On Multiparameter q-Noncentral Stirling and Bell Numbers

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Abstract

The notion of multiparameter q-noncentral Stirling numbers is introduced by means of a triangular recurrence relation. Some properties for these q-analogues such as vertical and horizontal recurrence relations, horizontal generating functions, explicit formula, orthogonality and inverse relations are established. Moreover, we introduce the multiparameter Bell numbers and Bell polynomials, their connection to factorial moments and their respective q-analogues.

Keywords: Stirling numbers, multiparameter noncentral Stirling numbers, Bell numbers, factorial moments, q-analogues.

1 Introduction

The classical Stirling numbers of the first and second kind, denoted by $s(n,k)$ and $S(n,k)$ respectively, were first introduced by James Stirling in [23]. The study of different generalizations and extensions of Stirling numbers were popular among many mathematicians. Among them were M. Koutras [16], A. Broder [2], B. El-Desouky [11], T. Cacoullous [3], L. Carlitz [5, 6], L. Hsu and P. Shiue [13], H. Yu [24] and the references therein. In [13], Hsu and Shiue defined a pair of inverse relations that unifies the

Stirling numbers of the first and second kind, and all other generalizations by the other mentioned authors. They used the symbols $\{S^1(n,k), S^2(n,k)\}$ to denote these Stirling numbers which some call the unified generalization of Stirling numbers. On the other hand, the weighted Stirling pair, denoted by $\{S(n,k;\alpha,\beta,t), S(n,k;\beta,\alpha,-t)\}$, by Yu in [24] are exactly the same pair of numbers with the ones used by Hsu and Shiue except for the manner in which these pairs were defined respectively.

Some generalizations of the classical Stirling numbers were based on the one discovered by Katriel [14] in 1974 that the classical Stirling numbers of the second kind $S(n,k)$ appeared as coefficients of the normal ordering expressions in the boson annihilation a and creation operator a^+ satisfying the commutation relation

$$aa^+ - a^+a = 1 \tag{1}$$

of the Weyl algebra. That is,

$$(a^+a)^n = \sum_{k=0}^{n} S(n,k)(a^+)^k a^k. \tag{2}$$

In line with this, a generalization was obtained by Blasiak et. al. [1] in 2004 which is given by

$$[(a^+)^r a^s)]^n = (a^+)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^+)^k a^k. \tag{3}$$

This was further generalized by Mendez et. al. [19] in 2005 as

$$(a^+)^{r_n} a^{s_n} \ldots (a^+)^{r_2} a^{s_2} (a^+)^{r_1} a^{s_1} = (a^+)^{d_n} \sum_{k=s}^{ns} S_{\bar{r},\bar{s}}(k)(a^+)^k a^k$$

where $\bar{r} = (r_1, r_2, \ldots, r_n)$, $\bar{s} = (s_1, s_2, \ldots, s_n)$ and $d_n = \sum_{i=1}^{n}(r_i - s_i)$. Recently, Mansour et. al. [18] generalized the commutation relation in (1) as

$$UV - VU = hV^s$$

where h is the Planck constant and obtained the following generalization of Stirling numbers

$$(VU)^n = \sum_{k=1}^{n} S_{s,h}(n,k) V^{s(n-k)+k} U^k.$$

Another generalizations of the classical Stirling numbers are the ones defined by El-Desouky in [11]. These are the multiparameter noncentral

Stirling numbers of the first and second kind which are defined as

$$(t)_n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(t/\alpha)_k, \tag{4}$$

$$(t/\alpha)_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k, \tag{5}$$

respectively, where $\bar{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ and

$$(t/\alpha)_n = \prod_{j=0}^{n-1}(t-\alpha_j), \ (t/\alpha)_0 = 1. \tag{6}$$

These numbers possessed some properties including the vertical generating function and the following triangular recurrence relations

$$s(n+1,k;\bar{\alpha}) = s(n,k-1;\bar{\alpha}) + (\alpha_k - n)s(n,k;\bar{\alpha}), \tag{7}$$

$$S(n,k;\bar{\alpha}) = S(n-1,k-1;\bar{\alpha}) + (k-\alpha_{n-1})S(n-1,k;\bar{\alpha}). \tag{8}$$

