On graceful unicyclic wheels[1]

S. M. Hegde, Shivarajkumar
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka Surathkal
e-mail: *smhegde@nitk.ac.in*, *shivaraj_04@yahoo.co.in*

Abstract

A digraph D with e edges is labeled by assigning a distinct integer value $\theta(v)$ from $\{0, 1, ..., e\}$ to each vertex v. The vertex values, in turn, induce a value $\theta(u, v)$ on each edge (u, v) where $\theta(u, v) = \theta(v) - \theta(u) mod(e + 1)$. If the edge values are all distinct and nonzero, then the labeling is called a *graceful labeling* of a digraph.
In 1985, Bloom and Hsu conjectured that " ***All unicyclic wheels are graceful*** ". In this paper we prove the conjecture.

Keywords: Graceful labeling of graphs (digraphs), unicyclic wheels

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1. Introduction.

Several practical problems in real life situations have motivated the study of labelings of a graph $G = (V, E)$, which are required to obey a variety of conditions. There is an enormous literature built up on several kinds of labelings of graphs over the past four decades or so. An interested reader can refer to Gallian [2].

Graph labelings, where the vertices and/or edges are assigned values subject to certain conditions, have often been motivated by practical problems, but they are also of interest in their own right.

The labeling discussed in this paper has three ingredients:
(i) a set of number S from which the labels are chosen;
(ii) a rule that assigns a value to each edge;
(iii) a condition that these values must satisfy;

An undirected graph with e edges is gracefully numbered if each vertex v is assigned distinct value $\theta(v)$ from the set $\{0, 1, ..., e\}$ in such a way that the set of edge numbers equals $\{1, 2, ..., e\}$, where edge uv is numbered by

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$\theta(uv) = |\theta(u) - \theta(v)|$. A graph is said to be a *graceful (undirected) graph* if it can be gracefully numbered.

This concept was extended to a digraph by Bloom and Hsu [1] as follows: A digraph D with e edges is labeled by assigning a distinct integer value $\theta(v)$ from $\{0, 1, ..., e\}$ to each vertex v. The vertex values, in turn, induce a value $\theta(u, v)$ on each edge (u, v) where $\theta(u, v) = \theta(v) - \theta(u) mod(e+1)$. If the edge values are all distinct and nonzero, then the labeling is called a *graceful labeling of a digraph*.

An undirected graph consisting of a vertex (the "hub") joined to each vertex of the cycle (the rim) is termed a wheel and is denoted by $\overrightarrow{W_n}$, consisting of $(n+1)$ vertices and $2n$ edges.

A directed wheel is termed *outspoken*, if all spokes point out from the hub to the rim. Similarly, a wheel is *inspoken*, if all spokes point form the rim to the hub. If the rim of a directed wheel is unidirectional then the wheel is called *unicyclic*.

Proposition 1 [1]: An outspoken unicyclic wheel $\overrightarrow{W_q}$ is graceful if and only if the inspoken wheel $-\overrightarrow{W_q}$ is graceful.

Proposition 2 [1]: Let p be an odd prime number and α be a primitive element of Z_p. If $(\alpha^2 - 1) \equiv 2k + 1(mod\ p)$ for some k, then the outspoken and the inspoken unicyclic wheels $\overrightarrow{W_q}$ and $-\overrightarrow{W_q}$ respectively are graceful for $q = (p-1)/2$.

Proposition 3 [1]: Let p be an odd prime number, let $q = (p-1)/2$ and let α be a primitive element of Z_p. If $(\alpha^2 - 1) \equiv 2k + 1(mod\ p)$ for some k, and if $p \equiv 3(mod 4)$, then the inspoken and outspoken wheels $\overrightarrow{W_q}$ and $-\overrightarrow{W_q}$ respectively are graceful.

Bloom and Hsu [1] mentioned that "for $n \leq 11$, all unicyclic wheels $\overrightarrow{W_n}$ are known to be graceful except for $n = 6$ and $n = 10$. Can graceful numberings for these be found? And more generally, will the results for unicyclic wheels be as straight forward as for undirected wheels, i.e., is the following conjecture true?".

Conjecture: All unicyclic wheels are graceful.

Alison Marr [3] has given some results on the number of non equivalent graceful labeling for $\overrightarrow{W_n}$, $n < 11$.

Figure-1 gives a graceful labeling for $\overrightarrow{W_6}$ and $\overrightarrow{W_{10}}$.

