On Strongly Multiplicative Graphs

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Abstract

We give a characterization of strongly multiplicative graphs. First, we introduce some necessary conditions for a graph to be strongly multiplicative. Second, we discuss the independence of these necessary conditions. Third, we show that they are altogether not sufficient for a graph to be strongly multiplicative. Fourth, we add another necessary condition. Fifth, we prove that this necessary condition is stronger than the mentioned necessary conditions except one. Finally, we conjecture that the condition itself is stronger than all of them.

1 Introduction

Beineke and Hegde [3] call a graph G with n vertices strongly multiplicative if its vertices can be labeled with distinct integers $1, 2, ..., n$ such that, the labels induced on the edges by the product of their end-vertex labels are all distinct. They prove the following graphs are strongly multiplicative: trees, cycles, wheels, K_n if and only if $n \leq 5$, $K_{r,r}$ if and only if $r \leq 4$ and $P_n \times P_m$. They then consider the maximum number of edges a strongly multiplicative graph of n vertices can have. Denoting this number by $\lambda(n)$, they show: $\lambda(4r) \leq 6r^2, \lambda(4r+1) \leq 6r^2+4r, \lambda(4r+2) \leq 6r^2+6r+1$ and $\lambda(4r+3) < 6r^2+10r+3$. Adiga, Ramashekar, and Somashekara [2] give the bound $\lambda(n) \leq \frac{n(n+1)}{2} + n - 2 - \sum_{i=2}^{n} \frac{i}{p(i)}$, where $p(i)$ is the smallest prime dividing i. For large values of n this is a better upper bound for $\lambda(n)$ than the one given by Beineke and Hegde. It remained an open problem to find a nontrivial lower bound for $\lambda(n)$ until R. Kafshgarzaferani succeeded in finding a formula for $\lambda(n)$ in [9].

Beineke and Seoud [4] study the strong multiplicativity for small powers of paths and cycles.

Seoud and Zid [10] prove the following graphs are strongly multiplicative: wheels, rK_n for all r and n at most 5, rK_n for $r \geq 2$ and $n = 6$ or 7, rK_n for $r \geq 3$ and $n = 8$ or 9, $K_{4,r}$ for all r, and the corona of P_n and K_m for all n and $2 \leq m \leq 8$.

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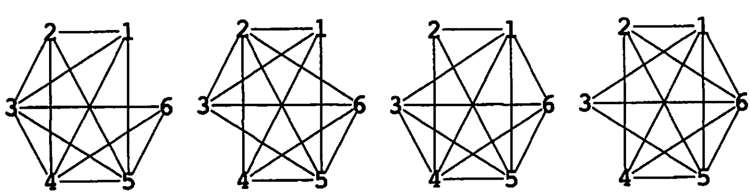
Acharya, Germina, and Ajitha [1] prove that $K_2+\overline{K_t}$, quadrilateral snakes, Petersen graphs, ladders and unicyclic graphs are strongly multiplicative. They define a graph with q edges and a strongly multiplicative labeling to be hyper strongly multiplicative if the induced edge labels are given by the set $\{2, 3, ..., q + 1\}$.
They show that every hyper strongly multiplicative graph has exactly one nontrivial component that is either a star or has a triangle and every graph can be embedded as an induced subgraph of a hyper strongly multiplicative graph.
Throughout this paper, we use the basic notations and conventions in graph theory as in [7], and in number theory as in [8], [6] and [11]. We use $| A |$ to denote the size of the set A, i.e., its number of elements, $\prod_{i=1}^{i=n} A_i$ to denote the cartesian product of the sets $A_1, A_2, ..., A_n$, and $A - B$ to denote the usual difference between the sets A and B. All graphs here are simple, i.e., containing no loops or multiple edges.

