Minimum Metric Dimension of Illiac Networks

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Abstract

Let $M = \{v_1, v_2 \ldots v_n\}$ be an ordered set of vertices in a graph G. Then $(d(u, v_1), d(u, v_2) \ldots d(u, v_n))$ is called the *M-coordinates* of a vertex u of G. The set M is called a *metric basis* if the vertices of G have distinct M-coordinates. A minimum metric basis is a set M with minimum cardinality. The cardinality of a minimum metric basis of G is called *minimum metric dimension*. This concept has wide applications in motion planning and in the field of robotics. In this paper, we have solved the minimum metric dimension problem for Illiac networks.

Keywords: Illiac network, metric basis, minimum metric dimension

1 Introduction

A circulant undirected graph denoted by $G(n; \pm\{1, 2 \ldots j\}), 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, for $n \geq 3$, is defined as a graph with vertex set $V = \{0, 1 \ldots n-1\}$ and edge set $E = \{(i, j) : |j - i| \equiv s(\mathrm{mod}\, n),\ s \in \{1, 2 \ldots j\}$ [18]. See Figure 1. It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2 \ldots \lfloor \frac{n}{2} \rfloor\})$ is the complete graph K_n.

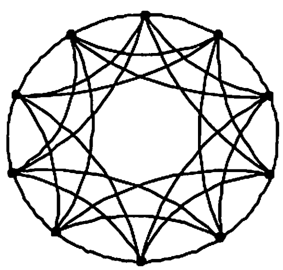

Figure 1: A circulant graph $G(10; \pm\{1, 2, 3\})$

The circulant graphs are an important class of topological structures of interconnection networks. For example, undirected circulant networks arise in the context of Mesh Connected Computer suited for parallel processing of data, such as the well-known ILLIAC type computers.

ILLIAC was the name given to a series of supercomputers built at the University of Illinois at Urbana-Champaign. The ILLIAC IV was one of the first attempts at a massively parallel computer; first large-scale array computer, with a computation speed of 200 million instructions per second, about 300 million operations per second and 1 billion bits per second of I/O transfer via a unique combination of parallel architecture and the overlapping or "pipe-lining" structure of its 64 processing elements. In all, 5 computers were built in this series between 1951 and 1974. Design of the ILLIAC VI began in early 2005.

The Illiac interconnection network consists of n^2 processors that could be depicted as the elements of an $n \times n$ matrix, each processor is directly connected through an undirected link to its immediate neighbours in its row and column and additional wrap-around connections exist. The Illiac network with 16 processors is shown in Figure 2(a) can be represented as a circulant graph $G(16; \pm\{1,4\})$ as shown in Figure 2(b). Generally the Illiac network with n^2 processors can be represented as a circulant graph $G(n^2; \pm\{1,n\})$ [18].

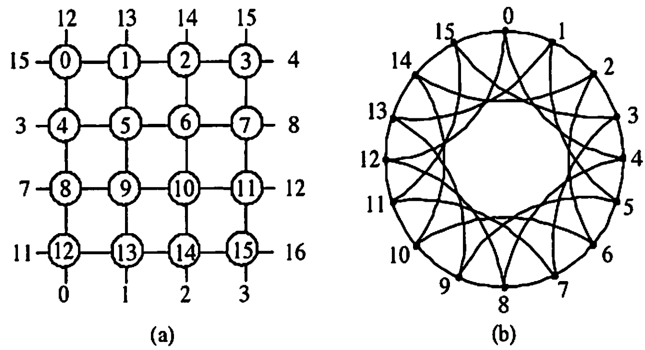

Figure 2: (a) An Illiac network with 16 processors (b) Circulant graph $G(16; \pm\{1,4\})$

2 An Overview of the Paper

Let $M = \{v_1, v_2 \ldots v_n\}$ be an ordered set of vertices in a graph G. Then $(d(u,v_1), d(u,v_2) \ldots d(u,v_n))$ is called the M-coordinates of a vertex u of G. The set M is called a *metric basis* if the vertices of G have distinct M-coordinates. A *minimum metric basis* is a set M with minimum cardinality. The cardinality of a minimum metric basis of G is called *minimum metric dimension* and is denoted by $\beta(G)$ [9]. The *minimum metric dimension* (mmd) problem is to find a minimum metric basis. If M is a metric basis then it is clear that for each pair of vertices u and v of $V \setminus M$, there is a vertex $w \in M$ such that $d(u,w) \neq d(v,w)$.

