Unsolvable Block-Transitive Automorphism Groups of $2-(v,31,1)$ Designs *

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Abstract

This paper is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be a $2-(v,31,1)$ design and $G \leq Aut(\mathcal{D})$ be block-transitive and point-primitive. If G is unsolvable, then $Soc(G)$, the socle of G, is not ${}^2F_4(q)$.

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1 Introduction

This paper is part of a project to classify groups and $2-(v,k,1)$ designs where the group acts transitively on the blocks of the design. A $2-(v,k,1)$ design $\mathcal{D} = (\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. Traditionally one defined $v =: |\mathcal{P}|$ and $b =: |\mathcal{B}|$. Our interest is in the situation where there is a group G of automorphisms that acts transitively on $\mathcal{B}$. This implies in particular that every block has same number k of points (where $2 < k < v$). It is not hard to see that every point lies on same number r of blocks. The numbers v, b, k, r are known as the parameters of $\mathcal{D}$.

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Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v,k,1)$ design $\mathcal{D}$. Then G is said to be block-transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point-transitive (point-primitive) on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag-transitive on $\mathcal{D}$ if G is transitive on the set of flags. By a theorem of Block (see [1]) our block-transitive automorphism group G will be transitive also on points.

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. The classification of block-transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [2]). In [3] Camina and Siemons classified $2-(v,4,1)$ designs with a block-transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block-transitive, unsolvable group of automorphisms (see [4]). Tong and Li classified $2-(v,5,1)$ designs with a block-transitive, solvable group of automorphisms in [5]. Han and Li [6] classified $2-(v,5,1)$ designs with a block-transitive, unsolvable group of automorphisms. Liu classified $2-(v,k,1)$ (where $k = 6,7,8,9,10$) designs with a block-transitive, solvable group of automorphisms in [7]. In [8], Han and Ma classified $2-(v,11,1)$ designs with a block transitive classical simple group of automorphisms. Dai and Zhao classified $2-(v,13,1)$ designs with block-transitive, unsolvable group of automorphisms whose socle is $Sz(q)$ in [9]. In this article we consider $2-(v,31,1)$ designs with a block-transitive, unsolvable group of automorphisms and prove the following theorem.

Main Theorem. Let $\mathcal{D}$ be a $2-(v,31,1)$ design, $G \leq Aut(\mathcal{D})$ be block-transitive and point-primitive. If G is unsolvable, then the socle of G is not isomorphic to ${}^2F_4(q)$.

Before starting the body of the article we introduce some notation. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and G be an automorphism group of $\mathcal{D}$ that acts transitively on blocks. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. In addition, G^B denotes the permutation group induced by the action of G_B on the points of B. Then $G^B \cong G_B/G_{(B)}$.

The second section describes several preliminary results concerning the Ree groups ${}^2F_4(q)$ and $2-(v,k,1)$ designs. In the third section we give the proof of the theorem.

2 Preliminary Results

The Ree groups ${}^2F_4(q)$ are the fixed points of a certain automorphism of the Chevalley groups of type F_4 over a finite field $F = GF(q)$, where $q = 2^{2n+1}$, $n \geq 0$. Ree [10] showed that the groups ${}^2F_4(q)$ are simple if $q > 2$, while Tits [11] showed that ${}^2F_4(2)$ is not simple but possesses a simple subgroup of index 2. In this paper we treat that ${}^2F_4(q)$ are simple, that is, $q > 2$ and $n \geq 1$. Let $a = 2n+1$ and $T = {}^2F_4(q)$. Then $q = 2^a$ and the order of T is $q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$.

There are two important parameters of a $2-(v,k,1)$ design, the number b of blocks and the number r of all blocks through a point. In fact we have $bk = vr$ and $bk(k-1) = v(v-1)$. Thus $r = (v-1)/(k-1)$. We can show that $b \geq v$ and so $k \leq r$. If $k = r$ then $v = k^2 - k + 1$; if $r \geq k+1$, then $v \leq k^2$.

We use a result of W. Fang and H. Li [12]. Define the following constants:

$$b_1 = (b,v),\ b_2 = (b,v-1),\ k_1 = (k,v),\ \text{and}\ k_2 = (k,v-1).$$

Using the basic equalities for $2-(v,k,1)$ design, we get the Fang-Li Equations:

$$k = k_1k_2,\ b = b_1b_2,\ r = b_2k_2,\ \text{and}\ v = b_1k_1.$$

We shall state a number of basic results which will be used repeatedly throughout the paper.

