A new sufficient condition for graphs to be (a, b, k)-critical graphs *†

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Abstract

Let a, b and k be nonnegative integers with $2 \leq a < b$ and $b \equiv 0 \ (mod\ a-1)$, and let G be a graph of order n with $n \geq \frac{(a+b-1)(2a+b-4)-a+1}{b} + k$. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, it is proved that G is an (a, b, k)-critical graph if

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + b + bk}{a+b-1}.$$

Furthermore, it is shown that the result in this paper is best possible in some sense.

Keywords: graph, neighborhood, minimum degree, $[a, b]$-factor, (a, b, k)-critical graph

2010 Mathematics Subject Classification: 05C70

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*Receive date: September 2, 2009

†Supported by the National Natural Science Foundation of China (Grant No. 11371009) and the National Social Science Foundation of China (Grant No. 14AGL001).

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1 Introduction

The graphs considered here are finite undirected graphs which have neither loops nor multiple edges. We refer the readers to [1] for the terminologies not defined here. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. The minimum degree of G is denoted by $\delta(G)$. The neighborhood $N_G(x)$ of x is the set of all vertices in $V(G)$ adjacent to x and for $X \subseteq V(G)$ we write $N_G(X) = \cup_{x \in X} N_G(x)$. For $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$, respectively. Let S and T be two disjoint subsets of $V(G)$, we denote the number of edges from S to T by $e_G(S, T)$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges.

Let a, b and k be nonnegative integers with $1 \le a \le b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for each $x \in V(G)$ (where of course d_F denotes the degree in F). And if $a = b = r$, then an $[a, b]$-factor of G is called an r-factor of G. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = r$, then an (a, b, k)-critical graph is simply called an (r, k)-critical graph. In particular, a $(1, k)$-critical graph is simply called a k-critical graph.

Favaron [2] investigated the properties of k-critical graphs. Cai and Favaron [3] got a toughness condition for a graph to be a $(2, k)$-critical graph. Liu and Yu [4] studied the characterization of (r, k)-critical graphs. Liu and Wang [5] obtained a necessary and sufficient condition for a graph to be an (a, b, k)-critical graph. Li [6] showed a degree condition for graphs to be (a, b, k)-critical graphs. Li [7] got two sufficient conditions for graphs to be (a, b, k)-critical graphs. Zhou [8] obtained an independence number and connectivity condition for a graph to be an (a, b, k)-critical graph. Zhou [9] gave a neighborhood condition for graphs to be (a, b, k)-critical graphs. Zhou and Jiang [10] showed a binding number condition for a graph to be an (a, b, k)-critical graph.

The following results on k-factors and (a, b, k)-critical graphs are known.

Theorem 1 [11] *Let $k \ge 2$ be an integer and G a graph of order n with $n \ge 4k - 6$. If k is odd, then n is even and G is connected. Let G satisfy*

$$|N_G(X)| \ge \frac{|X| + (k-1)n - 1}{2k - 1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) \geq \frac{k-1}{2k-1}(n+2).$$

Then G has a k-factor.

Theorem 2 [9] *Let a, b, k be nonnegative integers such that $2 \leq a < b$, and let G be a graph of order n with $n \geq \frac{(a+b-1)(2a+b-5)+b+1}{b} + \frac{bk}{b-1}$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \qquad if \qquad |X| \geq \left\lfloor \frac{(b(n-1)-bk)n}{(a+b-1)(n-1)} \right\rfloor; \quad or$$

$$|N_G(X)| \geq \frac{(a+b-1)(n-1)}{b(n-1)-bk}|X| \qquad if \qquad |X| < \left\lfloor \frac{(b(n-1)-bk)n}{(a+b-1)(n-1)} \right\rfloor.$$

Then G is an (a, b, k)-critical graph.

