On Randić index and the matching number

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Abstract

The Randić index of a graph G, denoted by $R(G)$, is defined as the sum of $1/\sqrt{d(u)d(v)}$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. Denote by $\mu(G)$ the matching number, i.e., the number of edges in a maximum matching of G. A conjecture of AutoGraphiX on the relation between the Randić index and the matching number of a connected graph G is: for any connected graph of order $n \geq 3$ with Randić index $R(G)$ and matching number $\mu(G)$,

$$R(G) - \mu(G) \leq \sqrt{\left\lfloor \frac{n+4}{7} \right\rfloor \left\lfloor \frac{6n+2}{7} \right\rfloor} - \left\lfloor \frac{n+4}{7} \right\rfloor,$$

with equality if and only if G is a complete bipartite graphs $K_{p,q}$ with $p = \mu(G) = \lfloor \frac{n+4}{7} \rfloor$, which was proposed by Aouchiche et al.. In this paper, we confirm this conjecture for some classes of graphs.

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1 Introduction

In 1975, Milan Randić [16] proposed a topological index R under the name "*branching index*", "*connectivity index*", suitable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons, which is now popularly as the *Randić index*. For a graph $G = (V, E)$, the *Randić index* $R(G)$ of G was defined as the sum of $1/\sqrt{d(u)d(v)}$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G, i.e.,

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

Randić noticed that there is a good correlation between the Randić index R and several physico–chemical properties of alkanes: boiling points, chromatographic retention times, enthalpies of formation, parameters in the Antoine equation for vapor pressure, surface areas, etc. Later, in 1998 Bollobás and Erdös [3, 4] generalized this topological index by replacing $-\frac{1}{2}$ with any real number α, which is called the general Randić index. For a comprehensive survey of its mathematical properties, see the book of Li and Gutman [8] or the survey of Li and Shi [10]. For terminology and notations not defined here, we refer the readers to [5].

Except for determining the maximal or minimal values of the Randić index among a given class of graphs, studying the relation between Randić index and some other graph invariants is also an interesting topic. There also some approaches on this topic, such as the Randić index and the minimum degree [2, 3], the maximum degree [1, 2], the diameter [2, 12], the average distance [12], the girth [2], the radius [2], the algebraic connectivity [2, 13], the chromatic number [11], the spectral radius [1], and so on.

In this paper, we will study the relation between Randić index and the matching number. Denote by $\mu(G)$ the matching number, i.e., the number of edges in a maximum matching of G. If G has a perfect matching, then $\mu(G) = n/2$. In [14], Lu et al. established a sharp lower bound of Randić index for trees with a given matching number. Pan et al. [15] generalized this result to the generalized Randić index for $\alpha \in [-1/2, 0)$. They also gave a sharp lower bound of the generalized Randić index for the trees with a given matching number for $\alpha > 0$. Recently, Chen et al. [6] gave a sharp lower bound of the generalized Randić index for conjugated trees for $\alpha \leq -1$. In [9], the authors characterized the structure of trees with the maximum value of the generalized Randić index for $\alpha > 1$ with a prescribed order and matching number and a sharp upper bound for $0 < \alpha < 1$ as well as the corresponding extremal trees. As one can see, usually the maximum problem is much harder than the minimum one.

In [1], the authors gave a conjecture on the relation of Randić index and the matching number.

Conjecture 1.1. For any connected graph of order $n \geq 3$ with Randić index $R(G)$ and matching number $\mu(G)$,

$$R(G) - \mu(G) \leq \sqrt{\left\lfloor \frac{n+4}{7} \right\rfloor \left\lfloor \frac{6n+2}{7} \right\rfloor} - \left\lfloor \frac{n+4}{7} \right\rfloor,$$

with equality if and only if G is a complete bipartite graphs $K_{p,q}$ with $p = \mu(G) = \lfloor \frac{n+4}{7} \rfloor$.

However, we have not seen any results on this conjecture. In this paper, we confirm this conjecture for some classes of graphs.

2 Main Results

Recall that for a graph G of order n, the Randić index $R(G) \leq \frac{n}{2}$, with equality if and only if G is a regular graph. Therefore, for a graph G with order n, if G has a perfect matching, then $R(G) - \mu(G) \leq \frac{n}{2} - \frac{n}{2} = 0$, which confirms Conjecture 1.1. If G is a regular bipartite graph, then from [5], we have that G has a perfect matching. Therefore, we have the following result.

