On the normalized Laplacian eigenvalues of graphs

Kinkar Ch. Das[a], A. Dilek Güngör[b,†]
and
S. Burcu Bozkurt[b,‡]

[a] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea.*
e-mail: kinkar@lycos.com

[b] *Selçuk Üniversity, Science Faculty, Department of Mathematics, 42031 Konya, Turkey.*
†e-mail: drdilekgungor@gmail.com
‡e-mail: srfburcu@gmail.com

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Abstract

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges. Further let $\lambda_i(L)$, $i = 1, 2, \ldots, n$, be the non-increasing eigenvalues of the normalized Laplacian matrix of the graph G. In this paper, we obtain the following result: For a connected graph G of order n, $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L)$ if and only if G is a complete graph K_n or G is a complete bipartite graph $K_{p,q}$. Moreover, we present lower and upper bounds for the normalized Laplacian spectral radius of a graph and characterize graphs for which the lower or upper bounds is attained.

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1 Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v_i \in V(G)$, the degree of v_i, denoted by d_i, is the number of vertices adjacent to v_i. The average of the degrees of the vertices adjacent to $v_i \in V(G)$ is denoted by m_i. We will use the notation $i \sim j$ to denote v_i and v_j are adjacent vertices. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of graph G and $A(G)$ is the $(0,1)-$ adjacency matrix of graph G.

The normalized Laplacian matrix of G is defined as $L = D(G)^{-\frac{1}{2}}L(G)D(G)^{-}$ where $D(G)^{-\frac{1}{2}}$ is the matrix which is obtained by getting $\left(-\frac{1}{2}\right) -$ power of each entry of $D(G)$. Throughout this paper let $\lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_n(L) = 0$ be the eigenvalues of L and call $\lambda_1(L)$ the normalized Laplacian spectral radius of G.

The normalized Laplacian has gotten increased attention in the last decade due to its connection with random walks. Chen et al. [2] and Li [5] established the interlacing results for the normalized Laplacian. S. Butler [1] gave an improved version of interlacing results for the normalized Laplacian. The monograph [3] provides a comprehensive survey of results on normalized Laplacian eigenvalues.

Now we shall see the normalized Laplacian spectrum of some special graphs. For graph G, $S(G)$ denotes the set of the normalized Laplacian eigenvalues. We have

$$S(K_{p,q}) = \Big\{2, \underbrace{1, 1, \ldots, 1}_{n-2}, 0\Big\} \; (p+q=n)\,,$$

and $$S(K_n) = \Big\{\underbrace{\frac{n}{n-1}, \frac{n}{n-1}, \ldots, \frac{n}{n-1}}_{n-1}, 0\Big\}\,,$$

are the normalized Laplacian spectrum for complete bipartite graph $K_{p,q}$, and complete graph K_n, respectively.

The rest of the paper is structured as follows. In Section 2 we give some known results which are used in the Section 3. In Section 3, we obtain: For a connected graph G of order n, $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}$ if and only if G is a complete graph K_n or G is a complete bipartite graph $K_{p,q}$. Also we give lower and upper bounds for the normalized Laplacian spectral radius of graphs and characterize those graphs for which the bounds are best possible.

2 Lemmas and results

In this section we give some known results.

Lemma 2.1. *[7] Let G be a graph with n vertices and normalized Laplacian matrix L without isolated vertices. Then*

$$\sum_{i=1}^{n} \lambda_i(L) = tr(L) = n\,, \tag{1}$$

$$\sum_{i=1}^{n} \lambda_i^2(L) = tr(L^2) = n + e_{-1}(G,G)\,, \tag{2}$$

and

$$\sum_{i=1}^{n}(1-\lambda_i(L))^2 = tr((I-L)^2) = e_{-1}(G,G)\ , \tag{3}$$

where

$$e_{-1}(G,G) = \sum_{i=1}^{n}\sum_{j:j\sim i}\frac{1}{d_i d_j} = 2\sum_{i\sim j}\frac{1}{d_i d_j}\,.$$