This problem has application in the field of robotics. A robot is a mechanical device which is made to move in space with obstructions around. It has neither the concept of direction nor that of visibility. But it is assumed that it can sense the distances to a set of landmarks. Evidently, if the robot knows its distances to a sufficiently large set of landmarks its position in space is uniquely determined.

The concept of metric basis and minimum metric basis has appeared in the literature under a different name as early as 1975. Slater in [16] and later in [17] had called a metric basis and a minimum metric basis as a *locating set* and a *reference set* respectively. Slater called the cardinality of a reference set as the *location number* of G. He described the usefulness of these ideas when working with sonar and loran stations. Chartrand et al. [6] have called a metric basis and a minimum metric basis as a *resolving set* and a *minimum resolving set*.

If G has p vertices then it is clear that $1 \leq \beta(G) \leq p-1$. Harary et al. [9] have shown that for the complete graph K_p, the cycle C_p and the complete bipartite graph $K_{m,n}$, the minimum metric dimensions are given by $\beta(K_p) = p-1$, $\beta(C_p) = 2$ and $\beta(K_{m,n}) = m+n-2$. This problem has been studied for grids [11], trees, multi-dimensional grids [10], Petersen graphs [3], De Bruijn graphs [12], Kautz networks [13], Benes and Butterfly networks [14], Honeycomb networks [15], Uniform Theta graphs and Quasi-uniform Theta graphs [4], Binary Tree Derived Architectures [2], and Hyper tree Derived Architectures [5].

The mmd problem is NP-complete for general graphs [8]. Recently Manuel et al. [14] have proved that the mmd problem is NP-complete for bipartite graphs by a reduction from 3-SAT, thus narrowing down the gap between the polynomial classes and NP-complete classes of the mmd problem. The method adopted is due to Khuller et. al. [10].

In this paper, the minimum metric dimension problem for Illiac networks has been solved.

The following theorem is crucial to our results in this paper.

Theorem 1 *[10] Let G be a graph with minimum metric dimension 2 and let $\{a,b\} \subset V$ be a metric basis in G. The following are true:*
1. *There is a unique shortest path P between a and b.*
2. *The degrees of a and b are at most 3.*
3. *Every other node on P has degree at most 5.* ■

3 Minimum Metric Dimension

Let $G(n^2; \pm\{1,n\})$, $n > 3$ be an Illiac network. As a graph, G has n^2 vertices and $2n^2$edges. Further G contains $n+1$ edge-disjoint cycles, one is Hamiltonian and the rest are cycles on n vertices [18]. We denote the Hamiltonian cycle by C. Assume that C is drawn as a circle in the plane. The geometric diameter of C through a is called the *mirror* through a. In Figure 3, the mirror through a is shown by a broken line.

We now determine the minimum metric dimension of the Illiac networks.

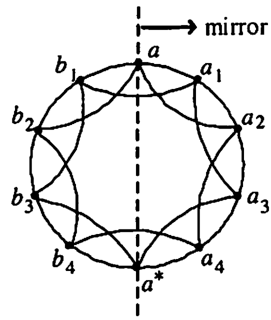

Figure 3: $G(10; \pm\{1, 2\})$

Theorem 2 *Let $G = G(n^2; \pm\{1, n\})$ be an Illiac network. Then $\beta(G) = 3$, for $n > 3$.*

Proof. Since G is 4-regular, in view of Theorem 1, G cannot have a metric basis of cardinality 2. Hence $\beta(G) > 2$. To prove the theorem we exhibit a metric basis M of cardinality 3. We consider two cases.

Case(i): *n is odd*

Fix a vertex a in C. Let the vertices of G be $a, a_1, a_2, \ldots a_t, b_1, b_2 \ldots b_t$, where $t = (n^2 - 1)/2$. Then the mirror through a divides $V(G) \setminus \{a\}$ into two sets S_1 and S_2, where $S_1 = \{a_1, a_2 \ldots a_t\}$ and $S_2 = \{b_1, b_2 \ldots b_t\}$. See Figure 4.

