Degree Conditions for Graphs to Be Fractional k-Covered Graphs *†

Sizhong Zhou ‡
School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
People's Republic of China

Abstract

Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq \max\{10, 4k-3\}$, and $\delta(G) \geq k+1$. If G satisfies $\max\{d_G(x), d_G(y)\} \geq \frac{n+1}{2}$ for each pair of nonadjacent vertices x, y of G, then G is a fractional k-covered graph. The result is best possible in some sense, and it is an improvement and extension of C. Wang and C. Ji's result (C. Wang and C. Ji, Some new results on k-covered graphs, Mathematica Applicata 11(1)(1998), 61–64).

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1 Introduction

We consider only finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. Set $N_G[x] = N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from

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‡Corresponding author.E-mail address: zsz_cumt@163.com(S. Zhou)

G by deleting vertices in S together with the edges incident to vertices in S. If $G[S]$ has no edges, then S is called independent. Let S and T be disjoint subsets of $V(G)$. We denote the number of edges joining S and T by $e_G(S,T)$. We write $\delta(G)$ for the minimum degree of G.

Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) \le f(x)$ for each $x \in V(G)$. Then a spanning subgraph F of G is called a (g,f)-factor if $g(x) \le d_F(x) \le f(x)$ holds for each $x \in V(G)$. Let a and b be two integers with $0 \le a \le b$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f)-factor is an $[a,b]$-factor. An $[a,b]$-factor is called a k-factor if $a = b = k$. A graph G is called a k-covered graph if for any $e \in E(G)$ there exists a k-factor containing e. Let $h : E(G) \to [0,1]$ be a function. Let $k \ge 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, we call $G[F_h]$ a fractional k-factor of G with indicator functional h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional k-covered if for each edge e of G, there exists a fractional k-factor $G[F_h]$ such that $h(e) = 1$. If $k = 1$, then a fractional k-covered graph is called a fractional 1-covered graph. The other terminologies and notations can be found in [1].

Many authors have investigated factors of graphs [2–7]. Liu and Zhang [8] obtained a toughness condition for graphs to have fractional k-factors. Zhou [9,10] gave some results about fractional k-factors of graphs. Li, Yan and Zhang [11] showed an isolated toughness condition for graphs to be fractional k-covered graphs.

The following results on k-factors, fractional k-factors and fractional k-covered graphs are known.

Theorem 1 [2] *Let k be an integer such that $k \ge 3$, and let G be a 2-connected graph of order n with $n \ge 4k-3$, kn even, and $\delta(G) \ge k+1$. If G satisfies $\max\{d_G(x), d_G(y)\} \ge \frac{n+1}{2}$ for each pair of nonadjacent vertices x, y of G, then G is a k-covered graph.*

Theorem 2 [8] *Let $k \ge 2$ be an integer. A graph G with $|V(G)| \ge (k+1)$ has a fractional k-factor if $t(G) \ge k - \frac{1}{k}$.*

Theorem 3 [9] *Let k be an integer such that $k \ge 1$, and let G be a connected graph of order n such that $n \ge 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \ge k$. If $|N_G(x) \cup N_G(y)| \ge \max\{\frac{n}{2}, \frac{1}{2}(n+k-2)\}$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.*

Theorem 4 [11] *Let G be a graph, and let k be an integer with $k \ge 2$. If* *the minimum degree $\delta(G) \geq k+1$ and the isolated toughness $I(G) > k$, then G is a fractional k-covered graph.*

In this paper, we give a new sufficient condition for a graph to be a fractional k-covered graph. The main result is the following theorem, which is an improvement and extension of Theorem 1.

Theorem 5 *Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq \max\{10, 4k-3\}$, and $\delta(G) \geq k+1$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n+1}{2}$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional k-covered graph.

2 Proof of Theorem 5

For any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$, we define $\varepsilon(S,T)$ as follows,

(1) $\varepsilon(S,T) = 2$, if S is not independent.

(2) $\varepsilon(S,T) = 1$, if S is independent and $e_G(S, V(G) \setminus (S \cup T)) \geq 1$, or there exists an edge $e = uv$, such that $u \in S, v \in T$ and $d_{G-S}(v) = k$.