In addition to these, one can easily obtain the following explicit formula

$$S(n,k;\bar{\alpha}) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j/\alpha)_n. \tag{9}$$

by rewriting (5) into the following form

$$(k/\alpha)_n = \sum_{j=0}^{n} S(n,j;\bar{\alpha})j!\binom{k}{j}.$$

In a recent paper of Cakić et. al. [4], the pair $\{s(n,k;\bar{\alpha}), S(n,k;\bar{\alpha})\}$ was redefined in terms of the differential operators which is parallel to that in (2). The purpose of which is to establish explicit formulas for the numbers. With this definition, the multiparameter noncentral Stirling numbers of the second kind are seen to be closely related to those numbers defined in (3).

The study of q-analogues of some well-known identities and numbers has been the interest of several mathematicians for many years ago. For instance, a q-analogue of the known *binomial inversion formula*

$$f_k = \sum_{j=0}^{k}\binom{k}{j}g_j \Longleftrightarrow g_k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f_j \tag{10}$$

is given by

$$f_k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q g_j \Longleftrightarrow g_k = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{n-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f_j \tag{11}$$

which is known to be the *q-binomial inversion formula* (see [8]). The number

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \frac{[k]_{j,q}}{[j]_q!} = \prod_{i=1}^{j} \frac{q^{k-i+1}-1}{q^i-1}$$

is a q-analogue of the binomial coefficient which is also known as the *q-binomial coefficient* or *Gaussian polynomial* (with $q \neq 1$) where

$$[k]_{j,q} = \prod_{i=0}^{j-1} [k-i]_q = \frac{(q^k-1)(q^{k-1}-1)(q^{k-2}-1)\cdots(q^{k-j+1}-1)}{(q-1)^k}$$

is the *q-factorial of k of order j* and $[j]_q! = [1]_q[2]_q[3]_q \cdots [j]_q$ is the *q-factorial of j* with $[j]_q = \frac{q^j-1}{q-1}$ is the *q-real number.*

For the classical Stirling numbers, L. Carlitz [7] was the first one to define their q-analogues and was discussed thoroughly by H. Gould in [12]. In line with this, Corcino et. al. in [10] established a q-analogue for the unified Stirling numbers in [13] and this q-analogue was further investigated by Corcino and Barrientos in [9]. On the other hand, Katriel [15] defined a q-analogue of $S(n,k)$ using the following normal ordering relation

$$(VU)^n = \sum_{k=0}^{n} S_q(n,k) V^k U^k$$

such that $UV - qVU = 1$. This was generalized further in [20][22][21] as

$$(V^r U^s)^n = V^{n(r-s)} \sum_{k=0}^{n} S_q^{r,s}(n,k) V^k U^k \tag{12}$$

and

$$V^{r_n} U^{s_n} V^{r_{n-1}} U^{s_{n-1}} \ldots V^{r_1} U^{s_1} = V^{d_n} \sum_{k=s_1}^{|s|} S^q_{\mathbf{r},\mathbf{s}}(n,k) V^k U^k \tag{13}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $d_n = \sum_{i=1}^{n} (r_i - s_i)$ such that, in the paper by Schork [22], the q-bosonic operators a_q^+ and a_q in [15] are being used to replace V and U in (12), respectively, while, in the paper by Mendez and Rodriguez [20], the operators X and D_q are being used to replace V and U in (13), respectively, where

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

and the q-analogue $S^q_{\mathbf{r},\mathbf{s}}(n,k)$ satisfies the following explicit formula

$$S^q_{\mathbf{r},\mathbf{s}}(n,k) = \frac{1}{[k]!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{n} [j + d_{i-1}]_{s_i, q}. \tag{14}$$

In this paper, we define a q-analogue for the multiparameter noncentral Stirling numbers $s(n,k;\bar{\alpha})$ and $S(n,k;\bar{\alpha})$ by means of triangular recurrence relations. Some properties such as vertical and horizontal recurrence relations, horizontal generating functions, explicit form, orthogonality and inverse relations for these q-analogues are established as well as some additional properties for the classical case. The study of *multiparameter Bell numbers* and *Bell polynomials* with their respective q-analogues are also introduced.