We begin with $M = \{a\}$. The vertices at distance i from a are a_i, $a_{jn+(i-j)}$, $a_{jn-(i-j)}$, b_i, $b_{jn+(i-j)}$, $b_{jn-(i-j)}$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor, 1 \leq j \leq i$, and $a_{jn-(i-j)}$, $a_{jn+(i-j)}$, $b_{jn-(i-j)}$, $b_{jn+(i-j)}$ for $\lfloor \frac{n}{2} \rfloor < i \leq n-1$, $i - \lfloor \frac{n}{2} \rfloor \leq j \leq \lfloor \frac{n}{2} \rfloor$. We augment M by including $a_{\lfloor \frac{n}{2} \rfloor}$. So, $M = \{a, a_{\lfloor \frac{n}{2} \rfloor}\}$.

Then, for $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, the vertices $a_{ns+k-\lceil \frac{n}{2} \rceil}$, $a_{ns-k+\lceil \frac{n}{2} \rceil}$ in S_1 and the vertices b_{ns+k-n}, b_{ns-k+1} in S_2 are equidistant from both a and $a_{\lfloor \frac{n}{2} \rfloor}$ and hence have the same M-coordinates. Now, we include $a_{t-\lfloor \frac{n}{2} \rfloor+1}$ in M.

We compute the distances of these equidistant pairs in S_1 from $a_{t-\lfloor \frac{n}{2} \rfloor+1}$.

$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, a_{ns+k-\lceil \frac{n}{2} \rceil}) = n - k - s + 1$

$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, a_{ns-k+\lceil \frac{n}{2} \rceil}) = n - k - s - 1$.

This implies that $d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, a_{ns+k-\lceil \frac{n}{2} \rceil}) \neq d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, a_{ns-k+\lceil \frac{n}{2} \rceil})$ for $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, i.e., M distinguishes pairs of vertices in S_1.

We next consider the pairs of vertices in S_2 having same M-coordinates and compute the distances from $a_{t-\lfloor \frac{n}{2} \rfloor+1}$.

For $s = 1, b_{ns+k-n} = b_k$ and $b_{ns-k+1} = b_{n-k+1}$. Thus

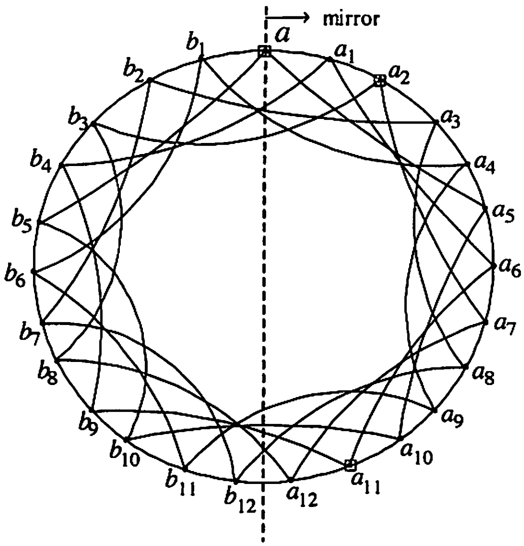

Figure 4: $G\{25; \pm\{1,5\}\}$ with $M = \{a, a_2, a_{11}\}$

$$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_k) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & k = 1 \\ \lceil \frac{n}{2} \rceil + k & 2 \le k < \lfloor \frac{n}{2} \rfloor \\ \lceil \frac{n}{2} \rceil + k - 1 & k = \lfloor \frac{n}{2} \rfloor \end{cases}$$

and

$$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_{ns-k+1}) = \begin{cases} \lceil \frac{n}{2} \rceil & k = 1 \\ \lceil \frac{n}{2} \rceil + k - 3 & 2 \le k < \lfloor \frac{n}{2} \rfloor \end{cases}$$

Similarly, for $2 \le s \le \lfloor \frac{n}{2} \rfloor$,

$$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_{ns+k-n}) = \begin{cases} \lceil \frac{n}{2} \rceil - s + 3 & k = 1 \\ \lceil \frac{n}{2} \rceil + k - s + 2 & 2 \le k < \lfloor \frac{n}{2} \rfloor \\ \lceil \frac{n}{2} \rceil + k - s & k = \lfloor \frac{n}{2} \rfloor \end{cases}$$

and

$$d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_{ns-k+1}) = \begin{cases} \lceil \frac{n}{2} \rceil - s + 1 & k = 1 \\ \lceil \frac{n}{2} \rceil + k - s - 2 & 2 \le k \le \lfloor \frac{n}{2} \rfloor \end{cases}$$

Therefore, $d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_{ns+k-n}) \neq d(a_{t-\lfloor \frac{n}{2} \rfloor+1}, b_{ns-k+1})$ for $1 \le s \le \lfloor \frac{n}{2} \rfloor$

and $1 \le k \le \lfloor \frac{n}{2} \rfloor$.