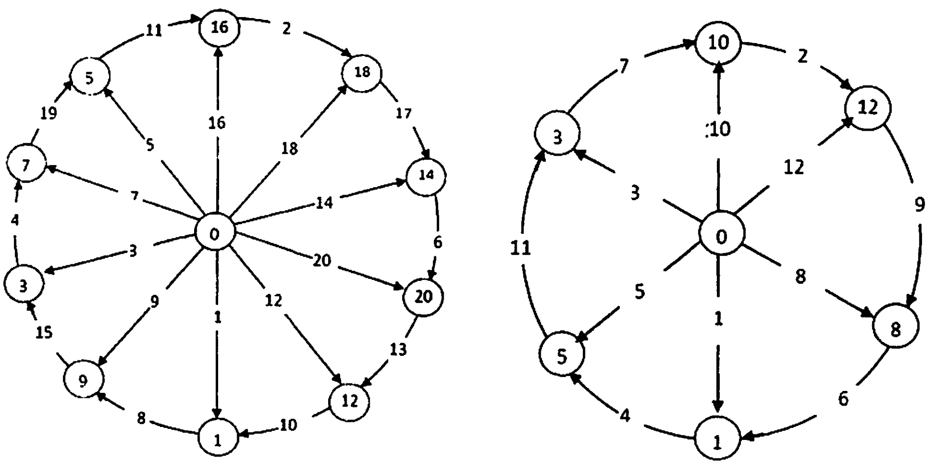

Figure-1

2. Graceful labeling of unicyclic wheels.

In this section we prove the above conjecture.

Theorem 1.1. *If an outspoken unicyclic wheel* $\overrightarrow{W_n}$ *is graceful then* $b_1 + b_2 + \ldots + b_n \equiv 0(mod\ 2n+1)$, *where* $b_1, b_2, \ldots, b_n$ *are the labels of the spokes.*

Proof. Let $\overrightarrow{W_n}$ be graceful with a_0 as the label of the central vertex and $a_1, a_2, \ldots, a_n$ as the labels of the vertices of the cycle. Let the edge values be $b_1, b_2, \ldots, b_{2n}$. Then by the definition of graceful labeling, we get $2n$ linear equations with $(n+1)$ unknowns as follows,

$$
\begin{aligned}
-a_0 + a_1 &= b_1 \\
-a_0 + a_2 &= b_2 \\
-a_0 + a_3 &= b_3 \\
&\vdots \\
-a_0 + a_n &= b_n \\
a_2 - a_1 &= b_{n+1} \\
a_3 - a_2 &= b_{n+2} \\
&\vdots \\
a_n - a_{n-1} &= b_{2n-1} \\
a_1 - a_n &= b_{2n}.
\end{aligned}
$$

From the above system of equations, we get
$b_{n+1} + b_{n+2} + ... + b_{2n} = (a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1}) + (a_1 - a_n)$.
Therefore $b_{n+1} + b_{n+2} + ... + b_{2n} = 0$ . ....(i)
Also we know that, $b_1 + b_2 + ... + b_{2n} = 2n(2n+1)/2 \equiv 0(mod\ 2n+1)$.
From (i), we have $b_{n+1} + b_{n+2} + ... + b_{2n} = 0$.
Therefore, $b_1 + b_2 + \ + b_n \equiv 0(mod\ 2n+1)$. □

Theorem 1.2. *The outspoken unicyclic wheel $\overrightarrow{W_n}$ is graceful.*

Proof. Case (i): When $n \equiv 0(mod\ 4)$.

Define a function l as follows:

$$
l(v_i) = \begin{cases} i & \text{if } i = 1, 3, ..., (n/2) - 1. \\ n + 1 - i & \text{if } i = 2, 4, ..., n/2. \\ n + 1 + i & \text{if } i = (n/2) + 1, (n/2) + 3, ..., n - 1. \\ 2(n+1) - i & \text{if } i = (n/2) + 2, (n/2) + 4, ..., n. \\ 0 & \text{if } i = 0. \end{cases}
$$

Suppose $l(v_i) = l(v_j)$, $i \neq j$.
Here we have 3 different cases.

i) $i,\ j \leq n/2$,
Suppose i and j are even then $l(v_i) = l(v_j)$ if and only if $v_i = v_j$.
Similarly, if i and j are odd then $l(v_i) = l(v_j)$ if and only if $v_i = v_j$.

Let i be even and j be odd.
Then $n+1-i = j$.
i.e., $i + j = n + 1$ a contradiction since $i,\ j \leq n/2$.

If i is odd and j is even we get a similar contradiction.

ii) If $i \leq n/2$ and $j > n/2$.
Then $l(v_i) < n$ for any i and $l(v_j) > n$ for any j. Therefore, $l(v_i) < l(v_j)$.
This is a contradiction to the hypothesis.

iii) $i, j > n/2$, where i is even and j is odd.
Then $2n + 2 - i = n + 1 + j$.
i.e., $i + j = n + 1$.
But if $i, j > n/2$, then $i + j \geq n + 3$, a contradiction.
Therefore, $l(v_i) \neq l(v_j)$ for $i \neq j$.

Now we shall prove all edge labels are distinct.

We know that $l(v_0) = 0 \ \therefore\ l(v_i) - l(v_0) = l(v_i)$ which are all distinct for i , $1 \leq i \leq n$.
i.e., the labels on the spokes are distinct.

Now we have to prove that $l(v_{i+1}) - l(v_i)$ are distinct from all $l(v_k)$,where $1 \leq k \leq n$.

