belong to $\mathcal{S}_2$. Let the further points of $OY \cap \mathcal{S}_2$ and $OX \cap \mathcal{S}_2$ be $A = (0,a)$, $B = (0,b)$, and $C = (c,0)$, $D = (d,0)$, respectively. Then $\{a,b,c,d\} \cap \{0,1\} = \emptyset$.

  Without loss of generality we may assume that the lines AC and BD are 4-secants; their equations are $X/c + Y/a = 1$ and $X/d + Y/b = 1$, respectively. Then the remaining points of $\mathcal{S}_2$ must be $PY \cap AC = K = (1, a - a/c)$, $PY \cap BD = L = (1, b - b/d)$, $RX \cap AC = M = (c - c/a, 1)$ and $RX \cap BD = N = (d - d/b, 1)$. Hence the lines OE and PR are bisecants. Consider the unique trisecant through O. It must contain one point from the set $\{K, L\}$ and one point from the set $\{M, N\}$. But none of the lines KM and LN contains O, thus without loss of generality we may assume, that the line KN is the trisecant through O. Hence

$$a - \frac{a}{c} = \frac{1}{d - \frac{d}{b}} \iff \frac{a(c-1)}{c} = \frac{b}{d(b-1)}. \tag{3}$$

  In the same way we get that the unique trisecants through the points P, R and E must be the lines MB, LC and DA, respectively. The equation of the line joining the points $(s,0)$ and $(0,t)$ is $X/s + Y/t = 1$, thus from these collinearity conditions we get the following equations:

$$c - \frac{c}{a} + \frac{1}{b} = 1 \iff \frac{c(a-1)}{a} = \frac{b-1}{b}, \tag{4}$$

$$\frac{1}{c} + b - \frac{b}{d} = 1 \iff \frac{b(d-1)}{d} = \frac{c-1}{c}, \tag{5}$$

$$\frac{1}{d} + \frac{1}{a} = 1 \iff d = \frac{a}{a-1}. \tag{6}$$

From the last equation we get $(d-1)/d = 1/a$, hence Equations (5) and (3) give $b = (2a-1)/a$ and $bc = 1$. Finally from Equations (5) and (4) we get $c = (a+1)/a$. Hence

$$\frac{2a-1}{a} \cdot \frac{a+1}{a} = 1, \quad \text{thus} \quad a^2 + a - 1 = 0.$$

But this equation has no root in GF(7), so there is no semiarc of this type in PG(2, 7).

- Case B_4. Each point of $\mathcal{S}_2 \cap \ell$ is contained in two trisecants. Thus the number of trisecants of $\mathcal{S}_2$ through the points of $\mathcal{S}_2 \cap \ell$ is 10. Let x'_2 and x'_3 be the number of bisecants and trisecants of $\mathcal{R}$, respectively. Then counting in two different ways the ordered triples (A, B, e) where both A and B are points in $\mathcal{R}$ and e is a line incident with both of them, we get $2x'_2 + 6x'_3 = 42$. On the other hand, each trisecant of $\mathcal{S}_2$ containing a point of ℓ corresponds to a bisecant of $\mathcal{R}$. Since the number of trisecants of $\mathcal{S}_2$ is 15, the other 5 trisecants of $\mathcal{S}_2$ must be trisecants also for $\mathcal{R}$, thus $x'_2 \geq 10$ and $x'_3 = 5$. Hence $42 = 2x'_2 + 6x'_3 \geq 20 + 30$, contradiction. So there is no semiarc of this type.

- Case B_5. There are no 4-secants through the points of $\ell \cap \mathcal{S}_2$. Hence each of the three 4-secants meets $\mathcal{R}$ in four points. But the union of the three 4-secants contains at least $4 + 3 + 2 = 9$ distinct points and $\mathcal{R}$ contains only seven points. So there is no semiarc of this type.

Thus only Case B_2 can appear. Now $\mathcal{S}_2$ has three 4-secants, say ℓ_1, ℓ_2 and ℓ_3. Let $\mathcal{M}$ be the set of points of intersections of the 4-secants. The number of 4-secants through any point of $\mathcal{S}_2$ is at most two, hence there are four possibilities.

