When is the Direct Product of Generalized Mycielskians a Cover Graph?

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Abstract

A graph is said to be a cover graph if it is the underlying graph of the Hasse diagram of a finite partially ordered set. The direct product $G \times H$ of graphs G and H is the graph having vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{(g_i, h_s)(g_j, h_t) \mid g_i g_j \in E(G) \text{ and } h_s h_t \in E(H)\}$. We prove that the direct product $\mathsf{M}_m(G) \times \mathsf{M}_n(H)$ of the generalized Mycielskians of G and H is a cover graph if and only if G or H is bipartite.

Keyword: cover graph; direct product; generalized Mycielskian; circular chromatic number.

1 Introduction

All graphs are assumed to be finite and simple in this paper. Let uv denote the edge joining the vertices u and v. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G if there is any, and ∞ if G possesses no cycles. The *chromatic number* $\chi(G)$ is the least number of colors to be assigned to the vertices of G so that adjacent vertices receive distinct colors. The Hasse diagram of a finite partially ordered set depicts the covering relation of elements; its underlying graph is called a *cover graph*. The problem of characterizing cover graphs goes back to Ore [10]. It is an NP-complete problem to decide whether a graph G is a cover graph (see [3] and [9]).

Let the graph G be endowed with an acyclic orientation D, i.e., there are no directed cycles with respect to D. An arc of D is called *dependent* if its reversal creates a directed cycle in D. Let $d_{\min}(G)$ be the minimum number of dependent arcs over all acyclic orientations of G. Pretzel [11] proved that $d_{\min}(G) = 0$ is equivalent to G being a cover graph. It follows immediately that a triangle is not a cover graph. The following is a well-known sufficient condition to identify a cover graph. It was obtained in Aigner and Prins [1] and first appeared in Pretzel and Youngs [12]. A simple proof is included in Fisher et al. [5].

Theorem 1 *A graph G is a cover graph if $\chi(G) < g(G)$.*

A *homomorphism* from a graph G to a graph H is a mapping from $V(G)$ to $V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. We denote

$G \to H$ if there is a homomorphism from G to H. Note that $G_1 \to G_2$ and $G_2 \to G_3$ imply $G_1 \to G_3$ for graphs G_1, G_2, and G_3. Lih et al. [8] proved the following.

Lemma 2 *Assume that $G \to H$. If H is a cover graph, then G is a cover graph.*

The *direct product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{(g_i, h_s)(g_j, h_t) \mid g_i g_j \in E(G) \text{ and } h_s h_t \in E(H)\}$. Other names for the direct product include tensor product, categorical product, Kronecker product, cardinal product, relational product, weak direct product, etc. The following is an outstanding conjecture of Hedetniemi [6] involving the direct product.

Conjecture 3 $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

The reader is referred to Imrich and Klavžar [7] for more information on direct products. It is easy to see that $G \times H \to G$ and $G \times H \to H$. The following theorem is a consequence of Lemma 2.

Theorem 4 *The direct product $G \times H$ is a cover graph if G or H is a cover graph.*

Since G is isomorphic to the subgraph of $G \times G$ induced by the set $\{(v, v) \mid v \in V(G)\}$, we have the following consequence.

Corollary 5 *The direct product $G \times G$ is not a cover graph if G is not a cover graph.*

Theorem 6 *We have the following two cases.*

1. *Suppose that $g(G) = g(H) = 3$. Then $G \times H$ is not a cover graph.*

2. *Suppose that $g(G) > 3 \geqslant \chi(H)$. Then $G \times H$ is a cover graph.*

Proof. If $g(G) = g(H) = 3$, then $G \times H$ contains a triangle; that is, $G \times H$ is not a cover graph. If $g(G) > 3 \geqslant \chi(H)$, then we have that

$g(G \times H) > 3 \geqslant \chi(H)$. Since $G \times H \to H$, $\chi(H) \geqslant \chi(G \times H)$. By Theorem 1, $G \times H$ is a cover graph. ∎

Probabilistic arguments ([2] for example) show that it is common to have triangle-free graphs that are not cover graphs. Take the direct product of such a graph with a triangle. Theorem 6 leaves open possibility that this direct product could be a cover graph even if each of the two factors is not a cover graph. In the next section we shall show that some direct product is not a cover graph when each of the two components has girth 4, chromatic number 4, and is not a cover graph. We shall also answer the question posed in the title. A general problem remains to be completely solved is to determine whether the product $G \times H$ is not a cover graph while both G and H are not cover graphs.

