Unicyclic graphs with equal total and total outer–connected domination numbers

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Abstract

Let $G = (V, E)$ be a graph without an isolated vertex. A set $D \subseteq V(G)$ is a ***total dominating set*** if D is dominating and the induced subgraph $G[D]$ does not contain an isolated vertex. The total domination number of G is the minimum cardinality of a total dominating set of G. A set $D \subseteq V(G)$ is a ***total outer–connected dominating set*** if D is total dominating and the induced subgraph $G[V(G) - D]$ is a connected graph. The total outer–connected domination number of G is the minimum cardinality of a total outer–connected dominating set of G. We characterize all unicyclic graphs with equal total domination and total outer–connected domination numbers.

Keywords: total domination number, total outer–connected domination number, unicyclic graphs
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1 Definitions

Here we consider simple undirected and connected graphs $G = (V, E)$. The *distance* $d_G(u, v)$ between two vertices u and v in G is the length of a shortest $(u - v)$ path in G. If D is a set and $u \in V(G)$, then $d_G(u, D) = \min\{d_G(u, v) : v \in D\}$. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. The degree of a vertex v is $d_G(v) = |N_G(v)|$ and a vertex of degree 1 is called *a leaf*. A vertex, which is a neighbour of a leaf, is called *a support vertex*. For example the path on two vertices, P_2, contains two leaves and two support vertices. Denote by $S(G)$ the set of all support vertices of G. If a support vertex is adjacent to more than one leaf, then we call it a *strong support vertex.* For notational convenience we denote the set of all support vertices and leaves of G by $J(G)$.

A set $D \subseteq V(G)$ is a *2-packing* in G if $d_G(u, v) \geq 3$ for every $u, v \in D$. A subset D of $V(G)$ is *dominating* if every vertex of $V(G) - D$ has a neighbour in D.

If G is without an isolated vertex, then a set $D \subseteq V(G)$ is a *total dominating set of* G if for every vertex $v \in V(G)$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a total dominating set in G is the *total domination number* denoted $\gamma_t(G)$. A minimum total dominating set of a graph G is called a $\gamma_t(G)$-set. The total domination number of a graph was introduced by Cockayne, Dawes and Hedetniemi [1] and is now well studied in the theory of domination.

If G is without an isolated vertex, then a set $D \subseteq V(G)$ is a *total outer-connected dominating set of* G if D is total dominating set of G and the subgraph induced by $V(G) - D$ is connected. The minimum cardinality of a total outer-connected dominating set in G is the *total outer-connected domination number* denoted $\gamma_{tc}(G)$. A minimum total outer-connected dominating set of a graph G is called a $\gamma_{tc}(G)$-set. The total-outer connected domination number of a graph was defined recently by Cyman [2] and further studied, for example, by Hattingh and Joubert [5]. As an immediate consequence of this definitions we have $\gamma_t(G) \leq \gamma_{tc}(G)$. For an application, we consider a computer network in which a core group of fileservers has the ability to communicate directly with every computer outside the core group. In addition, each fileserver is directly linked to at least one other fileserver and every two computers outside the core group may communicate with each other without intervention of any of the fileservers to protect the fileservers from overloading. A smallest core group with these properties is a γ_{tc}-set for the graph representing the network.

A unicyclic graph is a graph that contains precisely one cycle. For any graph theoretical parameters λ and μ, we define G to be (λ, μ)-graph if $\lambda(G) = \mu(G)$. In this paper we provide a constructive characterization of all (γ_t, γ_{tc})-unicyclic graphs. For any unexplained terms and symbols see [4].

2 Preliminary results

In our characterization of (γ_t, γ_{tc})-unicyclic graphs we assume that G is without isolates. First we present the following observations. The first one has been taken from [6].

Observation 1 [6] *Let T be a tree that is not a star. Then there exists a $\gamma_t(T)$-set that contains no leaf.* ■

Moreover, it is no problem to observe that if G is a graph, then each support vertex is in $\gamma_t(G)$-set.