1. $|\mathcal{M}| = 1$ and $\mathcal{M} \cap \mathcal{S}_2 = \emptyset$,
2. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 1$,
3. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 2$,
4. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 3$.

An exhaustive computer search shows that there are no examples in cases 1 and 2, and there are examples in cases 3 and 4. An example of the third case is the following. Let

$$\mathcal{S}_2 \cap \ell_1 = \{(1:0:0),(1:0:4),(1:0:5),(1:0:6)\},$$
$$\mathcal{S}_2 \cap \ell_2 = \{(1:0:0),(0:1:0),(1:5:0),(1:6:0)\},$$
$$\mathcal{S}_2 \cap \ell_3 = \{(0:1:0),(0:1:2),(0:1:3),(0:1:5)\},$$
$$\mathcal{S}_2 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \{(1:1:1),(1:5:1)\}.$$

An example of the fourth case is the following. Let

$$\mathcal{S}_2 \cap \ell_1 = \{(1:0:0),(0:0:1),(1:0:1),(1:0:5)\},$$
$$\mathcal{S}_2 \cap \ell_2 = \{(1:0:0),(0:1:0),(1:2:0),(1:3:0)\},$$
$$\mathcal{S}_2 \cap \ell_3 = \{(0:0:1),(0:1:0),(0:1:4),(0:1:5)\},$$
$$\mathcal{S}_2 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \{(1:1:3),(1:1:6),(1:4:3)\}.$$

□

Table 3 contains the projectively non-equivalent 2-semiarcs in PG(2, 7), the number of their i-secants, x_i, and the description of the stabilizer groups in PGL(3, 7). We denote the group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ by G_{42}.

5 The algorithm

The algorithm used for the classification of 2-semiarcs in $PG(2, q)$ is a modification of the one presented in [3, 37]. When possible, the search is helped by the structural constraints proven in Section 2.

In this case the algorithm works on *admissible sets*, i.e. sets such that each point lies on at least two tangent lines, instead of working on partial solutions. In fact, the property of being a 2-semiarc is not a hereditary feature, i.e. a feature conserved by all the subsets, so the weaker hereditary feature of being an admissible set has been used. It is weaker in the sense that it allows to prune very few branches of the search space with respect to the cases when considering arcs and $(k, 3)$-arcs. This and the fact that 2-semiarcs are in general larger than arcs and $(k, 3)$-arcs make the problem computationally harder than the ones faced in [36, 37].

Note also that, in general, not all the admissible sets can be extended to 2-semiarcs.

The exhaustive search has been feasible because projective properties among admissible sets have been exploited to avoid obtaining too many isomorphic copies of the same 2-semiarc and to avoid searching through parts of the search space isomorphic to previously searched ones.

The algorithm starts constructing a tree structure containing a representative of each class of non-equivalent admissible sets of size less than or equal to a fixed threshold h. If the threshold h were equal to the actual size of the putative 2-semiarcs, the algorithm would be orderly, that is capable of constructing each goal configuration exactly once [42].

However, in the present case, the construction of the tree with the threshold h equal to the size of the putative 2-semiarcs would have been too space and time consuming. For this reason a hybrid approach has been adopted. The obtained non-equivalent admissible sets of size h have been extended using a backtracking algorithm trying to determine 2-semiarcs of the desired size. In the backtracking phase, the information obtained during the classification of the admissible sets has been further exploited to prune the search tree. In fact the points that would have given admissible sets equivalent to already obtained ones have been excluded from the backtracking steps.

A simple parallelization technique, based on data distribution, has been used to divide the load of the computation in a multiprocessor computer. In our searches we used a 3.3 Ghz Intel Exacore with 16 Gb of memory.

6 Results for $8 \leq q \leq 13$

In Table 4, the number of non-equivalent examples of 2-semiarcs in PG$(2,q)$, $q \leq 9$, is given. The two examples of 2-semiarcs of size 8 in PG$(2,8)$ are obtained by deleting two points from the hyperoval (two points of the conic or one point of the conic and the nucleus).

The following non-existence results are obvious corollaries of Propositions 2.6 and 2.7.