Hence, any two vertices in G are at unequal distances from at least one element of M. Thus $M = \{a, a_{\lfloor \frac{n}{2} \rfloor}, a_{t-\lfloor \frac{n}{2} \rfloor+1}\}$ is a metric basis for G and consequently $\beta(G) = 3$.

Case(ii): n is even.

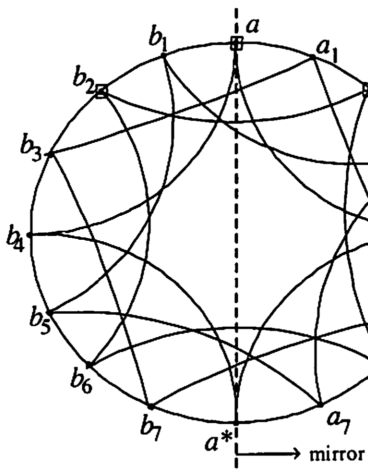

Figure 5: $G\{16; \pm\{1,4\}\}$ with $M = \{a, a_2, b_2\}$

For $a \in C$, let a^* denote the vertex in C such that $d(a, a^*)$ equals the diameter of C. Let $a, a_1, a_2 \ldots a_t, a^*, b, b_1, b_2 \ldots b_t$ be the vertices of G, where $t = \left\lfloor \frac{n^2-1}{2} \right\rfloor$. The mirror through a passes through a^* and it divides the set $V(G) \setminus \{a, a^*\}$ into two sets $S_1 = \{a_1, a_2 \ldots a_t\}$ and $S_2 = \{b_1, b_2 \ldots b_t\}$ so that the vertex a_i in S_1 becomes the mirror image of the vertex b_i in S_2, for $i = 1$ to t. See Figure 5.

As in Case(i), let $M = \{a\}$. The vertices $a_i, a_{jn+(i-j)}, a_{jn-(i-j)}, b_i, b_{jn+(i-j)}, b_{jn-(i-j)}$ for $1 \le i \le \frac{n}{2}$ and $1 \le j \le i$ (when $i = n/2$ and $j = n/2, a^* = a_{n^2/2} = b_{n^2/2}$), and $a_{jn-(j+(n-i-1))}, a_{jn-(i-j)}, b_{jn-(j+(n-i-1))}, b_{jn-(i-j)}$, for $\frac{n}{2} < i \le n-1$ and $(i+1) - \frac{n}{2} \le j \le \frac{n}{2}$ are at distance i from a. Hence it is necessary to augment M.

Let $M = \{a, a_{\frac{n}{2}}\}$. Then any two vertices in $S_1(S_2)$ have distinct M-coordinates. But, we find that the pairs $(a_{ns-\frac{n}{2}+k}, b_{ns-\frac{n}{2}-(k-1)})$ for $1 \le s < \frac{n}{2}$, $1 \le k \le \frac{n}{2}$; (a_{ns+k}, b_{ns-k}) for $1 \le s < \frac{n}{2}$, $1 \le k < \frac{n}{2}$; $(a_{ns-k}, b_{ns-\frac{n}{2}+(k-s+1)})$ for $s = \frac{n}{2}$, $1 \le k < \frac{n}{2}$; $(a^* (= a_{t+1}), b_{t-(n-2)})$ have the same M-coordinates.

Include $b_{\frac{n}{2}}$ in M. We now compute the distances of these pairs of vertices having same M-coordinates from $b_{\frac{n}{2}}$.

For $1 \le s < \frac{n}{2}$, $1 \le k \le \frac{n}{2}$,

$d(b_{\frac{n}{2}}, a_{ns-\frac{n}{2}+k}) = k + s$ and $d\left(b_{\frac{n}{2}}, b_{ns-\frac{n}{2}-(k-1)}\right) = k + s - 2$.