The following observation concerns the total outer-connected domination number of a graph.

Proposition 2 [3] *If G is a graph with $\gamma_{tc}(G) \leq n(G) - 2$, then each leaf and each support vertex belong to every minimum total outer-connected dominating set of G.* ∎

In [3] are constructively characterized all (γ_t, γ_{tc})-trees as follows. Let $\mathcal{O}$ be the following operation defined on a tree T.

- **Operation** $\mathcal{O}$. Assume $x \in V(T) - J(T)$. Then add a path (y_1, y_2, y_3) and the edge xy_1.

Let $\mathcal{T}$ be the family of trees such that $\mathcal{T} = \{T : T$ is obtained from P_6 by a finite sequence of operations $\mathcal{O}\} \cup \{P_2, P_3\}$. Fig. 1 gives an example of a tree belonging to $\mathcal{T}$.

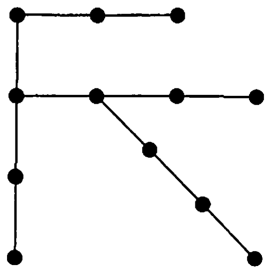

Figure 1: Tree T belonging to the family $\mathcal{T}$

It is no problem to observe, that if a tree T with $n(T) \geq 3$ belongs to the family $\mathcal{T}$, then each vertex of $S(T)$ is of degree 2, $S(T)$ is a 2-packing and $S(T)$ is a dominating set of T. Hence $\gamma_t(T) = |J(T)|$.

Theorem 3 [3] *A tree T is a (γ_t, γ_{tc})-tree if and only if T belongs to the family $\mathcal{T}$.* ∎

3 Unicyclic graphs

Now we constructively characterize all connected unicyclic graphs for which $\gamma_t(G) = \gamma_{tc}(G)$. To this aim define $\mathcal{C}$ to be the family of all graphs G for which exists a tree T belonging to the family $\mathcal{T}$, such that G may be obtained from T by one of the operations listed below.

- **Operation** $\mathcal{O}_1$. Assume $u, v \in V(T) - J(T)$. Then add the edge uv (see Fig. 2).

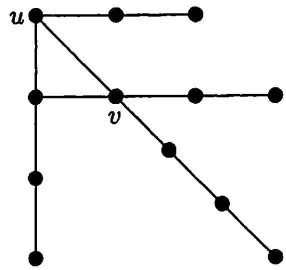

Figure 2: Graph G obtained from a tree $T \in \mathcal{T}$ by Operation $\mathcal{O}_1$.

- **Operation** $\mathcal{O}_2$. Assume $u, v \in S(T)$ and let u' and v' be the leaves adjacent to u and v, respectively. Then identify u with v and u' with v' (see Fig. 3).

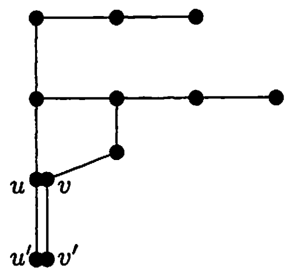

Figure 3: Operation $\mathcal{O}_2$.

- **Operation** $\mathcal{O}_3$. Assume $u, v \in S(T)$ and let u' and v' be the leaves adjacent to u and v, respectively. Then identify u with v' and v with u' (see Fig. 4).

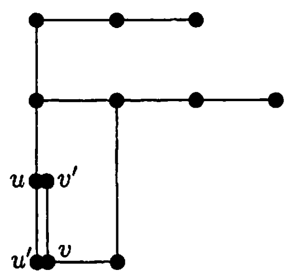

Figure 4: Operation $\mathcal{O}_3$.

Additionally, let the cycles C_3, C_5 and C_6 belong to $\mathcal{C}$ and observe that C_4 may be obtained from $P_6 \in \mathcal{T}$ by Operation $\mathcal{O}_3$.