Corollary 6.1. *In* PG$(2,9)$ *there are no* 2*-semiarcs of size* 10 *or* 11.

In Tables 5 and 6 the description of the stabilizer of the non-equivalent examples of 2-semiarcs $\mathcal{S}_2$ in PG$(2,8)$ and PG$(2,9)$ is presented.

In Table 7 (resp. 8) the 2-semiarcs in PG$(2,8)$ (resp. PG$(2,9)$) having stabilizer of size larger than 16 are listed (x_i indicates the number of i-secants of $\mathcal{S}_2$ and ω denotes an element satisfying the equation $\omega^3+\omega^2+1 =$

0 (resp. $\omega^2 - 2\omega - 1 = 0$)). We use the following notations:

$$\begin{aligned}
G_8 &= \mathbb{Z}_2 \times \mathbb{Z}_4,\\
G_{16}^1 &= \mathrm{D}_4 \times \mathbb{Z}_2,\\
G_{16}^2 &= \mathbb{Z}_4 \rtimes \mathbb{Z}_4,\\
G_{18} &= \mathcal{S}_3 \times \mathbb{Z}_3,\\
G_{24}^1 &= \mathrm{A}_4 \times \mathbb{Z}_2,\\
G_{24}^2 &= \mathrm{D}_4 \times \mathbb{Z}_3,\\
G_{24}^3 &= \mathcal{S}_3 \times \mathbb{Z}_4,\\
G_{36} &= \mathbb{Z}_6 \times \mathrm{S}_3,\\
G_{42} &= (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2,\\
G_{96}^1 &= ((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2,\\
G_{96}^2 &= \mathrm{D}_8 \times \mathrm{S}_3,\\
G_{144} &= ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2,\\
G_{168} &= \mathbb{Z}_7 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_8).
\end{aligned}$$

By our experimental results we are able to prove the following.

Theorem 6.2. *In* PG(2, 11) *there exist* 2*-semiarcs of size*

$$k \in \{11, 12, 14 - 26\}.$$

In PG(2, 13) *there exist* 2*-semiarcs of size* $k \in \{13, 27 - 30\}$.

Note that there exists a unique 2-semiarc of size 11 (resp. 13) in PG(2, 11) (resp. PG(2, 13)) and its stabilizer is $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$ (resp. $(\mathbb{Z}_{13} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_3$), according to Theorem 2.5. We also proved by an exhaustive computer search that there exists a unique 2-semiarc of size 12 in PG(2, 11) and its stabilizer is S_4.

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Table 3: Classification of the 2-semiarcs in PG(2, 7)