Proof: The (i, j)th entry of $L = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}$ is

$$\begin{cases} 1 & \text{if } i=j\,, \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } i\sim j\,, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have $\sum_{i=1}^{n}\lambda_i(L) = tr(L) = n$. Since $\lambda_i^2(L)$, $i=1,2,\ldots,n$, are the eigenvalues of L^2, we have

$$\sum_{i=1}^{n}\lambda_i^2(L) = tr(L^2) = n + \sum_{i=1}^{n}\sum_{j:j\sim i}\frac{1}{d_i d_j} = n + e_{-1}(G,G)\,.$$

Now,

$$\begin{aligned} \sum_{i=1}^{n}(1-\lambda_i(L))^2 &= n - 2\sum_{i=1}^{n}\lambda_i(L) + \sum_{i=1}^{n}\lambda_i^2(L) \\ &= e_{-1}(G,G) \qquad \text{by (1) and (2)} \\ &= tr((I-L)^2)\,. \end{aligned}$$

Hence the proof of the lemma is complete. □

Lemma 2.2. *[3] Let G be a graph with n vertices and normalized Laplacian eigenvalues* $\lambda_1(L)\geq\lambda_2(L)\geq\cdots\geq\lambda_n(L)=0$. *Then*

$$0\leq\lambda_i(L)\leq 2\ .$$

Moreover, $\lambda_1(L) = 2$ *if and only if a connected component of* G *is bipartite and nontrivial.*

Lemma 2.3. *[3] (i): For a graph* G *of order* n*, we have*

$$\sum_{i=1}^{n} \lambda_i(L) \leq n\,,$$

with equality holding if and only if G *has no isolated vertices.*

(ii) : For a graph which is not a complete graph, we have

$$\lambda_{n-1} \leq 1\ .$$

Lemma 2.4. *[6] Let* A *be an* $n \times n$ *complex matrix, and suppose that its eigenvalues are all real and ordered:*

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1\,.$$

Then

$$\frac{\sum_{i=1}^{n} \lambda_i}{n} + \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} \left(\lambda_i - \frac{\sum_{i=1}^{n} \lambda_i}{n}\right)^2} \leq \lambda_1$$

with equality holding if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$*.*

Lemma 2.5. *[3] Let* G *be a graph with* n *vertices and normalized Laplacian eigenvalues* $\lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_n(L) = 0$*. Then*

$$\lambda_1(L) \geq \frac{n}{n-1}\,. \tag{4}$$

Lemma 2.6. *[4] Let* A *be a* $p \times p$ *symmetric matrix and let* A_k *be its leading* $k \times k$ *submatrix; that is,* A_k *is the matrix obtained from* A *by deleting its* *last $p-k$ rows and columns. Then, for $i=1,2,\ldots,k$,*

$$\lambda_{p-i+1}(A) \leq \lambda_{k-i+1}(A_k) \leq \lambda_{k-i+1}(A), \tag{5}$$

where $\lambda_i(A)$ is the i-th largest eigenvalue of A.

3 Bound on the spectral radius of the normalized Laplacian matrix of a graph

In this section we give a lower bound on the spectral radius of the normalized Laplacian matrix of graphs. First we give the following lemma:

Lemma 3.1. *Let G be a graph of order n without isolated vertices. Then $\lambda_1(L) = \lambda_2(L) = \cdots = \lambda_{n-1}(L)$ if and only if G is a complete graph K_n.*

Proof: If G is a complete graph then $\lambda_1(L) = \lambda_2(L) = \cdots = \lambda_{n-1}(L) = \frac{n}{n-1}$ holds.

Conversely, let $\lambda_1(L) = \lambda_2(L) = \cdots = \lambda_{n-1}(L)$ holds. If G is not a complete graph then by Lemma 2.3, $\lambda_{n-1}(L) \leq 1$ and hence $\sum_{i=1}^{n} \lambda_i(L) \leq n-1$, a contradiction, by Lemma 2.1. Hence we get the result. □

Theorem 3.2. *Let G be a graph with n vertices. Then*

$$\lambda_1(L) \geq 1 + \sqrt{\frac{1}{n(n-1)} e_{-1}(G,G)}, \tag{6}$$

where

$$e_{-1}(G,G) = \sum_{i=1}^{n} \sum_{j:j\sim i} \frac{1}{d_i d_j} = 2\sum_{i\sim j} \frac{1}{d_i d_j}\ .$$

Moreover, the equality holds in (6) if and only if G is a complete graph K_n.