2 Multiparameter q-Noncentral Stirling Numbers of the First Kind

For simplicity, we use $[x] = [x]_q$ to denote the q-real number x in the following sections. To attain our objectives, there is a need to define some new identities which are useful in the sequel. Now, as a q-analogue for (6), the *multiparameter q-generalized factorial* will be defined as

$$([t]/[\alpha])_{q,n} = \prod_{j=0}^{n-1} ([t] - [\alpha_j]), \tag{15}$$

where $([t]/[\alpha])_{q,0} = 1$. It is also known that the transition of an expression to its q-analogue is not unique. Hence, we may define another q-analogue for the *falling factorial of t of order n* given by

$$(t)_n = t(t-1)(t-2)\cdots(t-n+1), \ (t)_0 = 1.$$

That is, we have

$$\langle [t] \rangle_n = [t]([t]-[1])([t]-[2])\cdots([t]-[n-1]), \tag{16}$$

where $\langle [t] \rangle_0 = 1$. Since it can be shown that $([t]/[\alpha])_{q,n} \to (t/\alpha)_n$ and $\langle [t] \rangle_n \to (t)_n$ as $q \to 1$, the two previously defined identities are proper q-analogues of their respective classical identities.

Definition 2.1. Let n and k be non-negative integers,

$$[\bar{\alpha}] = ([\alpha_0], [\alpha_1], \ldots, [\alpha_{n-1}])$$

where $[\alpha_0] < [\alpha_1] < \ldots < [\alpha_{n-1}]$ and the numbers α_j, $j = 0, 1, 2, \ldots, n-1$, be real numbers. A q-analogue of the numbers $s(n,k;\bar{\alpha})$, denoted by $s_q[n,k;[\bar{\alpha}]]$, is defined by

$$s_q[n+1,k;[\bar{\alpha}]] = s_q[n,k-1;[\bar{\alpha}]] + ([\alpha_k] - [n])\, s_q[n,k;[\bar{\alpha}]] \;\; for \;\; k \geq 1, \quad (17)$$

where $s_q[0,0;[\bar{\alpha}]] = 1$ and $s_q[n,k;[\bar{\alpha}]] = 0$ for $k > h$. We call $s_q[n,k;[\bar{\alpha}]]$ as *multiparameter q-noncentral Stirling numbers of the first kind.*

It is easy to verify that as $q \to 1$, $([\alpha_k] - [n]) \to (\alpha_k - n)$. Hence, the recurrence relation in (17) becomes the recurrence relation in (7), that is, the numbers $s_q[n,k;[\bar{\alpha}]]$ can be considered as a q-analogue of the numbers $s(n,k;\bar{\alpha})$. We note that the definition of the numbers $s_q[n,k;[\bar{\alpha}]]$ is in the form of a triangular recurrence relation. In the following theorem, we present a vertical recurrence relation for the numbers $s_q[n,k;[\bar{\alpha}]]$.

Theorem 2.2. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the first kind $s_q[n,k;[\bar{\alpha}]]$ satisfy the following vertical recurrence relation:*

$$s_q[n+1,k+1;[\bar{\alpha}]] = \sum_{j=k}^{n} \frac{\langle[\alpha_{k+1}]\rangle_{n+1}}{\langle[\alpha_{k+1}]\rangle_{j+1}} s_q[j,k;[\bar{\alpha}]]. \quad (18)$$

Proof. By repeated application of (17), we can easily obtain (18). □

As a consequence to this theorem, we have the following corollary which contains a vertical recurrence relation for the multiparameter noncentral Stirling numbers of the first kind. This can easily be obtained by taking the limit of (18) as $q \to 1$.

Corollary 2.3. *The vertical recurrence relation for the multiparameter noncentral Stirling numbers of the first kind is given by*

$$s(n+1,k+1;\bar{\alpha}) = \sum_{j=k}^{n} \frac{(\alpha_{k+1})_{n+1}}{(\alpha_{k+1})_{j+1}} s(j,k;\bar{\alpha}), \quad (19)$$

where $(\alpha_{k+1})_{n+1}$ is the falling factorial of α_{k+1} of order $n+1$.