$\lvert\mathcal{S}_2\rvert$	$\mathcal{S}_2$	x_0	x_1	x_2	x_3	x_4	x_5	G
7	1 1 1 0 0 1 1 1 2 0 1 0 3 6 1 3 0 0 1 2 4	22	14	21	0	0	0	G_{42}
9	0 1 0 1 0 1 1 1 1 1 0 1 3 0 6 1 5 1 5 0 0 2 1 3 1 2 5	15	18	18	6	0	0	$\mathbb{Z}_2$
9	0 1 0 1 0 1 1 1 1 1 0 1 3 0 4 0 6 1 5 0 0 2 1 1 4 5 1	15	18	18	6	0	0	$\mathbb{Z}_2$
9	0 1 0 1 0 1 1 1 1 1 0 1 3 0 6 4 6 1 5 0 0 2 1 3 0 5 1	15	18	18	6	0	0	$\mathbb{Z}_3$
9	0 1 0 1 0 1 1 1 1 1 0 1 3 0 4 3 6 1 5 0 0 2 1 1 3 5 1	15	18	18	6	0	0	$\mathbb{Z}_3$
9	1 1 1 0 0 1 0 1 1 1 3 0 1 0 3 1 5 2 1 5 0 0 1 2 5 1 6	15	18	18	6	0	0	$\mathbb{Z}_6$
9	1 0 1 0 1 0 1 1 1 4 1 0 1 3 0 6 1 1 3 5 0 0 2 1 3 1 5	15	18	18	6	0	0	S_3
10	1 1 0 0 1 1 0 1 1 1 1 0 1 0 3 5 1 6 1 6 1 0 0 1 2 0 5 6 2 5	12	20	15	10	0	0	$\mathbb{Z}_1$
10	1 1 1 0 0 1 1 0 1 1 1 3 0 1 0 3 5 1 2 1 1 5 0 0 1 2 0 5 1 3	12	20	15	10	0	0	$\mathbb{Z}_1$
10	1 1 0 0 1 1 0 1 1 1 1 0 1 0 3 3 1 0 4 6 1 0 0 1 2 6 5 1 2 5	12	20	15	10	0	0	$\mathbb{Z}_1$
10	1 1 1 0 0 1 1 1 0 1 1 3 0 1 0 3 0 5 1 2 1 5 0 0 1 2 6 0 5 1	12	20	15	10	0	0	$\mathbb{Z}_2$
10	1 1 1 0 0 1 1 1 0 1 1 4 0 1 0 3 0 1 1 6 1 1 0 0 1 2 3 0 5 5	12	20	15	10	0	0	$\mathbb{Z}_2$
10	0 1 0 1 0 1 1 1 1 1 1 0 1 3 0 0 2 3 1 5 5 0 0 2 1 6 6 5 1 5	12	20	15	10	0	0	$\mathbb{Z}_2$
10	1 1 1 0 0 1 1 0 1 1 1 6 0 1 0 3 1 1 4 6 1 3 0 0 1 2 0 5 2 5	12	20	15	10	0	0	$\mathbb{Z}_2$
10	1 1 0 0 1 1 1 0 1 1 1 0 1 0 2 3 3 1 0 6 1 0 0 1 2 2 6 5 1 5	12	20	15	10	0	0	$\mathbb{Z}_3$
10	0 1 1 0 1 0 1 1 1 1 1 0 0 1 3 0 4 6 1 1 5 0 1 0 2 1 2 5 1 3	12	20	15	10	0	0	$\mathbb{Z}_4$
10	0 1 0 0 1 0 1 1 0 1 1 0 1 0 3 1 1 5 1 2 5 0 0 1 3 3 1 5 2 2	10	20	25	0	0	2	D_4
10	1 1 0 0 1 1 1 0 1 1 1 0 1 0 3 0 5 1 5 2 1 0 0 1 2 6 4 5 1 6	12	20	15	10	0	0	D_6
10	0 1 1 0 1 0 1 1 1 1 1 0 6 1 3 0 3 0 1 5 5 0 4 0 2 1 3 4 1 2	12	20	15	10	0	0	$\mathcal{S}_4$
11	0 1 0 1 0 1 0 1 1 1 1 1 0 1 3 0 1 1 6 1 5 1 5 0 0 2 1 6 3 5 1 0 5	8	22	19	4	4	0	$\mathbb{Z}_1$
11	0 1 0 1 1 0 1 1 1 1 1 1 0 1 6 3 0 1 4 6 0 1 5 0 0 6 2 1 2 2 5 5 1	9	22	13	12	1	0	$\mathbb{Z}_1$
11	0 1 1 0 1 0 1 1 1 1 1 1 0 0 1 3 0 1 6 1 6 5 5 0 1 0 2 1 6 5 1 2 5	9	22	13	12	1	0	$\mathbb{Z}_1$
12	1 0 1 0 1 0 1 1 1 0 1 1 0 1 1 0 3 1 1 6 2 1 0 2 0 0 3 1 2 3 1 5 2 5 3 1	6	24	12	12	3	0	$\mathbb{Z}_1$
12	0 1 0 1 0 1 1 1 1 1 1 1 1 0 1 3 0 4 3 6 1 6 1 0 5 0 0 2 1 0 0 5 1 2 3 3	6	24	12	12	3	0	$\mathbb{Z}_3$
12	0 1 0 1 0 1 1 1 0 1 1 1 1 0 1 3 0 3 6 1 1 6 1 5 5 0 0 2 1 0 5 1 1 2 3 0	6	24	12	12	3	0	$\mathbb{Z}_3$