2 Some necessary conditions

2.1 Properties of strongly multiplicative graphs

Definition 2.1.1. [4] A simple graph G with n vertices is said to be strongly multiplicative if its vertices can be labeled with distinct integers $1, 2, ..., n$ such that the labels induced on the edges by the product of the end vertices are distinct. A graph which is not strongly multiplicative is said to be non-strongly multiplicative.
Definition 2.1.2. A maximal strongly multiplicative graph of n vertices is a strongly multiplicative graph such that adding any new edge yields a non-strongly multiplicative graph.
Remark The maximal strongly multiplicative graph is not unique (i.e., there are many non-isomorphic maximal strongly multiplicative graphs of the same number of vertices). So, we denote them by $R^k(n)$, where the first graph is denoted by $R^1(n)$, and so on ... .
Examples 2.1.1. The following graphs are the maximal strongly multiplicative graphs of 6 vertices.



Considering the four previous graphs without labeling, we see that the they are all isomorphic except the second graph.

Definition 2.1.3. [8] A positive integer $n \neq 1$ is said to be prime if it has no divisors other than 1 and n, while it is said to be composite if it is not prime.

Definition 2.1.4. [8] Let x be a non-negative real number. The Gauss's function $\pi(x)$ is defined to be the number of primes not exceeding x.

Theorem 2.1.1.(Condition 1) [9] The maximal strongly multiplicative graph of n vertices has $\lambda(n) = \frac{n(n-1)}{2} - \sum_{m=2}^{n} \sum_{k=1}^{m-1} \lfloor -\frac{\theta(m,k)}{\lfloor \sqrt{mk-1} \rfloor} - k + 1 \rfloor$, where $\theta(m,k) = \sum_{s=k+1}^{\lfloor \sqrt{mk-1} \rfloor} \lfloor \frac{\lfloor \frac{mk}{s} \rfloor}{\frac{mk}{s}} \rfloor$.

Theorem 2.1.2.(Condition 2) If G is a strongly multiplicative graphs of order n, then $\omega(G) \leq 1 + \sum_{i=1}^{\pi(n)} \lfloor \log_{p_i}(n) \rfloor$, where p_i is the ith prime number, and $\omega(G)$ is the clique number of G, namely the maximum order of a complete subgraph of G". .

Proof. The vertices whose labels are $1, 2, 2^2, ..., 2^{\lfloor \log_2(n) \rfloor}, 3, 3^2, ..., 3^{\lfloor \log_3(n) \rfloor}$, $5, 5^2, ..., 5^{\lfloor \log_5(n) \rfloor}, ..., p_{\pi(n)}, p_{\pi(n)}^2, ..., p_{\pi(n)}^{\lfloor \log_{p_{\pi(n)}}(n) \rfloor}$ can be adjacent in any strongly multiplicative graph of n vertices, since the product of any two of them gives different value from that of any two others (from the fundamental theorem of arithmetic), then they form a complete subgraph of a maximal strongly multiplicative graph. Now, we see that adding any new vertex whose label is s to these vertices yields omitting all vertices whose label is p_i satisfying that p_i divides s. So, $\omega(G) \leq 1 + \sum_{i=1}^{\pi(n)} \lfloor \log_{p_i}(n) \rfloor$, where p_i is the i^{th} prime number.

Condition 3

If the minimum degree of a graph G of n vertices is greater than the largest minimum degree in all corresponding maximal strongly multiplicative graphs $\delta(n)$, then the graph is non-strongly multiplicative.

Condition 4

If G is a graph of n vertices, which has more than $t(n)$ vertices of degree $n-1$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal strongly multiplicative graphs of n vertices, then G is non-strongly multiplicative.

Lemma 2.1.3. [6]

If p is a prime number, then $np \neq ij$ for all $n, i, j < p$.

Lemma 2.1.4.

If p is a prime number, then $t(p) = t(p-1) + 1$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal strongly multiplicative graphs of n vertices, and $\delta(p) = \delta(p-1) + 1$, where $\delta(n)$ is the largest minimum degree in all maximal strongly multiplicative graphs of n vertices.