Proof: Since normalized Laplacian matrix of G is a real symmetric matrix, its eigenvalues are real numbers and ordered as $\lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_n(L) = 0$. By Lemma 2.1 and Lemma 2.4, we get the required result (6). Moreover, the equality holds in (6) if and only if $\lambda_1(L) = \lambda_2(L) = \cdots = \lambda_{n-1}(L)$, by Lemma 2.4, that is, if and only if G is a complete graph K_n, by Lemma 3.1. □

Remark 3.3. *The bound (6) is always better than the bound (4). Because*

$$\begin{aligned} e_{-1}(G,G) &= \sum_{i=1}^{n} \sum_{j:j\sim i} \frac{1}{d_i d_j} \\ &\geq \sum_{i=1}^{n} \frac{d_i}{d_i m_i} \quad \textit{by the Arithmetic-Harmonic mean inequality} \\ &\geq \frac{n}{n-1} \quad \textit{by } m_i \leq n-1 \,. \end{aligned}$$

Using Lemma 3.1, we characterize the graphs in the following theorem.

Theorem 3.4. *Let G be a connected graph of order $n > 2$. Then $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L)$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

Proof: If G is a complete graph K_n, then $\lambda_1(L) = \lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L) = \frac{n}{n-1}$ holds. If G is a complete bipartite graph $K_{p,q}$ $(p+q=n)$, then $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L) = 1$ holds.

Conversely, let $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L)$. We have to prove that $G \cong K_{p,q}$ or $G \cong K_n$. Now we consider two cases (i) $\lambda_1(L) = \lambda_2(L)$, (ii) $\lambda_1(L) \neq \lambda_2(L)$.

Case (i): $\lambda_1(L) = \lambda_2(L)$. In this case we have $\lambda_1(L) = \lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L) = \frac{n}{n-1}$. Then by Lemma 3.1, $G \cong K_n$.

Case (ii): $\lambda_1(L) \neq \lambda_2(L)$. For $n \leq 4$, one can easily see that G is isomorphic to $K_{1,2}$ or $K_{2,2}$. Otherwise, $n \geq 5$. In this case we have $G \ncong K_n$. By Lemma 2.2, we have $\lambda_1(L) \leq 2$. If $\lambda_1(L) < 2$, then

$$\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L) > 1 .$$

Since $G \ncong K_n$, by Lemma 2.3, $\lambda_{n-1}(L) \leq 1$, a contradiction. So we must have $\lambda_1(L) = 2$ and hence G is a bipartite graph, by Lemma 2.2. Now we can assume that $V(G) = A \cup B$ and $A \cap B = \Phi$, $|A| = p$, $|B| = q$. By Lemma 2.1, and using given condition, we can conclude that 1 is an eigenvalue of multiplicity $n-2$ of L, that is,

$$\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L) = 1.$$

Now suppose that G is not isomorphic to complete bipartite graph $K_{p,q}$. So there exists at least one pair of vertices $(v_j,\, v_l)$ such that $v_j \in A$ and $v_l \in B$ are not adjacent. Let vertex v_i be adjacent to v_l and also let vertex v_k be adjacent to v_j as G is connected and $n \geq 5$. So we must have $v_i \in A$ and $v_k \in B$ as G is bipartite. Now we consider two subcases (a) $v_iv_k \notin E$, (b) $v_iv_k \in E$.