Table 4: Number of classes of the 2-semiarcs in PG(2, q), $q \leq 9$

q	Size	# non-equivalent examples
4	4	1
	6	1
	7	1
5	5	1
	6	1
	9	1
7	7	1
	9	6
	10	12
	11	3
	12	3
8	8	2
	9	2
	10	1
	11	10
	12	26
	13	31
	14	29
	15	11
	16	2
9	9	1
	12	30
	13	59
	14	360
	15	925
	16	1149
	17	655
	18	162
	19	19
	20	3

Table 5: Stabilizers of the 2-semiarcs in PG(2, 8)

Size	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	S_3	$\mathbb{Z}_2^3$	$\mathbb{Z}_{12}$	Q_6	G_{18}	S_4	G_{24}^2	G_{24}^1	G_{42}	G_{96}^1	G_{168}
8														1		1
9						1				1						
10									1							
11	5	4	1													
12	8	9	1	1		1	1	1			1	1	2			
13	22	4	5													
14	14	8		6								1				
15	5	1	2		2	1										
16													1		1	

Table 6: Stabilizers of the 2-semiarcs in PG(2, 9)

Size	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	S_3	G_8	D_4	D_6	D_8	G_{16}^1	G_{16}^2	G_{18}	S_4	G_{24}^3	G_{36}	G_{96}^2	G_{144}
9																			1
12	9	6	1		3	4	2			2					1	1		1	
13	42	11		4	2														
14	308	48		3								1							
15	836	74	3		6	2	2			1					1				
16	1054	73		6	11			2	1		1		1						
17	583	59		10	1						1		1						
18	126	22	3		3		4			1				1	1		1		
19	10	5			2				2										
20		2									1								

Table 7: List of the 2-semiarcs in PG(2, 8) with $|G| > 16$

$\|S_2\|$	S_2	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	G
8	$1\ 0\ 0\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ \omega\ \omega^2\ \omega^3\ \omega^5$ $0\ 0\ 1\ 1\ \omega^5\ \omega\ \omega^4\ \omega^3$	29	16	28	0	0	0	0	G_{42}
8	$1\ 0\ 0\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ \omega\ \omega^3\ \omega^5\ \omega^6$ $0\ 0\ 1\ 1\ \omega^5\ \omega^6\ \omega^4\ \omega$	29	16	28	0	0	0	0	G_{168}
9	$1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ \omega\ \omega^2\ \omega^3\ \omega^6$ $0\ 0\ 1\ 1\ \omega\ \omega^5\ \omega^4\ \omega^6\ \omega^2$	25	18	27	3	0	0	0	G_{18}
12	$1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ \omega\ \omega^2\ \omega^3\ \omega^5$ $0\ 0\ 1\ 1\ \omega\ \omega^2\ \omega^3\ \omega^5\ \omega\ \omega^2\ \omega^3\ \omega^5$	11	24	36	0	0	0	2	S_4
12	$1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ \omega\ \omega^2\ \omega^4\ \omega^5$ $0\ 0\ 1\ 1\ \omega\ \omega^5\ \omega\ \omega^2\ \omega^5\ \omega^5\ \omega^5\ \omega^6$	13	24	30	0	6	0	0	G^2_{24}
12	$1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 1\ 0\ \omega\ \omega^2\ \omega^4\ \omega^5\ \omega^5$ $0\ 0\ 1\ 1\ \omega\ \omega^5\ \omega^2\ \omega^5\ \omega^5\ 1\ \omega\ \omega^6$	14	24	24	8	3	0	0	G^1_{24}
12	$1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 1\ 0\ \omega\ \omega\ \omega^2\ \omega^5\ \omega^6$ $0\ 0\ 1\ 1\ \omega\ \omega^5\ \omega^2\ \omega^4\ \omega^5\ \omega^5\ \omega^6\ \omega^2$	14	24	24	8	3	0	0	G^1_{24}
14	$1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ \omega^2\ \omega^3\ \omega^3\ \omega^5\ \omega^6$ $0\ 0\ 1\ 1\ 1\ \omega\ \omega^2\ \omega^5\ \omega^2\ \omega^4\ \omega\ \omega^6\ \omega^6\ 0$	6	28	25	12	0	0	2	G^2_{24}
16	$1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 0\ 1\ \omega\ \omega\ \omega\ \omega^2\ \omega^2\ \omega^4\ \omega^4\ \omega^5\ \omega^5$ $0\ 0\ 1\ 1\ \omega\ \omega^2\ \omega^2\ 0\ 1\ \omega^4\ \omega^2\ \omega^5\ \omega^5\ \omega^6\ \omega^4\ \omega^6$	5	32	0	32	4	0	0	G^1_{24}
16	$1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1$ $0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ \omega\ \omega\ \omega^2\ \omega^2\ \omega^3\ \omega^3\ \omega^5\ \omega^5$ $0\ 0\ 1\ 1\ \omega\ \omega^2\ \omega^5\ \omega^4\ 0\ 1\ \omega^2\ \omega^5\ 1\ \omega^6\ \omega^4\ \omega^6$	5	32	0	32	4	0	0	G^1_{96}