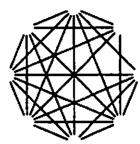
Proof. From the previous lemma, the vertex having the label p in a maximal strongly multiplicative graph of p vertices is adjacent with all the other vertices, hence $t(p) = t(p-1) + 1$, and $\delta(p) = \delta(p-1) + 1$.

2.2 Independence of some necessary conditions

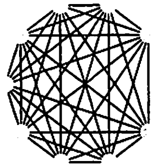
Now, we show that the four necessary conditions for strongly multiplicative graphs are independent, and they are altogether not sufficient for non-strong multiplicativity of a graph by using the following table.
Table 2.2.1.

n	$\lambda(n)$	$2+\sum_{i=1}^{\pi(n)}\lfloor \log_{p_i}(n)\rfloor$	$\delta(n)$	$t(n)$
2	1	3	1	2
3	3	4	2	3
4	6	5	3	4
5	10	6	4	5
6	13	6	4	3
7	19	7	5	4
8	24	8	5	4
9	31	9	6	5
10	36	9	6	4

Example 2.2.1. Only Condition 1 which proves that the following graph is non-strongly multiplicative.

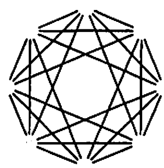


($n = 8$) For Condition 2 : $K_8 = K_{2+\sum_{i=1}^{\pi(8)}\lfloor \log_{p_i(8)}\rfloor} \nsubseteq G$. For Condition 3 : the minimum degree of the graph equals $5 = \delta(8)$. For Condition 4 : the number of vertices of degree $7 = 4 = t(8)$. But for Condition 1 : the number of edges of G $m = 25 > 24 = \lambda(8)$.
Example 2.2.2. Only Condition 2 which proves that the following graph is non-strongly multiplicative.



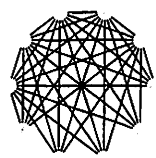
($n = 10$) For Condition 1 : the number of edges of G is $m = 36 = \lambda(10)$. For Condition 3 : the minimum degree of the graph equals $0 < 6 = \delta(10)$. For Condition 4 : the number of vertices of degree $9 = 0 < 4 = t(10)$. But For Condition 2 : $K_9 = K_{2+\sum_{i=1}^{\pi(10)} \lfloor \log_{p_i}(10) \rfloor} \subset G$.

Example 2.2.3. Only Condition 3 which proves that the following graph is non-strongly multiplicative.



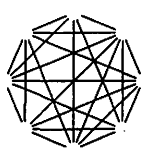
($n = 8$) For Condition 1 : the number of edges of G is $m = 24 = \lambda(8)$. For Condition 2 : $K_8 = K_{2+\sum_{i=1}^{\pi(8)} \lfloor \log_{p_i}(8) \rfloor} \nsubseteq G$. For Condition 4 : the number of vertices of degree $7 = 0 < 4 = t(8)$. But For Condition 3 : the minimum degree of the graph equals $6 > 5 = \delta(8)$.

Example 2.2.4. Only Condition 4 which proves that the following graph is non-strongly multiplicative.