Theorem 2.1. *If G is a regular bipartite graph of order $n \geq 4$, then Conjecture 1.1 holds.* □

Furthermore, we have the following result, since when $\mu(G) \geq \frac{7}{20}n$ and $n \geq 4$, we have

$$R(G) - \mu(G) \leq \frac{n}{2} - \frac{7}{20}n \leq \sqrt{\left\lfloor \frac{n+4}{7} \right\rfloor \left\lfloor \frac{6n+2}{7} \right\rfloor} - \left\lfloor \frac{n+4}{7} \right\rfloor.$$

Theorem 2.2. *If G is a graph of order $n \geq 4$ and $\mu(G) \geq \frac{7}{20}n$, then Conjecture 1.1 holds.* □

In the following, we will prove that Conjecture 1.1 holds for some classes of trees. Denote by $\mathcal{T}_{n,k}$ the class of trees of order n such that the neighbor of each pendent vertex has degree at least k. The following lemma will be useful in the proof of our main result.

Lemma 2.1 ([7]). *Let T be a tree of order n with an m-matching. If $n > 2m$, then there is an m-matching M and a pendent vertex v such that M does not saturate v.* □

Theorem 2.3. *For each tree $T \in \mathcal{T}_{n,k}$ with $k \geq 25$ and $n > 2\mu(T)$,*

$$R(G) - \mu(G) \leq \sqrt{\left\lfloor \frac{n+4}{7} \right\rfloor \left\lfloor \frac{6n+2}{7} \right\rfloor} - \left\lfloor \frac{n+4}{7} \right\rfloor.$$

Proof. Let $T \in \mathcal{T}_{n,k}$ with $k \geq 25$ and $n > 2\mu(T)$, and M be a maximum matching. By Lemma 2.1, let v be a pendent vertex which is not saturated by M. Suppose u is the neighbor of v. Denote by $d_1, d_2, \ldots, d_{k-1}$, respectively, the degrees of the neighbors of u except v. We will show the result by induction on n.

By some elementary calculations, we have

$$\begin{aligned} R(T) &= R(T-v) + \frac{1}{\sqrt{k}} + \sum_{i=1}^{k-1} \frac{1}{\sqrt{kd_i}} - \sum_{i=1}^{k-1} \frac{1}{\sqrt{(k-1)d_i}} \\ &= R(G-v) + \frac{1}{\sqrt{k}} - \frac{\sqrt{k}-\sqrt{k-1}}{\sqrt{k(k-1)}} \sum_{i=1}^{k-1} \frac{1}{\sqrt{d_i}}. \end{aligned}$$

Since $\sum_{i=1}^{k-1}(d_i - 1) \leq n-k-1$, we have $\sum_{i=1}^{k-1} d_i \leq n-1$. Therefore, we have

$$\begin{aligned} R(T) &\leq R(T-v) + \frac{1}{\sqrt{k}} - \frac{\sqrt{k}-\sqrt{k-1}}{\sqrt{k(k-1)}} \sum_{i=1}^{k-1} \frac{k-1}{\sqrt{n-1}} \\ &= R(T-v) + \frac{1}{\sqrt{k}} - \frac{(k-1)(\sqrt{k}-\sqrt{k-1})}{\sqrt{k(n-1)}}. \end{aligned}$$

Observe that the matching number of $T-v$ is the same as T, i.e., $\mu(T)$. By the induction hypothesis, we have

$$\begin{aligned} R(T) - \mu(T) &\leq R(T-v) + \frac{1}{\sqrt{k}} - \frac{(k-1)(\sqrt{k}-\sqrt{k-1})}{\sqrt{k(n-1)}} - \mu(T) \\ &\leq \sqrt{\left\lfloor \frac{n+3}{7} \right\rfloor \left\lfloor \frac{6n-4}{7} \right\rfloor} - \left\lfloor \frac{n+3}{7} \right\rfloor + \frac{1}{\sqrt{k}} - \frac{(k-1)(\sqrt{k}-\sqrt{k-1})}{\sqrt{k(n-1)}} \end{aligned}$$

By some elementary calculations, we have that

$$\sqrt{\left\lfloor\frac{n+3}{7}\right\rfloor\left\lfloor\frac{6n-4}{7}\right\rfloor}-\left\lfloor\frac{n+3}{7}\right\rfloor+\frac{1}{\sqrt{k}}-\frac{(k-1)(\sqrt{k}-\sqrt{k-1})}{\sqrt{k(n-1)}}$$

$$\leq\sqrt{\left\lfloor\frac{n+4}{7}\right\rfloor\left\lfloor\frac{6n+2}{7}\right\rfloor}-\left\lfloor\frac{n+4}{7}\right\rfloor$$

for $k \geq 25$.

Thus, the proof is complete. □

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