(3) $\varepsilon(S,T) = 0$, if neither (1) nor (2) holds.

Li, Yan and Zhang [12] obtained a necessary and sufficient condition for a graph to be a fractional k-covered graph, which is very useful in the proof of Theorem 5.

Lemma 2.1 [12] *A graph G is a fractional k-covered graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) \leq k\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

Proof of Theorem 5. Suppose that G satisfies the conditions of Theorem 5, but it is not a fractional k-covered graph. According to Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S,T) - 1, \qquad (1)$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le k\}$. In the following we consider three cases.

Case 1. $S = \emptyset$.

In this case, $\varepsilon(S,T) = 0$. In view of (1), we get

$$-1 \ge \delta_G(S,T) = d_G(T) - k|T| \ge (\delta(G) - k)|T| \ge |T| \ge 0,$$

a contradiction.

Case 2. $|S| = 1$.

In this case, $\varepsilon(S,T) \le 1$. By (1) we have

$$\begin{aligned} 0 &\ge \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\ge k|S| + d_G(T) - |T| - k|T| \\ &= k|S| + d_G(T) - (k+1)|T| \\ &\ge k|S| + (k+1)|T| - (k+1)|T| \\ &= k|S| = k \ge 3, \end{aligned}$$

this is a contradiction.

Case 3. $|S| \ge 2$.

In this case, $\varepsilon(S,T) \le 2$. We first prove the following claim.

Claim 1. $|T| \ge k+1$.

Proof. If $T = \emptyset$, then by (1) we have

$$\varepsilon(S,T) - 1 \ge \delta_G(S,T) = k|S| \ge |S| \ge \varepsilon(S,T),$$

which is a contradiction.

If $|T| = 1$, then from (1) we obtain

$$\begin{aligned} 1 &\ge \varepsilon(S,T) - 1 \ge \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\ge k|S| - k|T| \ge 2k - k = k \ge 3, \end{aligned}$$

it is a contradiction.

Hence, $|T| \ge 2$. In the following we assume that $|T| \le k$. Since $|T| \ge 2$, we have

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\ge |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T} (|S| + d_{G-S}(x) - k) \end{aligned}$$

$$\begin{aligned} &\geq \sum_{x \in T}(\delta(G) - k) \\ &= |T| \geq 2 \geq \varepsilon(S,T), \end{aligned}$$

which contradicts (1). This completes the proof of Claim 1.

According to Claim 1, $T \neq \emptyset$. Define

$$h_1 = \min\{d_{G-S}(x) : x \in T\}.$$

Choose $x_1 \in T$ such that $d_{G-S}(x_1) = h_1$. Furthermore, if $T \setminus N_T[x_1] \neq \emptyset$, we define

$$h_2 = \min\{d_{G-S}(x) : x \in T \setminus N_T[x_1]\}.$$

Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Thus, we have $0 \leq h_1 \leq h_2 \leq k$ by the definition of T.

Subcase 3.1. $T = N_T[x_1]$.

From Claim 1 and $T = N_T[x_1]$, we obtain $k \geq h_1 = d_{G-S}(x_1) \geq |T| - 1 \geq k$. Therefore, $h_1 = k$. According to the definition of h_1, we have

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| \\ &= k|S| + k|T| - k|T| \\ &= k|S| \geq |S| \geq \varepsilon(S,T). \end{aligned}$$

That contradicts (1).

Subcase 3.2. $T \setminus N_T[x_1] \neq \emptyset$.

It is easy to verify that

$$|S| \geq \frac{n+1}{2} - h_2. \tag{2}$$

Otherwise, $|S| < \frac{n+1}{2} - h_2$. That is, $|S| + h_2 < \frac{n+1}{2}$, then $d_G(x_2) \leq |S| + h_2 < \frac{n+1}{2}$ and $d_G(x_1) \leq |S| + h_1 \leq |S| + h_2 < \frac{n+1}{2}$. Since $x_1x_2 \notin E(G)$, that would contradict the hypothesis of Theorem 5.