2 Direct product of generalized Mycielskians

Let $G = (V_0, E_0)$ be a graph with vertex set $V_0 = \{\langle 0,0\rangle, \langle 0,1\rangle, \ldots, \langle 0, n-1\rangle\}$ and edge set E_0. For $m > 0$, the *generalized Mycielskian* $\mathsf{M}_m(G)$ of G has vertex set $V = V_0 \cup (\cup_{i=1}^{m} V_i) \cup \{u\}$, where $V_i = \{\langle i,j\rangle \mid 0 \leqslant j \leqslant n-1\}$ for $1 \leqslant i \leqslant m$, and edge set $E = E_0 \cup (\cup_{i=1}^{m} E_i) \cup \{\langle m,j\rangle u \mid 0 \leqslant j \leqslant n-1\}$, where $E_i = \{\langle i-1,j\rangle\langle i,k\rangle \mid \langle 0,j\rangle\langle 0,k\rangle \in E_0\}$ for $1 \leqslant i \leqslant m$. We call u the root in $\mathsf{M}_m(G)$. In the sequel, the second coordinates of vertices in $\mathsf{M}_m(G)$ are always taken modulo the order n. We note that $\mathsf{M}_1(G)$ is commonly known as the *Mycielskian* of G. It is easy to see that if H is a subgraph of G, then $\mathsf{M}_m(H)$ is a subgraph of $\mathsf{M}_m(G)$. The following theorem appeared in Lih et al. [8].

Theorem 7 *Let $n \geqslant 3$. The graph $\mathsf{M}_m(C_n)$ is a cover graph if and only if n is even.*

This can be further generalized into the following.

Theorem 8 *The graph $\mathsf{M}_m(G)$ is a cover graph if and only if G is bipartite.*

Proof. If G has no edge, then obviously $\mathsf{M}_m(G)$ is a cover graph. Let G be a bipartite graph with at least one edge. Then $\chi(\mathsf{M}_m(G)) = 3 < 4 \leqslant g(\mathsf{M}_m(G))$. Hence $\mathsf{M}_m(G)$ is a cover graph. If G is not bipartite, then G contains an odd cycle C with length at least 3. By Theorem 7, $\mathsf{M}_m(C)$ is not a cover graph. Hence, $\mathsf{M}_m(G)$, being a supergraph of $\mathsf{M}_m(C)$, is not a cover graph. ∎

The following two lemmas can be verified in a straightforward manner.

Lemma 9 *Let H_i be a subgraph of G_i for $i = 1$ and 2. Then $H_1 \times H_2$ is a subgraph of $G_1 \times G_2$.*

Lemma 10 *Let $H_i \to G_i$ for $i = 1$ and 2. Then $H_1 \times H_2 \to G_1 \times G_2$.*

The next lemma will be used in proving our main results.

Lemma 11 *Let $p \geqslant m \geqslant 1$ and $q \geqslant s \geqslant 1$. Then we have $\mathsf{M}_p(C_{2q+1}) \to \mathsf{M}_m(C_{2s+1})$.*

Proof. Let the cycle C_n with n vertices be denoted $\langle 0,0\rangle\langle 0,1\rangle\cdots\langle 0,n-1\rangle\langle 0,0\rangle$. The required homomorphism will be constructed in two stages.

Stage 1. Define a mapping $\sigma : V(\mathsf{M}_p(C_{2q+1})) \to V(\mathsf{M}_m(C_{2q+1}))$ as follows. Let $\sigma(u) = u$. We always assume that $0 \leqslant j \leqslant 2q$. Now let

$$\sigma(\langle i,j\rangle) = \begin{cases} \langle 0,j\rangle & \text{if } 0 \leqslant i \leqslant p-m, \\ \langle i-p+m,j\rangle & \text{if } p-m+1 \leqslant i \leqslant p. \end{cases}$$

To check that σ is a homomorphism from $\mathsf{M}_p(C_{2q+1})$ to $\mathsf{M}_m(C_{2q+1})$, let xy be an edge of $\mathsf{M}_p(C_{2q+1})$.