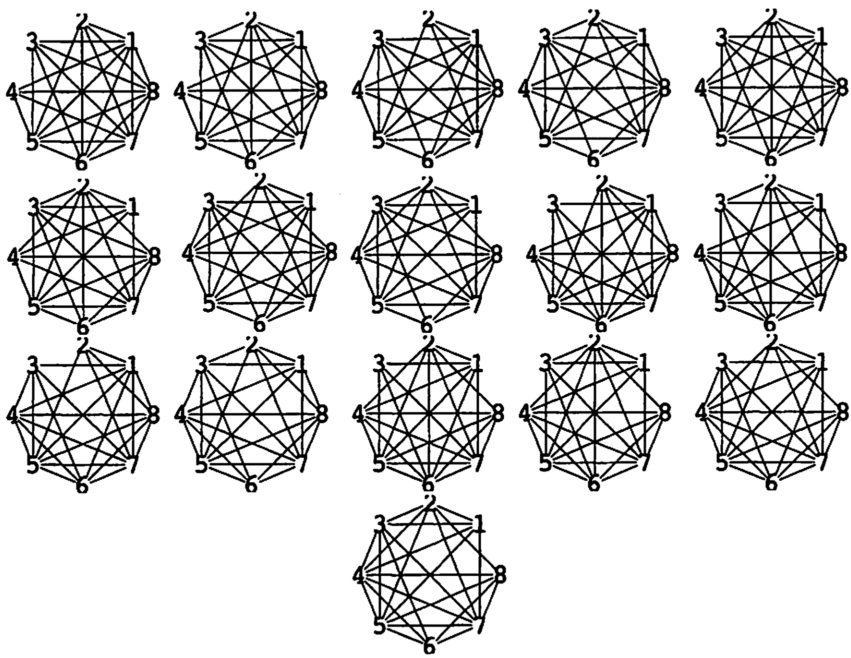


($n = 10$, $G = K_5 + \overline{K_5}$) For Condition 1 : the number of edges of G is $m = 35 < 36 = \lambda(10)$. For Condition 2 : $K_9 = K_{2+\sum_{i=1}^{\pi(10)} \lfloor \log_{p_i}(10) \rfloor} \nsubseteq G$. For Condition 3 : the minimum degree of the graph equals $5 < 6 = \delta(10)$. But For Condition 4 : the number of vertices of degree $9 = 5 > 4 = t(10)$.

Example 2.2.5. Here we give an example of a non-strongly multiplicative graph, but the four conditions fail to decide that it is non-strongly multiplicative, i.e., they are altogether not sufficient for non-strong multiplicativity of a graph.



($n = 8$) For Condition 1 : the number of edges of G is $m = 24 = \lambda(8)$. For Condition 2 : $K_8 = K_{2+\sum_{i=1}^{\pi(8)} \lfloor \log_{p_i}(8) \rfloor} \nsubseteq G$. For Condition 3 : the minimum degree of the graph equals $5 = \delta(8)$. For Condition 4 : the number of vertices of degree $7 = 1 < 4 = t(8)$. It remains to show that the graph G is non-strongly multiplicative.
The following graphs are all the maximal strongly multiplicative graphs of 8 vertices.



From the previous graphs we notice that since the minimum degree in some of them equals 4, it follows that our graph, which has minimum degree equals 5, is not a subgraph of those graphs. Also, in the remaining graphs, the number of vertices of degree at most 6 is less than 8, since the number of vertices of degree at most 6 in our graph equals 8. It follows that our graph is not a subgraph of any maximal strongly multiplicative graph of 8 vertices. Hence it is non-strongly multiplicative.

2.3 Other necessary conditions

Definition 2.3.1. [11] If n is a positive integer, the function $\tau(n)$ called the tau function, is defined to be the number of positive divisors of n, i.e.,

$\tau(n) :=\mid \{d \in \mathbb{N} : d \mid n\} \mid$
The tau function satisfies the following properties:
Theorem 2.3.1. [11] If $(m,n) = 1$, then $\tau(mn) = \tau(m)\tau(n)$, i.e., τ is multiplicative.
Theorem 2.3.2. [11] If p is a prime number, then
$\tau(p^{\alpha}) = \alpha + 1$
Corollary 2.3.3. [11] If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, where the p's are distinct prime numbers, then

$$\tau(n) = \prod_{i=1}^{k}(\alpha_i + 1)$$

Lemma 2.3.4. [6] If n has no divisors less than or equal to $\sqrt{n}$, then n is a prime number.
Definition 2.3.2. For Any two different positive integers $1 \le i, j \le n$, we define the set $V_{i*j}^{n} := \{(k,m) : i * j = k * m, \ and \ 1 \le k < m \le n\}$, ($i * j$ means the product of ij).
For example, if $n = 6$, $V_{1*6}^{6} = V_{2*3}^{6} = V_{6}^{6} = \{(1,6),(2,3)\}$, and $V_{2*6}^{6} = V_{3*4}^{6} = V_{12}^{6} = \{(2,6),(3,4)\}$.
The following lemma gives a formula to the order of the set V_{i*j}^{n}.
Lemma 2.3.5.