Theorem 3 [10] *Let G be a graph of order n, and let a, b and k be nonnegative integers such that $1 \leq a < b$. If the binding number bind$(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ and $n \geq \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, then G is an (a, b, k)-critical graph.*

Theorem 4 [12] *Let a, b and k be nonnegative integers with $1 \leq a < b$, and let G be a graph of order n with $n \geq \frac{(a+b)(a+b-2)}{b} + k$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b + bk - 2}{a+b-1}.$$

Then G is an (a, b, k)-critical graph.

In this paper, we obtain a new sufficient condition by using neighborhoods of independent sets for a graph to be an (a, b, k)-critical graph. The main result is the following theorem, which is an extension of Theorem 1 and an improvement of Theorem 4.

Theorem 5 *Let a, b and k be nonnegative integers with $2 \leq a < b$ and $b \equiv 0 \ (mod\ a-1)$, and let G be a graph of order n with $n \geq \frac{(a+b-1)(2a+b-4)-a+1}{b} + k$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a+b-1}$$

for every non-empty independent subset X of V(G), and

$$\delta(G) > \frac{(a-1)n+b+bk}{a+b-1}.$$

Then G is an (a, b, k)-critical graph.

In Theorem 5, if $k = 0$, then we obtain the following corollary.

Corollary 1 *Let a and b be integers such that* $2 \le a < b$ *and* $b \equiv 0 \pmod{a-1}$, *and let G be a graph of order n with* $n \ge \frac{(a+b-1)(2a+b-4)-a+1}{b}$. *Let G satisfy*

$$|N_G(X)| > \frac{(a-1)n+|X|-1}{a+b-1}$$

for every non-empty independent subset X of V(G), and

$$\delta(G) > \frac{(a-1)n+b}{a+b-1}.$$

Then G has an [a, b]-factor.

2 The Proof of Theorem 5

In order to prove Theorem 5, we depend heavily on the following lemma.

Lemma 2.1 [5] *Let a, b and k be nonnegative integers with* $a < b$, *and let G be a graph of order* $n \ge a+k+1$. *Then G is an (a, b, k)-critical graph if and only if for any* $S \subseteq V(G)$ *with* $|S| \ge k$

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - bk, \quad \text{or}$$

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \ge bk,$$

where $p_j(G-S)$ *denotes the number of vertices in* $G-S$ *with degree j,* $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a-1\}$ *and* $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

Proof of Theorem 5. Let G be a graph satisfying the hypothesis of Theorem 5, we prove the theorem by contradiction. Suppose that G is not an (a,b,k)-critical graph. Then by Lemma 2.1, there exists some subset S of $V(G)$ with $|S| \ge k$ such that

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \le bk-1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a-1\}$. Obviously, $T \ne \emptyset$ by (1). Set

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

In terms of the definition of T, we obtain

$$0 \le h \le a-1.$$

Since $T \ne \emptyset$, we may choose a vertex $t \in T$ with

$$h = \min\{d_{G-S}(t) : t \in T\}.$$

According to the choice of t, we have

$$\delta(G) \le d_G(t) \le d_{G-S}(t) + |S| = h + |S|,$$

that is,

$$|S| \ge \delta(G) - h. \tag{2}$$

Now in order to prove the theorem, we shall deduce some contradictions by the following three cases.

Case 1. $2 \le h \le a-1$.

Using (1) and $|S| + |T| \le n$, we have

$$\begin{aligned} bk-1 &\ge \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &\ge b|S| - (a-h)|T| \\ &\ge b|S| - (a-h)(n-|S|) \\ &= (a+b-h)|S| - (a-h)n, \end{aligned}$$

which implies

$$|S| \le \frac{(a-h)n + bk - 1}{a+b-h}. \tag{3}$$

On the other hand, we obtain from (2) and $\delta(G) > \frac{(a-1)n+b+bk}{a+b-1}$

$$|S| \ge \delta(G) - h > \frac{(a-1)n+b+bk}{a+b-1} - h.$$

Combining this with (3), we get

$$\frac{(a-1)n+b+bk}{a+b-1} - h < |S| \le \frac{(a-h)n+bk-1}{a+b-h}. \tag{4}$$

Let the LHS and RHS of (4) be denoted by A and B, respectively. Then (4) says that