If $x = \langle 0,j\rangle$ and $y = \langle 0,j+1\rangle$, then $\sigma(x)\sigma(y) = \langle 0,j\rangle\langle 0,j+1\rangle$.

If $x = \langle i,j\rangle$, $y = \langle i-1,j\pm 1\rangle$, and $1 \leqslant i \leqslant p-m$, then $\sigma(x)\sigma(y) = \langle 0,j\rangle\langle 0,j\pm 1\rangle$.

If $x = \langle i,j\rangle$, $y = \langle i-1,j\pm 1\rangle$, and $p-m+1 \leqslant i \leqslant p$, then $\sigma(x)\sigma(y) = \langle i-p+m,j\rangle\langle i-p+m-1,j\pm 1\rangle$.

If $x = \langle p,j\rangle$ and $y = u$, then $\sigma(x)\sigma(y) = \langle m,j\rangle u$.

We see that $\sigma(x)\sigma(y) \in E(\mathsf{M}_m(C_{2q+1}))$ in all cases. Hence, we have $\mathsf{M}_p(C_{2q+1}) \to \mathsf{M}_m(C_{2q+1})$.

Stage 2. Define a mapping $\tau : V(\mathsf{M}_m(C_{2q+1})) \to V(\mathsf{M}_m(C_{2s+1}))$ as follows. Let $\tau(u) = u$. When $0 \leqslant i \leqslant m$, let

$$\tau(\langle i,j\rangle) = \begin{cases} \langle i,j\rangle & \text{if } 0 \leqslant j \leqslant 2s, \\ \langle i,2s\rangle & \text{if } 2s+2 \leqslant j \leqslant 2q \text{ and } j \text{ is even}, \\ \langle i,2s-1\rangle & \text{if } 2s+1 \leqslant j \leqslant 2q-1 \text{ and } j \text{ is odd}. \end{cases}$$

To check that τ is a homomorphism from $\mathsf{M}_m(C_{2q+1})$ to $\mathsf{M}_m(C_{2s+1})$, let xy be an edge of $\mathsf{M}_m(C_{2q+1})$ having the form $x = \langle i,j\rangle$ and $y = \langle k,j+1\rangle$, where $i,k \in \{0,1,\ldots,m\}$, $k = i \pm 1$ if $i > 0$, and $k = 0$ if $i = 0$.

$$\tau(x)\tau(y) = \begin{cases} \langle i,j\rangle\langle k,j+1\rangle & \text{if } 0 \leqslant j \leqslant 2s-1, \\ \langle i,2s\rangle\langle k,2s-1\rangle & \text{if } 2s \leqslant j \leqslant 2q-2 \text{ and } j \text{ is even}, \\ \langle i,2s-1\rangle\langle k,2s\rangle & \text{if } 2s+1 \leqslant j \leqslant 2q-1 \text{ and } j \text{ is odd}, \\ \langle i,2s\rangle\langle k,0\rangle & \text{if } j = 2q. \end{cases}$$

Next if $x = \langle m,j\rangle$, $y = u$, and $0 \leqslant j \leqslant 2q$, then $\tau(x)\tau(y) = \langle m,h\rangle u$ for an appropriate $h \in \{0,1,\ldots,2s\}$.

We see that $\tau(x)\tau(y) \in E(\mathsf{M}_m(C_{2s+1}))$ in all cases. Hence, we have $\mathsf{M}_m(C_{2q+1}) \to \mathsf{M}_m(C_{2s+1})$. ∎

Theorem 12 *If m,n,s,t are all positive integers, then the direct product $\mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})$ is not a cover graph.*

Proof. Assume $p = \max\{m,n\}$ and $q = \max\{s,t\}$. By Lemma 10 and Lemma 11, we have $\mathsf{M}_p(C_{2q+1}) \times \mathsf{M}_p(C_{2q+1}) \to \mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})$. Then Lemma 2, Corollary 5, and Theorem 8 finish the proof. ∎

Theorem 13 *The direct product $\mathsf{M}_m(G) \times \mathsf{M}_n(H)$ is a cover graph if and only if G or H is bipartite.*

Proof. Without loss of generality, we may suppose that G is bipartite. Then $\mathsf{M}_m(G)$ is a cover graph by Theorem 8. Since $\mathsf{M}_m(G) \times \mathsf{M}_n(H) \to \mathsf{M}_m(G)$, $\mathsf{M}_m(G) \times \mathsf{M}_n(H)$ is a cover graph by Lemma 2.