$$\mid V_{i*j}^{n} \mid = \begin{cases} \lfloor \frac{\tau(i*j)}{2} \rfloor & , if \ n \ge i * j \\ \\ \lfloor \frac{\tau(i*j)}{2} \rfloor - \sum_{d \mid i*j} \alpha(d) & , if \ n < i * j \end{cases}$$

where $\alpha(x) = \begin{cases} 1 & , if \ x > n \\ \\ 0 & , if \ x \le n \end{cases}$

Proof. Since the set $V_{i*j}^{n} := \{(k,m) : i * j = k * m, \ and \ 1 \le k < m \le n\}$ and in the first case we have $n \ge i * j$, it follows that $\mid V_{i*j}^{n} \mid$ is equal to the number of all distinct pairs of divisors of $i * j$ except the repeated pair $(\sqrt{i*j}, \sqrt{i*j})$ in case of $\sqrt{i*j}$ is a positive integer, also the number $\tau(i*j)$ is always an even number except in the case when $i * j = m^2, m \in \mathbb{N}$. So, $\mid V_{i*j}^{n} \mid = \lfloor \frac{\tau(i*j)}{2} \rfloor$, if $n \ge i * j$, hence the first case is proved. In the second case we have $n < i * j$, it follows that $\mid V_{i*j}^{n} \mid$ is equal to the number of all pairs of the first case except those pairs in which there exists a divisor exceeds n. So, $\mid V_{i*j}^{n} \mid = \lfloor \frac{\tau(i*j)}{2} \rfloor - \sum_{d \mid i*j} \alpha(d)$; if $n < i * j$, hence the second case is proved.
Definition 2.3.3. A set theoretic operation
Let $A_1, A_2, ..., A_m$ be sets satisfying that for each i, $\mid A_i \mid > 1$, we define the operation $\overline{\prod_{i=1}^{m} A_i} := \{(\cup_{i=1}^{m} A_i) - \cup_{i=1}^{m}\{a_i\} : a_i \in A_i\}$.
Lemma 2.3.6. The operation $\overline{\prod_{i=1}^{m} A_i}$ is well defined.
Proof. The well definition of the operation $\overline{\prod_{i=1}^{m} A_i}$ is a direct consequence

of the well definition of the union and difference.

Remark. After calculating all the sets V^n_{i*j} satisfying that $| V^n_{i*j} |> 1$, and calculating $\overline{\prod_{2\leq i*j\leq n(n-1),|V^n_{i*j}|>1} V^n_{i*j}}$, we notice that, for each element $E \in \overline{\prod_{2\leq i*j\leq n(n-1),|V^n_{i*j}|>1} V^n_{i*j}}$, we construct a maximal strongly multiplicative graph by deleting all edges in the set E from the complete graph of n vertices.

Now, we calculate the degree of the vertex labeled i in the maximal strongly multiplicative graph associated with an element $E \in \overline{\prod_{2\leq i*j\leq n(n-1),|V^n_{i*j}|>1} V^n_{i*j}}$ by the following lemma.

Lemma 2.3.7. The degree of the vertex labeled i in the maximal strongly multiplicative graph associated with an element $E \in \overline{\prod_{2\leq i*j\leq n(n-1),|V^n_{i*j}|>1} V^n_{i*j}}$ is given by the following function $F(i, E) = n - 1 - \sum_{(r,s)\in E} \theta_i(r, s)$, where

$$\theta_i(r,s) = \begin{cases} 1 & ,if\ r = i\ or\ s = i \\ \\ 0 & ,otherwise \end{cases}$$

Definition 2.3.4. Let G be a given graph of n vertices. We define the following three sequences :

(1)the sequence of the distinct degrees of vertices in a maximal strongly multiplicative graph arranged in an ascending order, we call it the maximal degree sequence. In fact we have many different sequences of such type due to the existence of non-isomorphic maximal strongly multiplicative graphs. We denote it by $D_{R^k(n)} = (d^k_i)$, where d^k_i is the ith degree of vertices in the kth maximal strongly multiplicative graph.