We consider the following cases,

Case (a): Let $i \in \{1, 2, ..., n/2\}$.
Let i be odd .
Then $l(v_{i+1}) - l(v_i) = n + 1 - i - 1 - i = n - 2i$.

Suppose, $n - 2i = l(v_k)$, where $k \in \{1, 3, ..., (n/2) - 1\}$.
Then $n - 2i = k$.
i.e., $n = k - 2i$.
Since k is odd, which is a contradiction because n is even.
Therefore, $n - 2i \neq l(v_k)$ for any $k \in \{1, 3, ..., (n/2) - 1\}$.

Suppose $n - 2i = l(v_k)$, where $k \in \{2, 4, ..., (n/2)\}$.

Then $n-2i=n+1-k$.
i.e., $2i+1=k$, which is a contradiction since k is even and $2i+1$ is odd.
Therefore, $n-2i \neq l(v_k)$ for any $k \in \{2,4,...,(n/2)\}$.

Suppose $n-2i=l(v_k)$, where $k \in \{(n/2)+1,(n/2)+3,...,n-1\}$.
Then $n-2i=n+1+k$.
i.e., $-1-2i=k$.
i.e., $2n+1-1-2i=k$, which is a contradiction since k is odd.
Therefore, $n-2i \neq l(v_k)$ for any $k \in \{(n/2)+1,(n/2)+3,...,n-1\}$.

Suppose $n-2i=l(v_k)=2n+2-k$, where $k \in \{(n/2)+2,(n/2)+4,...,n\}$.
Then $-2i=n+2-k$.
i.e., $k=n+2+2i$.
i.e., $k \in \{n+4,n+8,...,2n\}$.
So $n+4 \leq k \leq 2n$ which is a contradiction since the maximum of k is n.
Therefore, $n-2i \neq l(v_k)$ for any $k \in \{(n/2)+2,(n/2)+4,...,n\}$.

Let i be even.
Then $l(v_{i+1})-l(v_i)=i+1-n-1+i=-n+2i$.
$=-n+2i+2n+1=n+2i+1$.

Suppose, $n+2i+1=l(v_k)$, where $k \in \{1,3,...,(n/2)-1\}$.
Then, $n+2i+1=k$.
i.e., $k \in \{n+5,n+9,,2n+1\}$.
So $k=0$ or $k \geq n+5$, which is a contradiction since $k \in \{1,3,...,(n/2)-1\}$.
Therefore, $n+2i+1 \neq l(v_k)$ for any $k \in \{1,3,...,(n/2)-1\}$.

Suppose $n+2i+1=l(v_k)$, where $k \in \{2,4,...,(n/2)\}$.
Then $n+2i+1=n+1-k$.
i.e., $k=(2n+1)-2i$, which is a contradiction since k is even.
Therefore, $n+2i+1 \neq l(v_k)$ for any $k \in \{2,4,...,(n/2)\}$.

Suppose $n+2i+1=l(v_k)$, where $k \in \{(n/2)+1,(n/2)+3,...,n-1\}$.
Then $n+2i+1=n+1+k$.
i.e., $k=2i$, a contradiction, since k is odd.
Therefore, $n+2i+1 \neq l(v_k)$, for any $k \in \{(n/2)+1,(n/2)+3,...,n-1\}$.

Suppose $n+2i+1=l(v_k)$, where $k \in \{(n/2)+2,(n/2)+4,...,n\}$.
Then $n+2i+1=2n+2-k$.
i.e., $2n+2-n-2i-1=k$.
i.e., $n+1-2i=k$.
i.e., $n=k-1+2i$.

Since k is even $k-1$ becomes odd, therefore $n = k - 1 + 2i$ is odd, a contradiction.
Therefore, $n + 2i + 1 \neq l(v_k)$ for any $k \in \{(n/2)+2, (n/2)+4, ..., n\}$.

Case (b): Let $i \in \{(n/2)+1, (n/2)+2, ..., n\}$.
Let i be odd.
Then $l(v_{i+1}) - l(v_i) = 2n + 2 - i - 1 - n - 1 - i$.
$= n - 2i$.

Suppose $n - 2i = l(v_k)$, where $k \in \{1, 3, ..., (n/2) - 1\}$.
Then $n - 2i = k$.
i.e., $n = k + 2i$, which is a contradiction since k is odd and n is even.
Therefore, $n - 2i \neq l(v_k)$ for any $k \in \{1, 3, ..., (n/2) - 1\}$.

Suppose $n - 2i = l(v_k)$, where $k \in \{2, 4, ..., (n/2)\}$.
Then $n - 2i = n + 1 - k$.
i.e., $k = 2i + 1$, which is a contradiction since k is even and $2i+1$ is odd.
Therefore, $n - 2i \neq l(v_k)$ for any $k \in \{2, 4, ..., (n/2)\}$.