Lemma 4 *If G belongs to the family $\mathcal{C}$, then $\gamma_t(G) = \gamma_{tc}(G)$.*

Proof. If G is a cycle belonging to $\mathcal{C}$, then the result is immediate. If not, assume first that G is obtained from a tree $T \in \mathcal{T}$ by Operation $\mathcal{O}_1$, e.g. G is obtained from T by adding the edge uv, where $\{u, v\} \subseteq V(T) - J(T)$. Then $S(G) = S(T)$ and $J(G) = J(T)$. Thus in G, similarly like in T, $S(G)$ is a 2-packing and $S(G)$ is a dominating set of G. Hence $\gamma_t(G) = 2|S(G)| = |J(G)|$. On the other hand, $J(G)$ is a total outer-connected dominating set of G, so

$$|J(G)| = \gamma_t(G) \le \gamma_{tc}(G) \le |J(G)|. \tag{1}$$

Thus $\gamma_t(G) = \gamma_{tc}(G)$.

Assume now that G is obtained from a tree $T \in \mathcal{T}$ by Operation $\mathcal{O}_2$, e.g. G is obtained from T by identifying u with v and u' with v', where $u, v \in S(T)$ and u', v' are the leaves adjacent in T to u and v, respectively. Denote by w the vertex obtained by identifying u and v and denote by w' the vertex obtained by identifying u' and v'. Then $S(G) = (S(T) \cup \{w\}) - \{u, v\}$ and $J(G) = (J(T) \cup \{w, w'\}) - \{u, v, u', v'\}$. However in G, similarly like in T, $S(G)$ is a 2-packing and $S(G)$ is a dominating set of G. Hence $\gamma_t(G) = 2|S(G)| = |J(G)|$. On the other hand, $J(G)$ is a total outer–connected dominating set of G, so the inequaity chain (1) cholds and thus $\gamma_t(G) = \gamma_{tc}(G)$.

Assume now that G is obtained from a tree $T \in \mathcal{T}$ by Operation $\mathcal{O}_3$, e.g. G is obtained from T by identifying u with v' and u' with v, where $u, v \in S(T)$ and u', v' are the leaves adjacent in T to u and v, respectively. Denote by w the vertex obtained by identifying u and v' and denote by w' the vertex obtained by identifying u' and v. Then $S(G) = S(T) - \{u, v\}$ and $J(G) = J(T) - \{u, v, u', v'\}$. Similarly like in T, $S(G)$ is a 2-packing. Moreover, $S(G) \cup \{w, w'\}$ is a dominating set of G, $d_G(w, S(G)) = 3$ and $d_G(w', S(G)) = 3$. Hence $\gamma_t(G) = 2|S(G)| + 2 = |J(G) \cup \{w, w'\}|$. On the other hand, $J(G) \cup \{w, w'\}$ is a total outer–connected dominating set of G, so

$$|J(G)| + 2 = \gamma_t(G) \le \gamma_{tc}(G) \le |J(G)| + 2.$$

Thus $\gamma_t(G) = \gamma_{tc}(G)$. ■

Lemma 5 *If G is connected unicyclic graph with $\gamma_t(G) = \gamma_{tc}(G)$, then G belongs to the family $\mathcal{C}$.*

Proof. Let G be a connected unicyclic graph, where $C_k = (v_1, \ldots, v_k)$ is the unique cycle of G. Assume first that each vertex of C_k is of degree 2. Then G is a cycle C_k for some $k \ge 3$. It is no problem to see that $\gamma_{tc}(C_k) = k - 2$ for $k \ge 4$ and $\gamma_{tc}(C_3) = 2$. On the other hand, $\gamma_t(C_k) < k - 2$ for $k \ge 7$. Thus it is possible to verify that if $\gamma_t(C_k) = \gamma_{tc}(C_k)$, then $k \in \{3, 4, 5, 6\}$.