Lemma 2.1 ([13]) *Let $G = {}^2F_4(q)$, where $q = 2^{2n+1}$ with $n \geq 1$ and M be maximal in G. Then M is conjugate to one of the subgroups in the table below.*

Table I

Strcture	Oder	Remarks
$P_1 = [q^{11}] : (PSL(2,q) \times (q-1))$	$q^{12}(q+1)(q-1)^2$	*parabolic*
$P_2 = [q^{10}] : ({}^2B_2(q) \times (q-1))$	$q^{12}(q-1)^2(q^2+1)$	*parabolic*
$SU(3,q) : 2$	$2q^3(q-1)\cdot$ $(q+1)^2(q^2-q+1)$	
$(Z_{q+1} \times Z_{q+1}) : GL(2,3)$	$48(q+1)^2$	
$(Z_{q+\epsilon\sqrt{2q}+1} \times Z_{q+\epsilon\sqrt{2q}+1}) : [96]$	$96(q+\epsilon\sqrt{2q}+1)^2$	if $\epsilon = -1$, $q > 8$
$Z_{q^2+\epsilon\sqrt{2}q^{\frac{3}{2}}+q+\epsilon\sqrt{2q}+1} : 12$	$12(q^2+\epsilon\sqrt{2}q^{\frac{3}{2}}+$ $q+\epsilon\sqrt{2q}+1)$	
$PGU(3,q) : 2$	$2q^3(q-1)(q+1)^2$	
${}^2B_2(q) \wr 2$	$2q^2(q^2+1)(q-1)$	
$B_2(q) : 2$	$2q^4(q^2-1)(q^4-1)$	
${}^2F_4(q_0)$	$q_0^{12}(q_0-1)(q_0^3+1)\cdot$ $(q_0^4-1)(q_0^6+1)$	$q = q_0^\delta$ and δ is a prime

Conversely, if H is conjugate to one of these groups, then $N_G(H)$ is maximal in G.

Lemma 2.2 ([14]) *Let $T = {}^2F_4(q)$ be an exceptional simple group of Lie type over $GF(q)$, and G be a group with $T \trianglelefteq G \leq Aut(T)$. Suppose that M*

is a maximal subgroup of G not containing T. Then one of the following holds:

(1) $|M| < q^{12}|G:T|$;

(2) $T \cap M$ *is a parabolic subgroup of* T;

(3) $T \cap M = L_3(3).2$ *or* $L_2(25)$, *if* $q = 2$.

Lemma 2.3 ([8]) *Let* G *and* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a group and a design, and* $G \leq Aut(D)$ *be block-transitive, point-primitive but not flag-transitive. Let* $Soc(G) = T$. *Then*

$$|T| \leq \frac{v}{\lambda}|T_\alpha|^2|G:T|,$$

where $\alpha \in \mathcal{P}$, λ *is the length of the longest suborbit of* G *on* $\mathcal{P}$.

Lemma 2.4 ([15]) *Let* $G = T : \langle x \rangle$ *and act block-transitively on a* $2-(v,k,1)$ *design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$. *Then* T *acts transitively on* $\mathcal{P}$.

3 Proof of the Main Theorem

For prove the Main Theorem, we prove the following two lemmas firstly.

Lemma 3.1 *Let* $\mathcal{D}$ *be a* $2-(v,31,1)$ *design,* G *be block-transitive, point-primitive but not flag-transitive. Then* $v = 930b_2 + 1$.

Proof. Since $k = 31$ and $k_1 = (k,v)$, $k_1 = 1$ or 31. If $k_1 = 31$, then $k|v$, by [14], G is flag-transitive, a contradiction. Hence we have $k_1 = 1$. Thus $v = k(k-1)b_2 + 1 = 930b_2 + 1$.

Lemma 3.2 *Let* D *be a* $2-(v,31,1)$ *design,* G *be block-transitive, point-primitive but not flag-transitive and* $Soc(G) = T$ *be even order. If* G *be unsolvable, then* $|T| \leq 466|T_\alpha|^2|G:T|$.

Proof. Let $B = \{1, 2, \cdots, 31\} \in \mathcal{B}$. Since G is unsolvable, then the following possibility for the structure of G_B, the rank and subdegree of G does not occur:

Type of G_B	Rank of G	Subdegree of G
$\langle 1 \rangle$	931	$1, \overbrace{b_2, b_2, \cdots, b_2}^{930}$

Otherwise, $|G^B|$ is odd and hence $|G|$ is odd, which contradicts the fact that $|T|$ is even. Thus $\lambda \geq 2b_2$. By Lemma 2.3 we have $\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda} \cdot |G:T| \leq \frac{v}{2b_2} \cdot |G:T|$. It follows by Lemma 3.1 that $\frac{|T|}{|T_\alpha|^2} \leq \frac{930b_2+1}{2b_2} \cdot |G:T| < 466 \cdot |G:T|$.

Now we can prove our Main Theorem stated in the Introduction.

Suppose that $Soc(G) = {}^2F_4(q) = T$. Thus ${}^2F_4(q) \trianglelefteq G \leq Aut({}^2F_4(q))$. We have $G = T : \langle x \rangle$, where $x \in Out(T)$. Let $o(x) = m$. Then $m|a$ and $|G| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$. By [16], G is not flag-transitive. Since G is point-primitive, G_α is the maximal subgroup of G. By Lemma 2.2 we have $|G_\alpha| < q^{12}|G : T|$, $G_\alpha \cap T$ is a parabolic subgroup of T, $M = L_3(3)$ or $L_2(25)$, if $q = 2$. We shall consider three cases to prove the Main Theorem.