Suppose $n - 2i = l(v_k)$, where $k \in \{(n/2)+1, (n/2)+3, ..., (n-1)\}$.
Then $n - 2i = n + 1 + k$.
i.e., $k = -1 - 2i$.
i.e., $k = 2n + 1 - 1 - 2i = 2n - 2i$, which is a contradiction since k is odd.
Therefore, $n - 2i \neq l(v_k)$ for any $k \in \{(n/2)+1, (n/2)+3, ..., (n-1)\}$.

Suppose $n - 2i = l(v_k)$, where $k \in \{(n/2)+2, (n/2)+4, ..., n\}$.
Then $n - 2i = 2n + 2 - k$.
i.e., $k = n + 2 + 2i$.
i.e., $k \in \{2n+4, 2n+8, ..., 3n\}$ implies $k \in \{3, 7, ..., n-1\}$ under modular arithmetic.
This is a contradiction since k is even.
Therefore, $n - 2i \neq l(v_k)$ for any $k \in \{(n/2)+2, (n/2)+4, ..., n\}$.

Note that $l(v_1) - l(v_n) \equiv n(mod\ 2n+1)$ and it is not equal to any of the vertex labels. ... (i)

Let i be even.
consider $l(v_{i+1}) - l(v_i) = n + 1 + i + 1 - 2n - 2 + i$.
$= -n + 2i$.
$= 2n + 1 - n + 2i$.
$= n + 1 + 2i$.

Suppose $n + 1 + 2i = l(v_k)$, where $k \in \{1, 3, ..., (n/2) - 1\}$.

Then $n+1+2i=k$.
i.e., $k \in \{2n+5, 2n+9, ..., 3n+1\}$ implies $k \in \{4, 8, ..., n\}$ under modular arithmetic.
This is a contradiction since k is odd.
Therefore, $n+1+2i \neq l(v_k)$ for any $k \in \{1, 3, ..., (n/2)-1\}$.

Suppose $n+1+2i = l(v_k)$, where $k \in \{2, 4, ..., (n/2)\}$.
Then $n+1+2i = n+1-k$.
i.e., $k = -2i = (2n+1) - 2i$, which is a contradiction since k is even.
Therefore, $n+1+2i \neq l(v_k)$ for any $k \in \{2, 4, ..., (n/2)\}$.

Suppose $n+1+2i = l(v_k)$, where $k \in \{(n/2)+1, (n/2)+3, ..., (n-1)\}$.
Then $n+1+2i = n+1+k$.
i.e., $k = 2i$, which is a contradiction since k is odd.
Therefore, $n+1+2i \neq l(v_k)$, for any $k \in \{(n/2)+1, (n/2)+3, ..., (n-1)\}$.

Suppose $n+1+2i = l(v_k)$, where $k \in \{(n/2)+2, (n/2)+4, ..., n\}$.
Then $n+1+2i = 2n+2-k$.
i.e., $n+1-2i = k$.
i.e., $n = k+2i-1$, which is a contradiction since $k+2i-1$ is odd and n is even.
Therefore, $n+1+2i \neq l(v_k)$, for any $k \in \{(n/2)+2, (n/2)+4, ..., n\}$.

Note that $l(v_{n/2+1}) - l(v_{n/2}) = n+1$ and it is not equal to any of the vertex labels. ... (ii)

Thus we have proved that $l(v_{i+1}) - l(v_i) \neq l(v_k)$, for any $1 \leq k \leq n$ and $0 \leq i \leq n$.

Now we have to prove that $l(v_{i+1}) - l(v_i)$ are distinct for all i.

Suppose $l(v_{i+1}) - l(v_i) = l(v_{j+1}) - l(v_j)$.

Case (a): Let $i, j \leq n/2$.
Suppose i is odd and j is odd.
Then $n+1-i+1-i = n+1-j+1-j$.
i.e., $-2i = -2j$.
i.e., $i = j$.

Suppose i is odd and j is even.
Then $n+1-i+1-i = j+1-n-1+j$.

i.e., $-2i - 2j = -2n - 2$.
i.e., $i + j = n + 1$, a contradiction.

Suppose i is even and j is odd.
Then $i + 1 - n - 1 + i = n + 1 - j - j$.
i.e., $2i + 2j = 2n + 1$.
i.e., $i + j = 0$, a contradiction to the hypothesis.

Suppose i is even and j is even.
Then $i + 1 - n - 1 + i = j + 1 - n - 1 + j$.
i.e., $2i = 2j$.
i.e., $i = j$.

Case (b): Let $i \leq n/2, j > n/2$.
Suppose i is odd and j is odd.
Then $n + 1 - i - 1 - i = 2n + 2 - j - 1 - n - 1 - j$.
i.e., $-2i = -2j$.
i.e., $i = j$.

Suppose i is odd and j is even.
Then $n + 1 - i - 1 - i = n + j + 2 - 2n - 2 + j$.
i.e., $2n = 2i + 2j$.
i.e., $n = i + j$, a contradiction.

Suppose i is even and j is even.
Then $i + 1 - n - 1 + i = n + 1 + j + 1 - 2n - 2 + j$.
i.e., $2i - n = -n + 2j$.
i.e., $i = j$.