Subcase (a) : $v_iv_k \notin E$. Let L' be the 4×4 submatrix of L obtained by deleting all the rows and columns except i, j, k and l from L. By Lemma 2.6, we have $\lambda_1(L) \geq \lambda_1'$ and $\lambda_2(L) \geq \lambda_2'$, where λ_1' and λ_2' are the largest and the second largest eigenvalues of $|L' - \lambda I| = 0$, that is,

$$\begin{vmatrix} 1-\lambda & 0 & 0 & -x \\ 0 & 1-\lambda & -y & 0 \\ 0 & -y & 1-\lambda & 0 \\ -x & 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

where $x = \frac{1}{\sqrt{d_i d_l}} > 0$ and $y = \frac{1}{\sqrt{d_j d_k}} > 0$. Now $|L' - \lambda I| = 0$ implies $\lambda = 1 + x,\ 1 + y,\ 1 - x,\ 1 - y$. So, $\lambda_1' > 1$ and $\lambda_2' > 1$, that is, $\lambda_1(L) > 1$ and $\lambda_2(L) > 1$.

Subcase (b) : $v_i v_k \in E$. Let L'' be the 4×4 submatrix of L obtained by deleting all the rows and columns except i, j, k and l from L. By Lemma 2.6, we have $\lambda_1(L) \geq \lambda_1''$ and $\lambda_2(L) \geq \lambda_2''$, where λ_1'' and λ_2'' are the largest and the second largest eigenvalues of $|L'' - \lambda I| = 0$, that is,

$$\begin{vmatrix} 1-\lambda & 0 & -x & -y \\ 0 & 1-\lambda & -z & 0 \\ -x & -z & 1-\lambda & 0 \\ -y & 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

where $x = \frac{1}{\sqrt{d_i d_k}} > 0$, $y = \frac{1}{\sqrt{d_i d_l}} > 0$ and $z = \frac{1}{\sqrt{d_j d_k}} > 0$. Now $|L'' - \lambda I| = 0$ implies

$$\lambda = 1 \pm \sqrt{\frac{1}{2}\left(x^2 + y^2 + z^2 \pm \sqrt{(x^2 + y^2 + z^2)^2 - 4y^2 z^2}\right)}.$$

So, $\lambda_1'' > 1$ and $\lambda_2'' > 1$, that is, $\lambda_1(L) > 1$ and $\lambda_2(L) > 1$.

From Subcase (a) and Subcase (b), we have $\lambda_2(L) > 1$, a contradiction. Hence $G \cong K_{p,q}$. □

Now we are ready to give an upper bound on the spectral radius of the normalized Laplacian matrix of a graph.

Theorem 3.5. *Let G be a connected graph of order n with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\lambda_1(L) \leq \sqrt{2 + e_{-1}(G,G)}\,, \tag{7}$$

where

$$e_{-1}(G,G) = \sum_{i=1}^{n} \sum_{j:j\sim i} \frac{1}{d_i d_j} = 2\sum_{i\sim j} \frac{1}{d_i d_j}\,.$$

Moreover, the equality holds in (7) if and only if $G \cong K_{p,q}$.

Proof: By Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=2}^{n-1} \lambda_i^2(L) &\geq \frac{1}{n-2}\left(\sum_{i=2}^{n-1} \lambda_i(L)\right)^2 & (8)\\ &= \frac{1}{n-2}\left(n - \lambda_1(L)\right)^2 \quad \text{by Lemma 2.1} & \\ &\geq n-2 \quad \text{by Lemma 2.2.} & (9) \end{aligned}$$

Since $\sum_{i=1}^{n} \lambda_i^2(L) = n + e_{-1}(G,G)$ and $\lambda_n(L) = 0$, we have

$$n + e_{-1}(G,G) - \lambda_1^2(L) \geq n-2\,.$$

From this we get the inequality in (7).

Now suppose that equality holds in (7). Then the equality holds in (8) and (9). From equality in (8), we get

$$\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_{n-1}(L)\,.$$

From equality in (9), we get

$$\lambda_1(L) = 2\,.$$

By Lemma 2.2, G is a connected bipartite graph. By Theorem 3.4, we conclude that $G \cong K_{p,q}$.

Conversely, one can easily see that the equality holds in (7) for complete bipartite graph $K_{p,q}$. $\square$

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