Therefore assume G is not a cycle. If $v_i \in V(C_k)$, then let $T(v_i)$ be the tree obtained from G by removing edges v_iv_{i+1} and $v_{i-1}v_i$ (where the indices are taken modulo k and added 1) and containing v_i. Let v_i be the root of $T(v_i)$. Let D_{tc} be a minimum total outer-connected dominating set of G.

Assume now, without loss of generality, that $d_G(v_1) \geq 3$ and denote by x any element of $V(T(v_1))$ which is not a leaf. Moreover, denote by $T(x)$ the subtree of $T(v_1)$ rooted in x. Then x is a cut-vertex and if additionaly x is in D_{tc}, then either $V(G)-D_{tc} \subseteq V(T(x))$ or $V(G)-D_{tc} \subseteq V(G)-V(T(x))$. Suppose $V(G) - D_{tc} \subseteq V(T(x))$. Then $V(T(v_2)) \cup \cdots \cup V(T(v_k)) \subseteq D_{tc}$. If $V(T(v_2)) \cup \cdots \cup V(T(v_k))$ contains a leaf, say u, then $D_{tc} - \{u\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossible. Therefore $d_G(v_2) = \cdots = d_G(v_k) = 2$. However then $D_{tc} - \{v_2\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, a contradiction.

Hence $V(G) - D_{tc} \subseteq V(G) - V(T(x))$. Then $V(T(x)) \subseteq D_{tc}$. If $T(x)$ contains more that one leaf, say x_1 and x_2 are leaves in $T(x)$, then $D_{tc} - \{x_1\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossible. Therefore $T(x)$ is a path. If $T(x)$ contains more than 3 vertices and u is the unique leaf of $T(x)$, then again $D_{tc} - \{u\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, a contradiction. Therefore $|V(T(x))| = 2$ and for this reason if $x \notin V(C_k)$, then $d_G(x) = 2$ because x is adjacent to exactly one leaf and one parent. Moreover, if $x \in V(C_k)$, then $d_G(x) = 3$.

Since each support vertex and each leaf is in D_{tc}, we conclude that $\gamma_{tc}(G) \geq 2|S(G)| = |J(G)|$. Further, if $v_i \in V(C_k)$ and $|V(T(v_i))| \geq 2$, then $V(T(v_i)) \cap D_{tc} = V(T(v_i) \cap J(G)$. Therefore, since each support vertex belongs to the total dominating set of G, we conclude that each support vertex has in every $\gamma_t(G)$-set exactly one neighbour and since $\gamma_t(G) = \gamma_{tc}(G)$, every two support vertices of G are at least distance 3 apart, e.g. $S(G)$ is a 2-packing.

1. **Assume** $V(C) \cap D_{tc} = \emptyset$. Then $V(C) \cap S(G) = \emptyset$ and since D_{tc} is dominating, $d_G(v_i) \geq 3$ for each $v_i \in V(C_k)$. Since removing an edge (which is not incident to a leaf) of a graph cannot decrease its total domination number, we obtain

$$\gamma_{tc}(G) = \gamma_t(G) \leq \gamma_t(G - v_1v_2) \leq \gamma_{tc}(G - v_1v_2). \quad (2)$$

It is possible to see that D_{tc} is also a total outer-connected dominating set of $G - v_1v_2$, so $\gamma_{tc}(G) \geq \gamma_{tc}(G - v_1v_2)$ and thus we have equalities throughout the inequality chain (2). In particular, $\gamma_t(G - v_1v_2) = \gamma_{tc}(G - v_1v_2)$ and since $G - v_1v_2$ is a tree, Theorem 3 implies that $G - v_1v_2$ belongs to the family $\mathcal{T}$. Obviously $v_1, v_2 \notin J(G - v_1v_2)$, so finally we conclude that G may be obtained from a tree belonging to the family $\mathcal{T}$ by Operation $\mathcal{O}_1$. Therefore G belongs to the family $\mathcal{C}$.