(2)the sequences $C_{R^k(n)} = (c^k_i)$, where c^k_i is defined to be the number of vertices of degree at most d^k_i in the kth maximal strongly multiplicative graph, we call them the maximal strongly multiplicative sequence.

(3)for the given graph G the graph sequences $B^k_G = (b^k_i)$, where b^k_i is defined to be the number of vertices of degree at most d^k_i in G.

Example 2.3.1. For $n = 6$, $V^6_{1*6} = V^6_{2*3} = V^6_6 = \{(1,6),(2,3)\}$, and $V^6_{2*6} = V^6_{3*4} = V^6_{12} = \{(2,6),(3,4)\}$. So, $\overline{\prod_{2\leq i*j\leq 6*5,|V^6_{i*j}|>1} V^6_{i*j}} =$ $\{\{(1,6),(2,6)\},\{(1,6),(3,4)\},\{(2,3),(2,6)\},\{(2,3),(3,4)\}\}$.

The corresponding graphs are $R^1(6)$, $R^2(6)$, $R^3(6)$,$R^4(6)$ are shown in Example 2.1.1., their distinct sequences are $D_{R^1(6)} = D_{R^3(6)} = D_{R^4(6)} = \{3,4,5\}, C_{R^1(6)} = C_{R^3(6)} = C_{R^4(6)} = \{1,3,6\}$,
$D_{R^2(6)} = \{4,5\}, C_{R^2(6)} = \{4,6\}$, and hence $t(6) = 3, \delta(6) = 4$.

Theorem 2.3.8. (Condition 5) Let G be a simple graph for which there exists i^k_0 such that $b^k_{i^k_0} < c^k_{i^k_0}$; for every k, then G is non-strongly multiplicative.

Proof. For a fixed k, suppose that there exists i^k_0 such that $b_{i^k_0} < c_{i^k_0}$, i.e., the number of vertices of degree at most $d_{i^k_0}$ in G is less than the number of vertices of degree at most $d_{i_0^k}$ in the corresponding kth maximal strongly multiplicative graph, which is equal to the number of the labels of those vertices in the corresponding kth maximal strongly multiplicative graph. Then, to distribute these labels on the vertices of G we must put them on vertices of degrees at most $d_{i_0^k}$, this implies that there exists at least one label, say r_0^k, on a vertex of degree at most $d_{i_0^k}$ in the corresponding kth maximal strongly multiplicative graph must be given to a vertex of degree more than $d_{i_0^k}$ in G, say v_0^k, then there exist three vertices w_0^k, u_0^k, z_0^k, where w_0^k is adjacent with v_0^k, and has label, say m_0^k, also, the two vertices u_0^k, z_0^k are adjacent having labels, say s_0^k and t_0^k, satisfying $r_0^k m_0^k = s_0^k t_0^k$. Hence the graph G is not a subgraph of the kth maximal strongly multiplicative graph. Since it happens for each k, it follows that G is not a subgraph of any maximal strongly multiplicative graph. Hence G is non-strongly multiplicative.

Now, we'll show that Condition 5 is stronger than conditions 1,3,4 in the sense that every non-strongly multiplicative graph by these conditions is non-strongly multiplicative by Condition 5.

Corollary 2.3.9. If G is a graph of n vertices and m edges such that, $m > \lambda(n)$, then for each k, there exists i_0^k such that $b_{i_0^k} < c_{i_0^k}$.