Table 8: List of the 2-semiarcs in PG(2,9) with $|G| > 16$

$\|\mathcal{S}_2\|$	$\mathcal{S}_2$	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	G
9	$1\,0\,0\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,\omega\,2\,\omega^5\,\omega^6\,\omega^7$ $0\,0\,1\,1\,\omega^5\,\omega\,\omega^3\,\omega^7\,\omega^2$	37	18	36	0	0	0	G_{144}
12	$1\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,1\,1\,\omega\,\omega^3\,2\,\omega^5\,\omega^6$ $0\,0\,1\,1\,\omega\,\omega^2\,\omega^3\,\omega^7\,\omega^6\,\omega^5\,\omega\,\omega$	24	24	36	4	3	0	G_{24}^3
12	$1\,0\,0\,1\,0\,1\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,0\,\omega\,\omega^2\,\omega^3\,2\,2\,\omega^6$ $0\,0\,1\,1\,\omega\,2\,\omega^7\,\omega^7\,\omega^3\,1\,\omega^2\,\omega$	25	24	30	12	0	0	S_4
12	$1\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,1\,0\,\omega\,\omega^3\,2\,\omega^5\,\omega^7$ $0\,0\,1\,1\,\omega\,\omega^5\,\omega^3\,\omega^7\,\omega\,\omega^2\,\omega^7\,\omega^6$	24	24	36	4	3	0	G_{96}^2
15	$1\,0\,0\,1\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,0\,1\,\omega\,\omega\,\omega^3\,\omega^3\,\omega^5\,\omega^5\,\omega^6\,\omega^7$ $0\,0\,1\,1\,\omega\,\omega^5\,\omega^6\,2\,\omega^7\,1\,\omega^6\,\omega\,\omega^5\,\omega^7\,2$	16	30	15	30	0	0	S_4
18	$1\,0\,0\,1\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,1\,1\,0\,\omega\,\omega\,\omega\,\omega\,\omega^3\,\omega^3\,\omega^5\,\omega^6\,\omega^7\,\omega^7$ $0\,0\,1\,1\,1\,\omega\,\omega^3\,\omega^3\,1\,\omega\,2\,\omega^7\,\omega^2\,\omega^3\,1\,1\,0\,\omega^2$	1	36	36	9	0	9	G_{18}
18	$1\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,1\,0\,1\,\omega\,\omega\,\omega\,\omega^3\,\omega^3\,\omega^3\,2\,\omega^6\,\omega^6\,\omega^7$ $0\,0\,1\,1\,\omega\,\omega^5\,1\,\omega^3\,2\,\omega^5\,\omega^7\,0\,\omega^3\,\omega^7\,1\,\omega\,2\,2$	6	36	15	22	12	0	S_4
18	$1\,0\,0\,1\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1$ $0\,1\,0\,1\,1\,0\,0\,1\,1\,\omega\,\omega\,\omega\,\omega^2\,\omega^3\,2\,2\,2\,\omega^5$ $0\,0\,1\,1\,\omega\,1\,\omega^7\,\omega^2\,\omega^3\,2\,\omega^6\,\omega^7\,1\,0\,0\,\omega^2\,\omega^7\,\omega^2$	4	36	27	6	18	0	G_{36}