Using the notation,

$$\{-[n]/[\alpha]\}_k = \prod_{m=0}^{k-1} (-[n] + [\alpha_m]), \quad (20)$$

with initial condition that $\{-[n]/[\alpha]\}_0 = 1$, we will now state the horizontal recurrence relation for the numbers $s_q[n,k;[\bar{\alpha}]]$.

Theorem 2.4. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the first kind $s_q[n,k;[\bar{\alpha}]]$ satisfy the following horizontal recurrence relation:*

$$s_q[n,k;[\bar{\alpha}]] = \sum_{j=0}^{n-k}(-1)^j \frac{\{-[n]/[\alpha]\}_{k+j+1}}{\{-[n]/[\alpha]\}_{k+1}} s_q[n+1,k+j+1;[\bar{\alpha}]]. \qquad (21)$$

Proof. Evaluating the right-hand side of (21) using (17) and reindexing the sum, we obtain the left-hand side of (21). □

If we take the limit of (21) as $q \to 1$, we have the following corollary

Corollary 2.5. *The horizontal recurrence relation for the multiparameter noncentral Stirling numbers of the first kind is given by*

$$s(n,k;\bar{\alpha}) = \sum_{j=0}^{n-k}(-1)^j \frac{\{-n/\alpha\}_{k+j+1}}{\{-n/\alpha\}_{k+1}} s(n+1,k+j+1;\bar{\alpha}), \qquad (22)$$

where $\{-n/\alpha\}_{k+j+1} = \prod_{m=0}^{k-1}(-n+\alpha_m)$.

The next theorem presents the horizontal generating function for the numbers $s_q[n,k;[\bar{\alpha}]]$. This is necessary in establishing the orthogonality and the inverse relations of $s_q[n,k;[\bar{\alpha}]]$ and $S_q[n,k;[\bar{\alpha}]]$.

Theorem 2.6. *Let t be a real number and n a non-negative integer. The multiparameter q-noncentral Stirling numbers of the first kind $s_q[n,k;[\bar{\alpha}]]$ satisfy the following horizontal generating function:*

$$\langle [t] \rangle_n = \sum_{k=0}^{n} s_q[n,k;[\bar{\alpha}]]([t]/[\alpha])_{q,k}. \qquad (23)$$

Proof. We prove this by induction on n. Clearly, (23) holds when $n = 0$. Now, assume that (23) also holds for $n > 0$. Then by Definition 2.1,

$$\begin{aligned}
&\sum_{k=0}^{n+1} s_q[n+1,k;[\bar{\alpha}]]([t]/[\alpha])_{q,k} \\
&= \sum_{k=0}^{n} \left\{ s_q[n,k;[\bar{\alpha}]]\,([t]-[\alpha_k]) + ([\alpha_k]-[n])\, s_q[n,k;[\bar{\alpha}]] \right\} ([t]/[\alpha])_{q,k} \\
&= ([t]-[n]) \sum_{k=0}^{n} s_q[n,k;[\bar{\alpha}]]([t]/[\alpha])_{q,k} \\
&= ([t]-[n])\,\langle [t] \rangle_n \\
&= \langle [t] \rangle_{n+1}\,.
\end{aligned}$$

This completes the proof. □

Again, when $q \to 1$, we obtain the following generating function.

Corollary 2.7. *The multiparameter noncentral Stirling numbers of the first kind satisfy the following generating function*

$$(t)_n = \sum_{k=0}^{n} s(n,k;\bar{\alpha})(t/\alpha)_k. \tag{24}$$

This generating function is useful in obtaining the orthogonality relation of multiparameter noncentral Stirling numbers. It is usually being paired with the corresponding generating function for $S(n,k;\bar{\alpha})$. However, the said orthogonality relation will be obtained in the next section as a limiting case, when $q \to 1$, of the orthogonality relation for multiparameter q-noncentral Stirling numbers.