2. **Assume now, without loss of generality, that** $v_1 \in V(C) \cap D_{tc}$ **and** $d_G(v_1) \geq 3$. Then $|V(T(v_1))| = 2$, $d_G(v_1) = 3$ and v_1 is a support vertex. Denote by x the unique leaf adjacent to v_1.

   Suppose $v_2 \in D_{tc}$. Then $D_{tc} - \{x\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, a contradiction. We conclude that x is the unique neighbour of v_1 belonging to D_{tc} and thus v_1 is the unique vertex of C_k belonging to D_{tc}. Suppose now that $y, v_1 \in N_G(v_2) \cap D_{tc}$. Then, since D_{tc} is outer–connected, $y \in V(T(v_2))$. Define

$$D' = \{u : u \in D_{tc} - V(T(v_2))\} \cup$$
$$\cup \; \{u : u \text{ is the parent of a vertex belonging to } D_{tc} \cap V(T(v_2))\}.$$

   It is no problem to see that $|D'| \leq |D_{tc}|$. Moreover, since $v_2 \in D'$, $D' - \{x\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossible. Therefore $N_G(v_2) \cap D_{tc} = \{v_1\}$ and, by symmetry, $N_G(v_k) \cap D_{tc} = \{v_1\}$. Thus each vertex of G, which is not a support, is a neighbour of exactly one support vertex. Therefore $\gamma_t(G) = \gamma_{tc}(G) = 2|S(G)| = |J(G)|$.

   Denote by G_1 the graph obtained from G by splitting v_1 and x, e.g. we remove from G the edge v_1v_2 and we add a path (v_1', x') and the edge v_2v_1' (see Fig. 5).

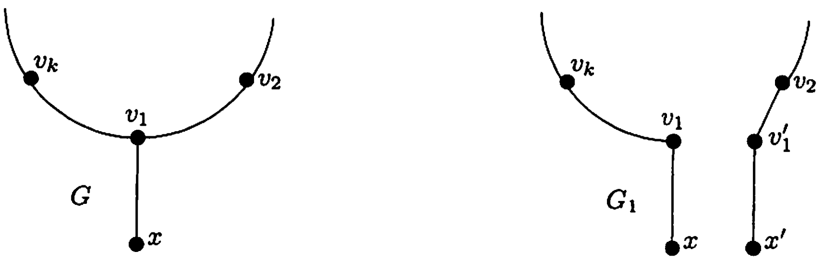

Figure 5: Splitting v_1 and x in G

   Of course $\gamma_t(G) \leq \gamma_t(G_1)$. By the construction of G_1 and since each support vertex of a graph is in minimum total dominating set and $S(G_1) = S(G) \cup \{v_1'\}$, we have $\gamma_t(G) \leq \gamma_t(G_1) - 1$. Suppose D_t is a $\gamma_t(G_1)$-set of cardinality $\gamma_t(G) + 1$. Then $v_1' \in D_t$ and v_1' has a neighbour in D_t. If $x' \in D_t$, then $D_t - \{x', v_1'\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, a contradiction. Thus assume $x' \notin D_t$. This implies $v_2 \in D_t$. If additionally $x \in D_t$, then

$D_t - \{x, v_1'\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$. Thus assume $x, x' \notin D_t$. Then $v_k \in D_t$ and $v_2 \in D_t$ and for this reason $D_t - \{v_1'\}$ would be a $\gamma_t(G)$-set. However, since each vertex of G, which is not a support, is a neighbour of exactly one support vertex, $D_t - \{v_1', v_2\}$ is also a total dominating set of G of cardinality smaller than $\gamma_t(G)$, a contradiction. We conclude that $\gamma_t(G) \leq \gamma_t(G_1) - 2$ and thus we obtain

$$\gamma_{tc}(G) = \gamma_t(G) \leq \gamma_t(G_1) - 2 \leq \gamma_{tc}(G_1) - 2. \qquad (3)$$