Suppose i is even and j is odd.
Then $i + 1 - n - 1 + i = 2n + 2 - j - 1 - n - 1 - j$.
i.e., $2i - n = n - 2j$.
i.e., $i + j = n$, a contradiction.

Case (c): Let $i > n/2, j > n/2$.
Suppose i is odd and j is odd.
Then $2n + 2 - i - 1 - n - 1 - i = 2n + 2 - j - 1 - n - 1 - j$.
i.e., $i = j$.

Suppose i is odd and j is even.
Then $2n + 2 - i - 1 - n - 1 - i = n + 1 + j + 1 - 2n - 2 + j$.
i.e., $n - 2i = -n + 2j$.
i.e., $2i + 2j = 2n$.

i.e., $i + j = n$, a contradiction.

Suppose i is even and j is even.
Then $n + 1 + i + 1 - 2n - 2 + i = n + 1 + j + 1 - 2n - 2 + j$.
i.e., $i = j$.

Suppose i is even and j is odd.
Then $n + 1 + i + 1 - 2n - 2 + i = 2n + 2 - j - 1 - n - 1 - j$.
i.e., $-n + 2i = n - 2j$.
i.e., $i + j = n$, a contradiction.

Also from equation (i) and (ii) we observe that $l(v_{i+1}) - l(v_i) \neq l(v_{j+1}) - l(v_j)$ for any i and j .
Thus, one can see that l is a graceful labeling of the outspoken wheel $\overrightarrow{W_n}$ when $n \equiv 0(mod 4)$.

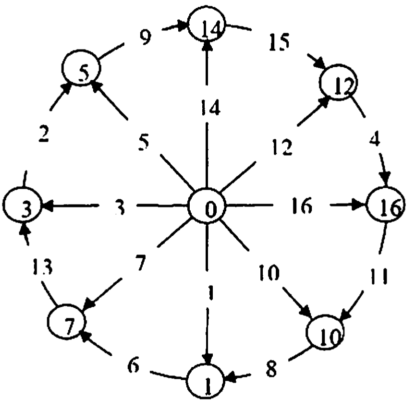

Figure 1: Graceful labeling of $\overrightarrow{W_8}$

Case (ii): When $n \equiv 2(mod 4)$.

Define a function l as follows:

$$l(v_i) = \begin{cases} i & \text{if } \ i = 1, 3, ..., (n/2). \\ n+1-i & \text{if } \ i = 2, 4, ..., (n/2)-1. \\ 2(n+1)-i & \text{if } \ i = (n/2)+1, (n/2)+3, ..., n. \\ n+1+i & \text{if } \ i = (n/2)+2, (n/2)+4, ..., n-1. \\ 0 & \text{if } \ i = 0. \end{cases}.$$

As proved in the case (i), we can prove that l is a graceful labeling of outspoken wheel $\overrightarrow{W_n}$ when $n \equiv 2(mod 4)$. Thus, the outspoken wheel is graceful for n even.

Case (iii): Let $n \equiv 1(mod 4)$.

Define a function l as follows:

$$l(v_i) = \begin{cases} \frac{3n-i}{2} & \text{if } \ i = 1, 3, ..., \frac{(n-3)}{2}. \\ n + \frac{i}{2} & \text{if } \ i = 2, 4, ..., \frac{(n-1)}{2}. \\ \frac{n+i-1}{2} & \text{if } \ i = \frac{(n+3)}{2}, \frac{(n+7)}{2}, ..., n. \\ \frac{2n-i-1}{2} & \text{if } \ i = \frac{(n+1)}{2}, \frac{(n+5)}{2}, ..., n-1. \\ 0 & \text{if } \ i = 0. \end{cases}.$$

Suppose $l(v_i) = l(v_j)$, $i \neq j$.

Here we have 3 different cases.

i) $i, j \leq (n-1)/2$.
Suppose i and j are even then $l(v_i) = l(v_j)$ if and only if $v_i = v_j$.
Similarly, if i and j are odd then $l(v_i) = l(v_j)$ if and only if $v_i = v_j$.

Let i be even and j be odd.

Then $n + i/2 = (3n - j)/2$.
i.e., $i + j = n$, which is a contradiction since the maximum of $i + j$ is $n - 2$.