Proof. Suppose that the degree of a vertex v_i of G is $\rho(i)$, and suppose that m, the number of edges of G, is equal to $1 + \lambda(n)$, and by deleting an edge we get a strongly multiplicative graph, i.e., G becomes a maximal strongly multiplicative graph. Suppose that the edge which causes G to be non-strongly multiplicative with respect to the kth maximal strongly multiplicative graph is one of g_k, h_k, connecting the vertices v_r^k, w_s^k and y_t^k, z_u^k respectively, having the labels r, s, t, u, such that $rs = tu$. Without any loss of generality, suppose that $r < s$, then the degree of v_r^k after removing the edge g_k is $\rho(r) = d_{i_r^k}$ and before removing g_k the degree is $d_{i_r^k} + 1$, then the number of vertices of degree at most $d_{i_r^k}$ in G is less than that number in the corresponding kth maximal strongly multiplicative graph, i.e., there exists $i_0^k = i_r^k$ such that $b_{i_0^k} < c_{i_0^k}$.

Corollary 2.3.10. If the minimum degree of the graph is greater than $\delta(n)$, which is defined in Condition 3, then for every k, there exists i_0 such that $b_{i_0} < c_{i_0}$.

Proof. Since the minimum degree of the graph is greater than $\delta(n)$, the largest minimum degree in all corresponding maximal strongly multiplicative graphs, then for every k, the number of vertices of degree at most d_1^k in the graph equals to zero which is less than the number of vertices of degree at most d_1^k in the corresponding kth maximal strongly multiplicative graph, and it is clear that $i_0 = 1$, which satisfies that for every k, $0 = b_{i_0} < c_{i_0}$.

Corollary 2.3.11. If a graph G of n vertices has more than $t(n)$ -which is defined in Condition 4- vertices of degree $n - 1$, then there exists i_0 such that $b_{i_0} < c_{i_0}$, for each k.

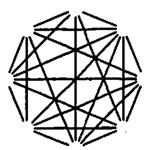
Proof. Suppose that the number of vertices of degree $n-1$ in G is greater than $t(n)$, which is defined in Condition 4, then the number of vertices of degree less than $n-1$ in G is $n-t(n)$, which is less than this number in all corresponding maximal strongly multiplicative graphs. Then there exists i_0 (the prelast term) such that $b_{i_0} < c_{i_0}$, for every k.

In the following table we give the distinct sequences of all maximal strongly multiplicative graphs of n vertices.

Table 2.3.1.

n	$D_{R^i(n)}$	$C_{R^i(n)}$
6	{3,4,5}	{1,3,6}
	{4,5}	{4,6}
7	{4,5,6}	{1,3,7}
	{5,6}	{4,7}
8	{4,5,6,7}	{1,3,4,8}
	{4,5,6,7}	{1,2,5,8}
	{5,6,7}	{2,6,8}
	{5,6,7}	{3,5,8}

Example 2.3.2. Condition 5 proves that the following graph is non-strongly multiplicative, while Condition 2 fails to decide that it is non-strongly multiplicative.



($n=8$) For Condition 2 : $K_8 = K_{2+\sum_{i=1}^{\pi(8)} \lfloor \log_{p_i}(8) \rfloor} \nsubseteq G$. But For Condition 5 : from table 2.3.1, the corresponding distinct sequences are $D_{R^1(8)} = \{4,5,6,7\}, C_{R^1(8)} = \{1,3,4,8\}$.

$D_{R^2(8)} = \{4,5,6,7\}, C_{R^2(8)} = \{1,2,5,8\}$.

$D_{R^3(8)} = \{5,6,7\}, C_{R^3(8)} = \{2,6,8\}$.

$D_{R^4(8)} = \{5,6,7\}, C_{R^4(8)} = \{3,5,8\}$.

Also, their corresponding graph sequences are $B_G^1 = \{0,1,7,8\}$,$B_G^2 = \{0,1,7,8\}$,$B_G^3 = \{1,7,8\}$,$B_G^4 = \{1,7,8\}$.

Conjecture 2.3.1. Condition 5 is stronger than Condition 2 in the sense that every non-strongly multiplicative graph by Condition 2 is a non-strongly multiplicative by Condition 5.

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