Thus $d\left(b_{\frac{n}{2}}, a_{ns-\frac{n}{2}+k}\right) \neq d\left(b_{\frac{n}{2}}, b_{ns-\frac{n}{2}-(k-1)}\right)$ for $1 \leq s < \frac{n}{2}$, $1 \leq k \leq \frac{n}{2}$.
For $1 \leq s < \frac{n}{2}$, $1 \leq k < \frac{n}{2}$,
$d\left(b_{\frac{n}{2}}, a_{ns+k}\right) = \frac{n}{2} - k + s + 1$ and $d\left(b_{\frac{n}{2}}, b_{ns-k}\right) = \frac{n}{2} - k + s - 1$.
This implies that $d\left(b_{\frac{n}{2}}, a_{ns+k}\right) \neq d\left(b_{\frac{n}{2}}, b_{ns-k}\right)$ for $1 \leq s < \frac{n}{2}, 1 \leq k < \frac{n}{2}$.
For $s = \frac{n}{2}$, $1 \leq k < \frac{n}{2}$,
$d\left(b_{\frac{n}{2}}, a_{ns-k}\right) = n - k$ and $d\left(b_{\frac{n}{2}}, b_{ns-\frac{n}{2}+(k-s+1)}\right) = n - k - 2$.
Hence $d\left(b_{\frac{n}{2}}, a_{ns-k}\right) \neq d\left(b_{\frac{n}{2}}, b_{ns-\frac{n}{2}+(k-s+1)}\right)$ for $s = \frac{n}{2}$, $1 \leq k < \frac{n}{2}$.
Similarly, $d\left(b_{\frac{n}{2}}, a^*\right) = n - 1$, $d\left(b_{\frac{n}{2}}, b_{t-(n-2)}\right) = n - 2$.
This implies that $d\left(b_{\frac{n}{2}}, a^*\right) \neq d\left(b_{\frac{n}{2}}, b_{t-(n-2)}\right)$. Thus $M = \left\{a, a_{\frac{n}{2}}, b_{\frac{n}{2}}\right\}$ is a minimum metric basis for G and consequently $\beta(G) = 3$. ■

Remark 1 *Theorem 2 is not true when $n = 3$. In fact, for $n = 3$, we have the following theorem.*

Theorem 3 $\beta(G\{9; \pm\{1,3\}\}) = 4$.

Proof. Let the vertices of the graph $G(9; \pm\{1,3\})$ be a, a_1, a_2, a_3, a_4, b_4, b_3, b_2, b_1 forming a Hamiltonian cycle C as shown in Figure 6.

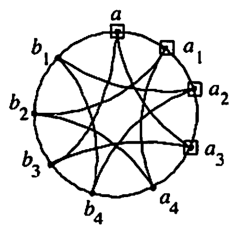

Figure 6: $G(9; \pm\{1,3\})$

We prove that no three vertices of the graph form a metric basis. Let M be any one of the 3-subsets: $\{a, a_1, a_2\}$, $\{a, a_1, a_3\}$, $\{a, a_1, a_4\}$, $\{a, a_1, b_4\}$, $\{a, a_2, a_4\}$, $\{a, a_2, b_4\}$ and $\{a, a_3, b_3\}$. Then $M = \{a, a_1, a_2\}$ is not a metric basis because a_3 and b_1 have the same M-coordinates. Similarly, the remaining 3-subsets are not metric bases, since the pairs a_2, a_4; b_1, b_3; a_2, a_4; a_1, a_3; a_1, a_3 again and b_2, b_4 are equidistant from each vertex of the corresponding 3-subset.

This implies that $\beta(G(9; \pm\{1,3\})) \neq 3$. As there are only two vertices with same M-coordinates for any chosen M, we include one more vertex in M. Without loss of generality, we take $M = \{a, a_1, a_2, a_3\}$. Hence $\beta(G) = 4$. ■

4 Conclusion

We have solved the minimum metric dimension problem for undirected Illiac networks $G(n^2; \pm\{1, n\})$. The problem has been considered already for $G(n; \{1, 2$

... $j\})$, $j = 2,3,4$ [1, 7]. Thus the problem remains open for $G(n; \{1,2 \dots j\})$, $j \geq 5$.

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