If i is odd and j is even we get a similar contradiction.

iii) Suppose $i \in \{1, 2, ..., (n-1)/2\}$ and $j \in \{(n+1)/2, (n+3)/2, ..., n\}$ .Then,
a) If i is odd and j is even.
Then $(3n - i)/2 = (n + j - 1)/2$.
i.e., $i + j = 2n + 1$.
i.e., $i + j \equiv 0(mod\ 2n + 1)$, a contradiction.

b) If i is even and j is even.
Then $n + i/2 = (n + j - 1)/2$.
i.e., $j - i = n + 1$.
Since $i < j$, the maximum value of $j - i$ is $n - 3$, which is a contradiction.

c) If i is odd and j is odd.
Then $(3n - i)/2 = (2n - j - 1)/2$.
i.e., $j - i = -n - 1 \equiv n(mod\ 2n + 1)$.
Since $i < j$, the maximum value of $j - i$ is $n - 1$, which is a contradiction.

d) If i is even and j is odd.
Then $n + i/2 = (2n - j - 1)/2$.
i.e., $i + j \equiv 2n(mod\ 2n + 1)$ which is a contradiction, since $i + j \leq (3n - 1)/2 < 2n$.

iii) $i, j \geq (n + 1)/2$.
Let i be odd and j be even.
Then $i + j = n$, a contradiction since $i + j \geq n + 2$.
If i is even and j is odd we get a similar contradiction.
Therefore, $l(v_i) \neq l(v_j)$ for $i \neq j$.

Now we shall prove all edge labels are distinct.

We know that $l(v_0) = 0$ $\therefore$ $l(v_i) - l(v_0) = l(v_i)$ which are all distinct for i , $1 \leq i \leq n$.
i.e., the labels on the spokes are distinct.

Now we have to prove that $l(v_{i+1}) - l(v_i)$ are distinct from all $l(v_k)$, where $1 \leq k \leq n$.
We consider following cases.

Case (a): Let $i \in \{1, 2, ..., (n-1)/2\}$.
Let i be even.
Then $l(v_{i+1}) - l(v_i) = (3n - i - 1)/2 - (n + i/2) = (n - 2i - 1)/2$.

Suppose $(n - 2i - 1)/2 = l(v_k)$, where $k \in \{1, 3, ..., (n-3)/2\}$.
Then $n - 2i - 1 = 3n - k$.
i.e., $k = 2i$.
Since k is odd, this is a contradiction because $2i$ is even.
Therefore, $(n - 2i - 1)/2 \neq l(v_k)$ for any $k \in \{1, 3, ..., (n-3)/2\}$.

Suppose $(n - 2i - 1)/2 = l(v_k)$, where $k \in \{2, 4, ..., (n-1)/2\}$.
Then, $k = -n - 2i - 1$.
i.e., $k \equiv n - 2i (mod\ 2n + 1)$, which is a contradiction since k is even and $n - 2i$ is odd.
Therefore, $(n - 2i - 1)/2 \neq l(v_k)$ for any $k \in \{2, 4, ..., (n-1)/2\}$.

Suppose $(n-2i-1)/2 = l(v_k)$, where $k \in \{(n+3)/2, (n+7)/2, ..., n-1\}$.
Then $k = -2i$.
i.e., $k = (2n + 1) - 2i$, which is a contradiction because $2(n - i) + 1$ is odd and k is even.
Therefore, $(n-2i-1)/2 \neq l(v_k)$ for any $k \in \{(n+3)/2, (n+7)/2, ..., n-1\}$.

Suppose $(n - 2i - 1)/2 = l(v_k)$, where $k \in \{(n+1)/2, (n+5)/2, ..., n\}$.
Then $k = n + 2i$.
i.e., $k \in \{n + 4, n + 8, ..., 2n - 1\}$.
So $n + 4 \leq k \leq 2n - 1$ which is a contradiction as the maximum of k is n.
Therefore, $(n - 2i - 1)/2 \neq l(v_k)$ for any $k \in \{(n+1)/2, (n+5)/2, ..., n\}$.

Let i be odd.
Then $l(v_{i+1}) - l(v_i) = n + (i+1)/2 - (3n - i)/2 = (3n + 3 + 2i)/2$.

Suppose $(3n + 3 + 2i)/2 = l(v_k)$, where $k \in \{2, 4, ..., (n-1)/2\}$.
Then $(3n + 3 + 2i)/2 = (n + k/2)$.
i.e., $k = n + 3 + 2i$.
i.e., $k \in \{n + 5, n + 9, ..., 2n\}$.
So $n + 5 \leq k \leq 2n$, which is a contradiction as the maximum of k is $(n-1)/2$.
Therefore, $(3n + 3 + 2i)/2 \neq l(v_k)$ for any $k \in \{2, 4, ..., (n-1)/2\}$.

Suppose $(3n + 3 + 2i)/2 = l(v_k)$, where $k \in \{1, 3, ..., (n-3)/2\}$.
Then $(3n + 3 + 2i)/2 = (3n - k)/2$.

i.e., $k = -3 - 2i \equiv 2(n - i - 1)(mod\ 2n + 1)$ which is a contradiction since k is odd.
Therefore, $(3n + 3 + 2i)/2 \neq l(v_k)$ for any $k \in \{1, 3, ..., (n - 3)/2\}$.

Suppose $(3n+3+2i)/2 = l(v_k)$, where $k \in \{(n+3)/2, (n+7)/2, ..., n-1\}$.
Then $(3n + 3 + 2i)/2 = (n + k - 1)/2$.
i.e., $k = 3 + 2i$, which is a contradiction as k is even.
Therefore, $(3n+3+2i)/2 \neq l(v_k)$ for any $k \in \{(n+3)/2, n+7)/2, ..., n-1\}$.