It is possible to see that $D_{tc} \cup \{v_1', x'\}$ is a total outer–connected dominating set of G_1, so $\gamma_{tc}(G) \geq \gamma_{tc}(G_1) - 2$ and thus we have equalities throughout the inequality chain (3). In particular, $\gamma_t(G_1) = \gamma_{tc}(G_1)$ and since G_1 is a tree, Theorem 3 implies that G_1 belongs to the family $\mathcal{T}$. Finally we conclude that G may be obtained from a tree belonging to the family $\mathcal{T}$ by Operation $\mathcal{O}_2$. Therefore G belongs to the family $\mathcal{C}$.

3. **Assume $v_1 \in V(C) \cap D_{tc}$ and $d_G(v_1) = 2$.** Then, without loss of generality, $v_2 \in D_{tc}$. By similar reasoning as in the previous case, we have $d_G(v_2) = 2$. **Assume additionally that $v_3 \notin D_{tc}$ and $v_k \notin D_{tc}$.** Then since D_{tc} is outer–connected, exactly two vertices of $V(C_k)$ belong to D_{tc}, namely v_1 and v_2. If $v_3 = v_k$, then, since G is not a cycle, $d_G(v_3) \geq 3$. If $V(T(v_3)) \cap D_{tc} \neq \emptyset$, then $(D_{tc} - \{v_1, v_2\}) \cup \{v_3\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, a contradiction. If $V(T(v_3)) \cap D_{tc} = \emptyset$, then define

$$\begin{aligned} D' &= \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_3))\} \\ &\cup \{v_1, v_2\}. \end{aligned}$$

It is no problem to see that $|D'| \leq |D_{tc}|$. Moreover, $(D' - \{v_1, v_2\}) \cup \{v_3\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe.

Thus let $v_3 \neq v_k$, that is $k \geq 4$. Suppose y_k and v_1 are two distinct elements of $N_G(v_k) \cap D_{tc}$ and y_3 and v_2 are two distinct elements of $N_G(v_3) \cap D_{tc}$. Then define

$$\begin{aligned} D' &= \{u : u \in D_{tc} - (V(T(v_k)) \cup V(T(v_3)))\} \cup \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_k))\} \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_3))\}. \end{aligned}$$

It is no problem to see that $|D'| \leq |D_{tc}|$. Moreover, since $v_k, v_3 \in D'$, $D' - \{v_1, v_2\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. Thus suppose $\{y_k, v_1\} \subseteq N_G(v_k) \cap D_{tc}$

and $N_G(v_3) \cap D_{tc} = \{v_2\}$. Then v_4 has a neighbour in $D_{tc} \cap V(T(v_4))$ (note that possibly $v_k = v_4$). Define

$$\begin{aligned} D' = &\{u : u \in D_{tc} - (V(T(v_4)) \cup V(T(v_k)))\} \cup \\ \cup &\{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_4))\} \\ \cup &\{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_k))\}. \end{aligned}$$

It is no problem to see that $|D'| \leq |D_{tc}|$ (also if $v_k = v_4$). Moreover, since $v_k, v_4 \in D'$, $D' - \{v_2\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. We conclude that v_1 is the unique neigbour of v_k belonging to D_{tc} and v_2 is the unique neigbour of v_3 belonging to D_{tc}. Thus each vertex of G, which is not a support, is a neighbour of exactly one vertex from $S(G) \cup \{v_1, v_2\}$. Therefore $\gamma_t(G) = \gamma_{tc}(G) = |J(G) \cup \{v_1, v_2\}|$.

Denote by G_2 the graph obtained from G by splitting with a twist v_1 and v_2, e.g. we remove from G the edge v_2v_3 and we add a path (v_1', v_2') and an edge v_3v_1' (see Fig. 6).