$$A - B < 0. \tag{5}$$

But, after some rearranging, we find that

$$\begin{aligned}
& (a+b-1)(a+b-h)(A-B) \\
= \; & (h-1)(bn-(a+b-1)(a+b-h)-bk+a-1) \\
& -(a-2)(a+b-1) \\
\geq \; & (h-1)(bn-(a+b-1)(a+b-2)-bk+a-1) \\
& -(a-2)(a+b-1) \\
\geq \; & (h-1)(b(\frac{(a+b-1)(2a+b-4)-a+1}{b}+k) \\
& -(a+b-1)(a+b-2)-bk+a-1)-(a-2)(a+b-1) \\
= \; & (h-1)(a-2)(a+b-1)-(a-2)(a+b-1) \\
= \; & (h-2)(a-2)(a+b-1),
\end{aligned}$$

that is,

$$(a+b-1)(a+b-h)(A-B) \geq (h-2)(a-2)(a+b-1). \tag{6}$$

From $2 \leq h \leq a-1$ and $2 \leq a < b$, we have $(h-2)(a-2)(a+b-1) \geq 0$. In terms of (6) and $2 \leq h \leq a-1$, we obtain

$$A - B \geq 0.$$

That contradicts (5).

Case 2. $h = 1$.

According to (2) and $\delta(G) > \frac{(a-1)n+b+bk}{a+b-1}$, we get

$$|S| > \frac{(a-1)n+b+bk}{a+b-1} - 1. \tag{7}$$

Subcase 2.1. $|T| \geq \frac{b(n-k-1)}{a+b-1} + 1$.

Using (7), we obtain

$$|S| + |T| > \frac{(a-1)n+b+bk}{a+b-1} - 1 + \frac{b(n-k-1)}{a+b-1} + 1 = n.$$

This inequality contradicts $|S| + |T| \leq n$.

Subcase 2.2. $|T| < \frac{b(n-k-1)}{a+b-1} + 1$.

In terms of (7), we have

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&\geq b|S| - (a-1)|T| \\
&> b(\frac{(a-1)n+b+bk}{a+b-1} - 1) - (a-1)(\frac{b(n-k-1)}{a+b-1} + 1) \\
&= bk - (a-1),
\end{aligned}$$

that is,

$$\delta_G(S,T) - bk \geq b|S| - (a-1)|T| - bk > -(a-1). \tag{8}$$

Since $b \equiv 0 \ (mod\ a-1)$ and $b|S| - (a-1)|T| - bk > -(a-1)$, we obtain

$$b|S| - (a-1)|T| - bk \geq 0.$$

Combining this with (1) and (8), we have

$$bk - 1 \geq \delta_G(S,T) \geq bk.$$

This is a contradiction.

Case 3. $h = 0$.

Set $I = \{x : x \in T, d_{G-S}(x) = 0\}$. Obviously, $I \neq \emptyset$ by $h = 0$, and I is independent. Thus, we have by the assumption of Theorem 5

$$|S| \geq |N_G(I)| > \frac{(a-1)n + |I| + bk - 1}{a+b-1}. \tag{9}$$

Using (9) and $|S| + |T| \leq n$, we obtain

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&\geq b|S| + |T| - |I| - a|T| \\
&= b|S| - (a-1)|T| - |I| \\
&\geq b|S| - (a-1)(n - |S|) - |I| \\
&= (a+b-1)|S| - (a-1)n - |I| \\
&> (a+b-1) \cdot \frac{(a-1)n + |I| + bk - 1}{a+b-1} - (a-1)n - |I| \\
&= bk - 1,
\end{aligned}$$

which contradicts (1).

From the argument above, we deduce the contradictions. Hence, G is an (a,b,k)-critical graph. This completes the proof of Theorem 5.