Suppose $(3n + 3 + 2i)/2 = l(v_k)$, where $k \in \{(n + 1)/2, (n + 5)/2, ..., n\}$.
Then $(3n + 3 + 2i)/2 = (2n - k - 1)/2$.
i.e., $k = n + 4 + 2i$.
i.e., $k \in \{n + 6, n + 10, ..., 2n + 1\}$.
So $n + 6 \leq k \leq 2n + 1$ which is a contradiction as the maximum of k is n.
Therefore, $(3n+3+2i)/2 \neq l(v_k)$ for any $k \in \{(n+1)/2, (n+5)/2, ..., n\}$.

Case (b): Let $i \in \{(n + 1)/2, (n + 3)/2, ..., n\}$.
Let i be odd.
Then $l(v_{i+1}) - l(v_i) = (n+i+1-1)/2 - (2n-i-1)/2 = (-n+2i+1)/2$.

Suppose $(-n + 2i + 1)/2 = l(v_k)$, where $k \in \{1, 3, ..., (n - 3)/2\}$.
Then $(-n + 2i + 1)/2 = (3n - k)/2$.
i.e., $k = 4n - 1 - 2i = 2n - 2i - 2$ and it is even.
Since k is odd, this is a contradiction.
Therefore, $(-n + 2i + 1)/2 \neq l(v_k)$ for any $k \in \{1, 3, ..., (n - 3)/2\}$.

Suppose $(-n + 2i + 1)/2 = l(v_k)$, where $k \in \{2, 4, ..., (n - 1)/2\}$.
Then $k = -3n + 1 + 2i \equiv -n + 2i + 2(mod\ 2n + 1)$.
i.e., $k \in \{3, 7, ..., n + 2\}$, a contradiction.
Therefore, $(-n + 2i + 1)/2 \neq l(v_k)$ for any $k \in \{2, 4, ..., (n - 1)/2\}$.

Suppose $(-n+2i+1)/2 = l(v_k)$, where $k \in \{(n+3)/2, (n+7)/2, ..., n-1\}$.
Then $k = -2n + 2i + 2$.
i.e., $k = 2i + 3$, a contradiction.
Therefore, $(-n+2i+1)/2 \neq l(v_k)$ for any $k \in \{(n+3)/2, (n+7)/2, ..., n-1\}$.

Suppose $(-n + 2i + 1)/2 = l(v_k)$, where $k \in \{(n + 1)/2, (n + 5)/2, ..., n\}$.
Then $k = 3n - 2i - 2 \equiv n - 2i - 3(mod\ 2n + 1)$, a contradiction as k is odd.
Therefore, $(-n+2i+1)/2 \neq l(v_k)$ for any $k \in \{(n+1)/2, (n+5)/2, ..., n\}$.

Let i be even.
Then $l(v_{i+1}) - l(v_i) = (2n-i-1-1)/2 - (n+i-1)/2 = (n-2i-1)/2$.

Suppose $(n-2i-1)/2 = l(v_k)$, where $k \in \{1,3,...,(n-3)/2\}$.
Then $(n-2i-1)/2 = (3n-k)/2$.
i.e., $k = 2n+2i+1 = 2i$, a contradiction.
Therefore, $(n-2i-1)/2 \neq l(v_k)$ for any $k \in \{1,3,...,(n-3)/2\}$.

Suppose $(n-2i-1)/2 = l(v_k)$, where $k \in \{2,4,...,(n-1)/2\}$.
Then $k = -n-2i-1 = n-2i$, which is a contradiction because $n-2i$ is odd and k is even.
Therefore, $(n-2i-1)/2 \neq l(v_k)$ for any $k \in \{2,4,...,(n-1)/2\}$.

Suppose $(n-2i-1)/2 = l(v_k)$, where $k \in \{(n+3)/2,(n+7)/2,...,n-1\}$.
Then $k = -2i$.
i.e., $k = (2n+1)-2i$, a contradiction.
Therefore, $(n-2i-1)/2 \neq l(v_k)$ for any $k \in \{(n+3)/2,(n+7)/2,...,n-1\}$.

Suppose $(n-2i-1)/2 = l(v_k)$, where $k \in \{(n+1)/2,(n+5)/2,...,n\}$.
Then $k = n+2i$. i.e., $k \in \{2n+3, 2n+7,...,3n-2\}$.
i.e., $k \in \{2,6,...,n-3\}$ under modular arithmetic, a contradiction, since k is odd.
Therefore, $(n-2i-1)/2 \neq l(v_k)$ for any $k \in \{(n+1)/2,(n+5)/2,...,n\}$.

Note that $l(v_1) - l(v_n) \equiv n(mod\ 2n+1)$ and it is not equal to any of the vertex labels. ...(i)
Also $l(v_{(n+1)/2}) - l(v_{(n-1)/2}) \equiv (3n+1)/2(mod\ 2n+1)$ and it is not equal to any of the vertex labels. ...(ii)

Now we have to prove that $l(v_{i+1}) - l(v_i)$ are distinct for all i.