Figure 6: Splitting with a twist v_1 and v_2 in G

It is no problem to see that $\gamma_t(G) \leq \gamma_t(G_2)$. Suppose D_t is a $\gamma_t(G_2)$-set of cardinality smaller than $\gamma_t(G) + 2$. Then $v_1, v_1' \in D_t$ and both v_1 and v_1' have a neighbour in D_t. If $v_2, v_2' \in D_t$, then $D_t - \{v_1', v_2'\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, which is impossible. If either v_k or v_3 in D_t, then $D = (D_t - \{v_1', v_2'\}) \cup \{v_2\}$ is a total dominating set of G of cardinality at most $\gamma_t(G)$. However, since each vertex belonging to $N_G(v_k) - \{v_1\}$ has a neighbour in $S(G)$ and each vertex belonging to $N_G(v_3) - \{v_2\}$ has a neighbour in $S(G)$, $D - \{v_k, v_3\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, which is impossible. If both v_k and v_3 in D_t, then $D = (D_t - \{v_1', v_2'\}) \cup \{v_2\}$ is a total dominating set of G of cardinality at most $\gamma_t(G) + 1$. However then $D - \{v_k, v_3\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, a contradiction.

Therefore,

$$\gamma_{tc}(G) = \gamma_t(G) \leq \gamma_t(G_1) - 2 \leq \gamma_{tc}(G_1) - 2. \tag{4}$$

It is possible to see that $D_{tc} \cup \{v_1', v_2'\}$ is a total outer-connected dominating set of G_2, so $\gamma_{tc}(G) \geq \gamma_{tc}(G_1) - 2$ and thus we have equalities throughout the inequality chain (4). In particular, $\gamma_t(G_2) = \gamma_{tc}(G_2)$ and since G_2 is a tree, Theorem 3 implies that G_2 belongs to the family $\mathcal{T}$. Finally, we conclude that G may be obtained from a tree belonging to the family $\mathcal{T}$ by Operation $\mathcal{O}_3$. Therefore G belongs to the family $\mathcal{C}$.

4. **Assume now** $\{v_1, v_2, v_3\} \subseteq V(C) \cap D_{tc}$. Then by similar reasoning as in previous cases, we have $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$. Since G is not a cycle, $v_k \neq v_3$. Thus $k \geq 4$. **Assume additionally that** $v_4 \notin D_{tc}$ **and** $v_k \notin D_{tc}$. Then since D_{tc} is outer-connected, exactly three vertices of $V(C)$ belong to D_{tc}, namely v_1, v_2 and v_3. If $v_4 = v_k$, then, since G is not a cycle, $d_G(v_4) \geq 3$. However then $(D_{tc} - \{v_2, v_3\}) \cup \{v_4\}$ is a total dominating set of G of cardinality smaller than $\gamma_t(G)$, which is impossible. Thus $k \geq 5$. Suppose $d_G(v_4) \geq 3$ and $|N_G(v_4) \cap D_{tc}| \geq 2$. Then $d_G(v_4) \geq 3$ and v_4 has a neighbour in $V(T(v_4)) \cap D_{tc}$, denoted v_4'. Moreover, $D_{tc} - \{v_3\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. Thus v_3 is the unique neighbour of v_4 belonging to D_{tc} and $v_4' \notin D_{tc}$. Hence v_4' has a son in $T(v_4)$ belonging to D_{tc}, denoted v_4''. Define

$$\begin{aligned} D' &= \{u : u \in D_{tc} - V(T(v_4))\} \cup \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_4))\} \end{aligned}$$

and observe, that as $\{v_1, v_2, v_3, v_4', v_4''\} \subseteq D'$, $D' - \{v_3\}$ is again a total dominating set of G of smaller cardinality than $\gamma_t(G)$. Therefore, $d_G(v_4) = 2$. Since G is not a cycle, $v_k \neq v_5$. However then $v_5 \notin D_{tc}$ and v_5 has a neighbour in $V(T(v_5)) \cap D_{tc}$, denoted v_5'. Observe, that since $v_5 \notin D_{tc}$, v_5' has a son in $T(v_5)$ belonging to D_{tc}. Define

$$\begin{aligned} D' &= \{u : u \in D_{tc} - V(T(v_5))\} \cup \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_5))\}. \end{aligned}$$