Conversely, if both G and H are not bipartite, then we may assume that C_{2s+1} is a subgraph of G and C_{2t+1} is a subgraph of H for some positive integers s and t. Hence $\mathsf{M}_m(C_{2s+1})$ is a subgraph of $\mathsf{M}_m(G)$ and $\mathsf{M}_n(C_{2t+1})$ is a subgraph of $\mathsf{M}_n(H)$. Then $\mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})$ is a subgraph of $\mathsf{M}_m(G) \times \mathsf{M}_n(H)$ by Lemma 9. Since $\mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})$ is not a cover graph, neither is $\mathsf{M}_m(G) \times \mathsf{M}_n(H)$ a cover graph. ∎

3 A quick application

To conclude this paper, we give a quick application of our main results. In recent years, there has been an intensive investigation into the circular chromatic number $\chi_c(G)$ of a graph G. Since $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$, the circular chromatic number is regarded as a refinement of the ordinary chromatic number. Zhu [14] and [15] provide comprehensive surveys on the circular chromatic number. The reader is referred to them for basic notions and results. Among a number of approaches to define the circular chromatic number, one is via the minimum imbalance of acyclic orientations.

Let D be an acyclic orientation of the graph G. For an undirected cycle C of G, we choose one of the two traversals of C as the positive direction. An arc is said to be *forward* if its orientation under D is along the positive direction of C, otherwise it is said to be *backward*. We use $(C, D)^+$ (or $(C, D)^-$) to denote the set of all forward (or backward) arcs of C with respect to D. The *imbalance* $\mathrm{Imb}(D)$ of D is defined to be

$$\max\left\{\max\left\{\frac{|C|}{|(C,D)^+|}, \frac{|C|}{|(C,D)^-|}\right\} \;\middle|\; C \text{ is a cycle of } G\right\}.$$

By convention, $\mathrm{Imb}(D) = 2$ if G has no cycles. A result of Goddyn et al. [4] implies that

$$\chi_c(G) = \min\{\mathrm{Imb}(D) \mid D \text{ is an acyclic orientation of } G\}.$$

We can generalize Theorem 1 in the context of circular chromatic number as follows.

Theorem 14 *A graph G is a cover graph if $\chi_c(G) < g(G)$.*

Proof. There exists some acyclic orientation D_0 of G with $\chi_c(G) = \mathrm{Imb}(D_0)$ by the result of Goddyn et al. Suppose that D_0 has a dependent arc. Then the reversal of that dependent arc will create a directed cycle with an underlying cycle C_0 of G. Hence $\max\{|C_0|/|(C_0, D_0)^+|, |C_0|/|(C_0, D_0)^-|\} \geqslant |C_0| \geqslant g(G)$. It follows that $\chi_c(G) = \mathrm{Imb}(D_0) \geqslant g(G)$ which contradicts our assumption. Therefore $d_{\min}(G) = 0$ and G is a cover graph. ∎

It follows from Theorems 8 and 14 that, $\chi_c(\mathsf{M}_m(C_{2s+1})) = 4$ for $s > 1$ since $\chi(\mathsf{M}_m(C_{2s+1})) = g(\mathsf{M}_m(C_{2s+1})) = 4$. Now let $s, t > 1$, and let $G = \mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})$. Since the girth of G is 4 and G is not a cover graph, we have $\chi_c(G) \geqslant 4$ by Theorem 14. On the other hand, $G \to \mathsf{M}_m(C_{2s+1})$ implies that $\chi_c(G) \leqslant \chi(G) \leqslant \chi(\mathsf{M}_m(C_{2s+1})) = 4$.

Corollary 15 *Let $s, t > 1$. Then*

$$\chi_c(\mathsf{M}_m(C_{2s+1}) \times \mathsf{M}_n(C_{2t+1})) = \min\{\chi_c(\mathsf{M}_m(C_{2s+1})), \chi_c(\mathsf{M}_n(C_{2t+1}))\}.$$

Note that the above Corollary also follows directly from the main argument of the proof of Theorem 12. It is also an immediate consequence of the following much stronger result established in Tardif [13]: $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$ if $\min\{\chi_c(G), \chi_c(H)\} \leqslant 4$.

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