Remark 1. Let us show that the condition $\delta(G) > \frac{(a-1)n+b+bk}{a+b-1}$ in Theorem 5 can not be replaced by $\delta(G) \geq \frac{(a-1)n+b+bk}{a+b-1}$. Let $b > a \geq 2, k \geq 0$ be three integers such that $b+k$ is even and $b \equiv 0 \ (mod\ a-1)$. Let $n = \frac{(a+b-1)(2a+b-4)-a+1+(a+2b-1)k}{b} \geq \frac{(a+b-1)(2a+b-4)-a+1}{b} + k$. Since $b \equiv 0$ $(mod\ a-1)$, n is an integer. Put $l = \frac{2a+b-4+k}{2}$ and $m = n - 2l = n - (2a + b - 4 + k)$. Set $G = K_m \bigvee lK_2$. Obviously, $\delta(G) = m + 1 = \frac{(a-1)n+b+bk}{a+b-1}$ and $|N_G(X)| \geq m + |X| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ for every non-empty independent subset X of $V(G)$. Let $S = V(K_m) \subsetneq V(G)$, $T = V(lK_2) \subsetneq V(G)$, then $|S| = m \geq k$, $|T| = 2l$. Thus, we obtain

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&= b|S| + |T| - a|T| = b|S| - (a-1)|T| \\
&= b(n - (2a+b-4+k)) - (a-1)(2a+b-4+k) \\
&= bn - (a+b-1)(2a+b-4+k) \\
&= bk - (a-1) < bk.
\end{aligned}$$

By Lemma 2.1, G is not an (a,b,k)-critical graph. In the above sense, the result in Theorem 5 is best possible.

Remark 2.[12] Let us show that the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ in Theorem 5 can not be replaced by $|N_G(X)| \geq \frac{(a-1)n+|X|+bk-1}{a+b-1}$. Let $b > a \geq 2, k \geq 0$ be three integers with $b \equiv 0 \ (mod\ a-1)$, and let $n = \frac{(a+b-1)^2}{a-1} + k$. Clearly, n is an integer. Since $b \equiv 0 \ (mod\ a-1)$ and $b > a \geq 2$, we have $b \geq 2(a-1)$. Thus, we obtain $n = \frac{(a+b-1)^2}{a-1} + k \geq \frac{2(a+b-1)^2}{b} + k > \frac{(a+b-1)(2a+b-4)-a+1}{b} + k$. Let $G = K_{a+b+k} \bigvee ((a+b)K_1 \cup (\frac{(a+b-1)^2}{a-1} - 2(a+b))K_2)$. It is easy to see that $|N_G(X)| \geq \frac{(a-1)n+|X|+bk-1}{a+b-1}$ for every non-empty independent subset X of $V(G)$ ($|N_G(X)| = \frac{(a-1)n+|X|+bk-1}{a+b-1}$ for $X = V((a+b)K_1)$) and $\delta(G) = a+b+k > \frac{(a-1)n+b+bk}{a+b-1}$. Set $r = \frac{(a+b-1)^2}{a-1} - 2(a+b)$. Let $S = V(K_{a+b+k}) \subsetneq V(G)$ and $T = V((a+b)K_1 \cup \{x_1, x_2, \cdots, x_r\}) \subsetneq V(G)$, where $\{x_1, x_2, \cdots, x_r\} \subset V(rK_2)$. Then $|S| = a+b+k \geq k$, $|T| = a+b+r = \frac{(a+b-1)^2}{a-1} - (a+b)$. Thus, we get

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&= b(a+b+k) + \frac{(a+b-1)^2}{a-1} - 2(a+b) \\
&\quad -a(\frac{(a+b-1)^2}{a-1} - (a+b)) \\
&= bk - 1 < bk.
\end{aligned}$$

According to Lemma 2.1, G is not an (a,b,k)-critical graph. In the above sense, the condition $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$ in Theorem 5 is best possible.

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