Case 3.1: $|G_\alpha| < q^{12}|G : T|$.

Since G is block-transitive, by Lemma 2.4, T is point-transitive. Hence $|G_\alpha| = |T_\alpha|m$ and so $|T_\alpha| < q^{12}$. It follows by Lemma 3.2 that

$$|T| < 466|T_\alpha|^2|G : T| < 466q^{24}|G : T| = 466q^{24}m.$$

It follows that

$$\frac{(q-1)(q^3+1)(q^4-1)(q^6+1)}{q^{12}} < 466m \leq 466 \cdot a.$$

Since $q^4 - 1 \geq q^3(q-1)$, we have

$$(q-1)^2 < 466a.$$

Recall that $a = 2n+1 \geq 3$, $q = 2^a$. We have

$$(2^a - 1)^2 < 466a. \tag{1}$$

Let $f(x) = (2^x - 1)^2 - 466x$. If $a \geq 6$, then $f'(a) = 2ln2 \cdot (2^a - 1) \cdot 2^a - 466 \geq f'(6) > 5123 > 0$. Hence $f(a) \geq f(6) = 1173$ and we have

$$(2^a - 1)^2 \geq 466a + 1173.$$

This, together with (1), gives a contradiction. Hence $a < 6$ and $a = 3, 5$.

Since $v = 930b_2 + 1$ is odd by Lemma 3.1 and $v = \frac{|T|}{|T_\alpha|}$, T_α contains a Sylow 2-subgroup of T. Together with Lemma 2.1, the only possibilities for T_α are cases where $T_\alpha \cong P_1$ and $T_\alpha \cong P_2$.

Case 3.1.1: $a = 3$

(1) $T_\alpha = P_1$

Then we have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{2^{36} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 37 \cdot 109}{2^{36} \cdot 3^2 \cdot 7^2} - 1 = 8741225024.$$

By Lemma 3.1 we have $v - 1 = 930b_2$ and so $930|8741225024$, a contradiction.

(2) $T_\alpha = P_2$

We have

$$v-1=\frac{|T|}{|T_\alpha|}-1=\frac{2^{36}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 13^2\cdot 19\cdot 37\cdot 109}{2^{36}\cdot 5\cdot 7^2\cdot 13}-1=1210323464,$$

which contradicts with the fact $930|(v-1)$.

Case 3.1.2: $a=5$

In this case we also can get contradictions in the same way as the case where $a=3$.

Case 3.2: $G_\alpha \cap T$ is a parabolic subgroup of T.

Looking at the list of maximal subgroups of ${}^2F_4(q)$ in Lemma 2.1, we can see that the parabolic subgroup of ${}^2F_4(q)$ is conjugate to P_1 or P_2 and hence $T_\alpha \cong P_1$ or $T_\alpha \cong P_2$.

If $T_\alpha \cong P_1$, then we have $v-1=|T:T_\alpha|-1=q^2(1+q+q^3+q^4+q^6+q^7+q^9)$. It follows by Lemma 3.1 that $3\cdot 5\cdot 31|1+q+q^3+q^4+q^6+q^7+q^9$, hence $3|1+q+q^3+q^4+q^6+q^7+q^9$. But

$$1+q+q^3+q^4+q^6+q^7+q^9\equiv\begin{cases}1 \pmod 3, & q\equiv 1 \pmod 3,\\ 2 \pmod 3, & q\equiv 2 \pmod 3,\end{cases}$$

which is a contradiction.

If $T_\alpha \cong P_2$, then we have $v-1=|T:T_\alpha|-1=q(1+q^2+q^3+q^5+q^6+q^8+q^9)$. By Lemma 3.1 we have $3\cdot 5\cdot 31|1+q^2+q^3+q^5+q^6+q^8+q^9$, hence $3|1+q^2+q^3+q^5+q^6+q^8+q^9$. But

$$1+q^2+q^3+q^5+q^6+q^8+q^9\equiv\begin{cases}1 \pmod 3, & q\equiv 1 \pmod 3,\\ 1 \pmod 3, & q\equiv 2 \pmod 3,\end{cases}$$

a required contradiction.

Case 3.3: $G_\alpha \cap T \cong L_3(3).2$ or $L_2(5)$.

In this case $q=2$ and $|T|=|{}^2F_4(2)|=2^{12}\cdot 3^3\cdot 5^2\cdot 13$. Hence we have $v=|T:T_\alpha|=2^6\cdot 5^2$ or $2^5\cdot 3$. It follows by Lemma 3.1 that $930|1599$ or $930|95$, which is a contradiction. This completes the proof of the Main Theorem.

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