Suppose $l(v_{i+1}) - l(v_i) = l(v_{j+1}) - l(v_j)$.

Case (a): Let $i, j \leq (n-1)/2$.
Suppose i is odd and j is odd.
Then $(3n+3+2i)/2 = (3n+3+2j)/2$.
i.e., $i = j$.

Suppose i is odd and j is even.
Then $(3n+3+2i)/2 = (n-2j-1)/2$.
i.e., $i+j = -n-2 = 2n+1-n-2 = n-1$, a contradiction.

Suppose i is even and j is odd.

Then $(n-2i-1)/2 = (3n+3+2j)/2$.
i.e., $i+j = -n-2 = 2n+1-n-2 = n-1$,a contradiction.

Suppose i is even and j is even.
Then $i = j$.

Case (b): Let $i \leq (n-1)/2$ and $j \geq (n+1)/2$
Suppose i is odd and j is odd.
Then $(3n+3+2i)/2 = (3n+3+2j)/2$.
i.e., $i = j$.

Suppose i is odd and j is even.
Then $(3n+3+2i)/2 = (n-2j-1)/2$.
i.e., $i+j = -n-2$.
i.e., $i+j = (2n+1)-n-2 = n-1$, a contradiction.

Suppose i is even and j is even.
Then $(n-2i-1)/2 = (n-2j-1)/2$.
i.e., $i = j$.

Suppose i is even and j is odd.
Then $(n-2i-1)/2 = (3n+3+2j)/2$.
i.e., $i+j = -n-2$.
i.e., $i+j = (2n+1)-n-2 = n-1$, a contradiction.

Case (c): Let $i \geq (n+1)/2$, $j \geq (n+1)/2$.
Suppose i is odd and j is odd.
Then $i = j$.

Suppose i is odd and j is even.
Then $(3n+3+2i)/2 = (n-2j-1)/2$.
i.e., $i+j = -n-2$.
i.e., $i+j = (2n+1)-n-2 = n-1$, a contradiction.

Suppose i is even and j is even.
Then $i = j$.

Suppose i is even and j is odd.
Then $(n-2i-1)/2 = (3n+3+2j)/2$.
i.e., $i+j = -n-2$.
i.e., $i+j = (2n+1)-n-2 = n-1$, a contradiction.

Also from equation (i) and (ii), we observe that $l(v_{i+1}) - l(v_i) \neq l(v_{j+1}) -$

$l(v_j)$ for any i and j.

Thus one can see that l is a graceful labeling of the outspoken wheel when $n \equiv 1(mod\ 4)$.

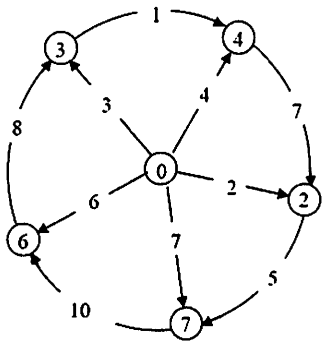

Figure 2: Graceful labeling of $\overrightarrow{W_5}$

Case (iv): Let $n \equiv 3(mod\ 4)$.

Define a function l as follows:

$$l(v_i) = \begin{cases} \frac{3n-i}{2} & \text{if } i = 1, 3, ..., \frac{(n-1)}{2}. \\ n + \frac{i}{2} & \text{if } i = 2, 4, ..., \frac{(n-3)}{2}. \\ \frac{n+i-1}{2} & \text{if } i = \frac{(n+1)}{2}, \frac{(n+5)}{2}, ..., n-1. \\ \frac{2n-i-1}{2} & \text{if } i = \frac{(n+3)}{2}, \frac{(n+7)}{2}, ..., n. \\ 0 & \text{if } i = 0. \end{cases}$$

As proved in the case(iii), we can prove that l is a graceful labeling of outspoken wheel $\overrightarrow{W_n}$ when $n \equiv 3(mod\ 4)$ .Thus, the outspoken wheel $\overrightarrow{W_n}$ is graceful for n odd.

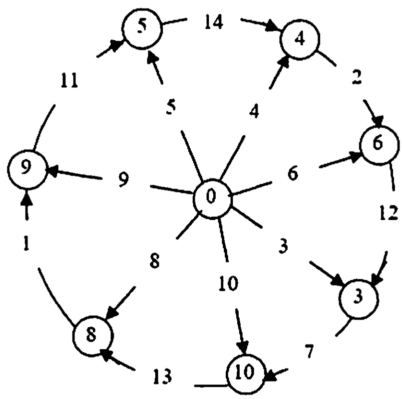

Figure 3: Graceful labeling of $\overrightarrow{W_7}$

Thus, the outspoken unicyclic wheel $\overrightarrow{W_n}$ is graceful.

□

Remark. By proposition 8.2 of Bloom and Hsu [1], it follows that the inspoken wheel is also graceful. Thus, all unicyclic wheels are graceful.

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