Subcase 3.2.1. $h_2 = 0$.

Clearly, $h_1 = 0$. By (1), (2) and $|S| + |T| \leq n$, we obtain

$$\begin{aligned} \varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| - k|T| \geq k|S| - k(n - |S|) \\ &= 2k|S| - kn \geq k(n+1) - kn \\ &= k > 2 \geq \varepsilon(S,T). \end{aligned}$$

This is a contradiction.

Subcase 3.2.2. $h_2 \geq 1$.

According to (2), $|S|+|T| \leq n$, $h_1 \leq h_2 \leq k$ and $|N_T[x_1]| \leq d_{G-S}(x_1)+1 = h_1+1$, we get

$$\begin{aligned}
\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\
&\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - k|T| \\
&= k|S| - (h_2 - h_1)|N_T[x_1]| - (k - h_2)|T| \\
&\geq k|S| - (h_2 - h_1)(h_1 + 1) - (k - h_2)(n - |S|) \\
&= (2k - h_2)|S| - (h_2 - h_1)(h_1 + 1) - (k - h_2)n \\
&\geq (2k - h_2)(\frac{n+1}{2} - h_2) - (h_2 - h_1)(h_1 + 1) - (k - h_2)n \\
&= h_2^2 + (\frac{n}{2} - 2k - \frac{3}{2})h_2 - h_1h_2 + h_1^2 + h_1 + k \\
&= (\frac{h_2 - 1}{2} - h_1)^2 + \frac{3}{4}h_2^2 + (\frac{n}{2} - 2k - 1)h_2 + k - \frac{1}{4} \\
&\geq \frac{3}{4}h_2^2 + (\frac{n}{2} - 2k - 1)h_2 + k - \frac{1}{4},
\end{aligned}$$

that is,

$$\delta_G(S,T) \geq \frac{3}{4}h_2^2 + (\frac{n}{2} - 2k - 1)h_2 + k - \frac{1}{4}. \tag{3}$$

If $k = 3$, then $n \geq 10$. Hence, we have by (3)

$$\delta_G(S,T) \geq \frac{3}{4}h_2^2 - 2h_2 + 3 - \frac{1}{4} > 1.$$

In view of the integrity of $\delta_G(S,T)$, we obtain

$$\delta_G(S,T) \geq 2 \geq \varepsilon(S,T).$$

This contradicts (1).

If $k \geq 4$, then $n \geq 4k - 3$. Therefore, from (3) we get

$$\delta_G(S,T) \geq \frac{3}{4}h_2^2 + (\frac{4k-3}{2} - 2k - 1)h_2 + k - \frac{1}{4} \geq \frac{3}{4}h_2^2 - \frac{5}{2}h_2 + 4 - \frac{1}{4} > 1.$$

According to the integrity of $\delta_G(S,T)$, we have

$$\delta_G(S,T) \geq 2 \geq \varepsilon(S,T).$$

Which contradicts (1). This completes the proof of Theorem 5.

Remark. Let us show that the condition $\max\{d_G(x), d_G(y)\} \geq \frac{n+1}{2}$ in Theorem 5 cannot be replaced by $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$. Let $t \geq 2$ and $k \geq 3$ be two integers. We construct a graph $G = ((kt-2)K_1 \cup K_2) \vee ktK_1$. Clearly, $\delta(G) = kt \geq 2k > k+1$, $n = |V(G)| = 2kt \geq 4k > 4k-3$ and

$$\max\{d_G(x), d_G(y)\} = \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of $((kt-2)K_1 \cup ktK_1) \subset G$. Let $S = V((kt-2)K_1 \cup K_2)) \subseteq V(G)$ and $T = V(ktK_1) \subseteq V(G)$. Then $|S| = kt$, $|T| = kt$ and S is not independent. Thus, we get $\varepsilon(S,T) = 2$ and

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= k^2t - k^2t = 0 < 2 = \varepsilon(S,T). \end{aligned}$$

According to Lemma 2.1, G is not a fractional k-covered graph. In the above sense, the result in Theorem 5 is best possible.

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