It is no problem to see that $|D'| \leq |D_{tc}|$. Moreover, as $\{v_1, v_2, v_3, v_5, v_5'\} \subseteq D'$ and $d_G(v_4) = 2$, $D' - \{v_3\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. We conclude that if $\gamma_t(G) = \gamma_{tc}(G)$ and G is not a cycle, then the case when $\{v_1, v_2, v_3\} \subseteq D_{tc}$ and $v_4, v_k \notin D_{tc}$ is impossible.

5. **Assume at last,** $\{v_1, v_2, v_3, v_4\} \subseteq V(C) \cap D_{tc}$. Then $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_4) = 2$. Since G is not a cycle, $v_k \neq v_4$. Thus $k \geq 5$. If $v_5 \in D_{tc}$, then $D_{tc} - \{v_3\}$ is a total dominating set of G of

smaller cardinality than $\gamma_t(G)$, which is impossilbe. Thus $v_5 \notin D_{tc}$ and similarly $v_k \notin D_{tc}$. Suppose $d_G(v_5) \geq 3$ and $|N_G(v_5) \cap D_{tc}| \geq 2$. Then $d_G(v_5) \geq 3$ and v_5 has a neighbour in $V(T(v_5)) \cap D_{tc}$, denoted v_5'. Observe, that $D_{tc} - \{v_4\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. Thus v_4 is the unique neighbour of v_5 belonging to D_{tc} and $v_5' \notin D_{tc}$. Hence v_5' has a son in $T(v_5)$ belonging to D_{tc}, denoted v_5''. Define

$$\begin{aligned} D' &= \{u : u \in D_{tc} - V(T(v_5))\} \cup \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_5))\}. \end{aligned}$$

and observe, that as $\{v_1, v_2, v_3, v_4, v_5', v_5''\} \subseteq D'$, $D' - \{v_4\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$. Therefore, $d_G(v_5) = 2$. Since G is not a cycle, $v_k \neq v_6$. However then $v_6 \notin D_{tc}$, $d_G(v_6) \geq 3$ and v_6 has a neighbour in $V(T(v_6)) \cap D_{tc}$, denoted v_6'. Observe, that since $v_6 \notin D_{tc}$, v_6' has a son in $T(v_6)$ belonging to D_{tc}. Define

$$\begin{aligned} D' &= \{u : u \in D_{tc} - V(T(v_6))\} \cup \\ &\cup \{u : u \text{ is the parent of a vertex from } D_{tc} \cap V(T(v_6))\}. \end{aligned}$$

It is no problem to see that $|D'| \leq |D_{tc}|$. Moreover, as $\{v_1, v_2, v_3, v_4, v_6, v_6'\} \subseteq D'$, $D' - \{v_4\}$ is a total dominating set of G of smaller cardinality than $\gamma_t(G)$, which is impossilbe. We conclude that if $\gamma_t(G) = \gamma_{tc}(G)$ and G is not a cycle, then the case when $\{v_1, v_2, v_3, v_4\} \subseteq D_{tc}$ is impossible.

■

Our last result gives a characterization of all (γ_t, γ_{tc})-unicyclic graphs. The straightforward proof is omitted.

Theorem 6 *Let G be a unicyclic graph. Then $\gamma_t(G) = \gamma_{tc}(G)$ if and only if exactly one connected component of G is a unicyclic graph belonging to the family $\mathcal{C}$ and each other connected compoment of G is a tree belonging to the family $\mathcal{T}$.* ■

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