3 Multiparameter q-Noncentral Stirling Numbers of the Second Kind

Definition 3.1. Let n and k be non-negative integers,

$$[\bar{\alpha}] = ([\alpha_0],[\alpha_1],\ldots,[\alpha_{n-1}])$$

where $[\alpha_0] < [\alpha_1] < \ldots < [\alpha_{n-1}]$ and the numbers α_j, $j = 0,1,2,\ldots,n-1$, be real numbers. A q-analogue of the numbers $S(n,k;\bar{\alpha})$, denoted by $S_q[n,k;[\bar{\alpha}]]$, is defined by the triangular recurrence relation

$$S_q[n,k;[\bar{\alpha}]] = S_q[n-1,k-1;[\bar{\alpha}]] + ([k]-[\alpha_{n-1}])\,S_q[n-1,k;[\bar{\alpha}]], \tag{25}$$

where $S_q[0,0;[\bar{\alpha}]] = 1$ and $S_q[n,k;[\bar{\alpha}]] = 0$ for $k > h$. We call $S_q[n,k;[\bar{\alpha}]]$ as *multiparameter q-noncentral Stirling numbers of the second kind.*

It is easily verified that by repeated application of (25), we obtain the following theorem.

Theorem 3.2. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the second kind $S_q[n,k;[\bar{\alpha}]]$ satisfy the following vertical recurrence relation:*

$$S_q[n+1,k+1;[\bar{\alpha}]] = \sum_{j=k}^{n} \frac{([k+1]/[\alpha])_{n+1}}{([k+1]/[\alpha])_{j+1}} S_q[j,k;[\bar{\alpha}]]. \tag{26}$$

Consequently, if we take the limit of (26) as $q \to 1$, we have the following corollary.

Corollary 3.3. *The vertical recurrence relation for the multiparameter Stirling numbers of the second kind is given by*

$$S(n+1,k+1;\bar{\alpha}) = \sum_{j=k}^{n} \frac{(k+1/\alpha)_{n+1}}{(k+1/\alpha)_{j+1}} S(j,k;\bar{\alpha}). \tag{27}$$

The notation

$$\{-[\alpha_n]\}_k = \prod_{m=0}^{k-1} (-[\alpha] + [m]) \tag{28}$$

will be used in the following theorem.

Theorem 3.4. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the second kind $S_q[n,k;[\bar{\alpha}]]$ satisfy the following horizontal recurrence relation:*

$$S_q[n,k;[\bar{\alpha}]] = \sum_{j=0}^{n-k} (-1)^j \frac{\{-[\alpha_n]\}_{k+j+1}}{\{-[\alpha_n]\}_{k+1}} s_q[n+1,k+j+1;[\bar{\alpha}]]. \tag{29}$$

Proof. The proof is established by simply evaluating the right-hand side of (29) using (25) to obtain $S_q[n,k;[\bar{\alpha}]]$. □

If we take the limit of (29) as $q \to 1$, we have

Corollary 3.5. *The horizontal recurrence relation for the multiparameter noncentral Stirling numbers of the second kind is given by*

$$S(n,k;\bar{\alpha}) = \sum_{j=0}^{n-k} (-1)^j \frac{\langle -\alpha_n \rangle_{k+j+1}}{\langle -\alpha_n \rangle_{k+1}} S(n+1,k+j+1;\bar{\alpha}). \tag{30}$$

where $\langle -\alpha_n \rangle_{k+j+1} = \prod_{m=0}^{k+j}(-\alpha + m)$ is the rising factorial of $-\alpha$ of order $k+j+1$.

The following horizontal generating function is essential in obtaining the explicit formula for the numbers $S_q[n,k;[\bar{\alpha}]]$ as well as in the establishment of the orthogonality and the inverse relations of $s_q[n,k;[\bar{\alpha}]]$ and $S_q[n,k;[\bar{\alpha}]]$.

Theorem 3.6. *Let t be a real number and n a non-negative integer. The multiparameter q-noncentral Stirling numbers of the second kind $S_q[n,k;[\bar{\alpha}]]$ satisfy the horizontal generating function*

$$([t]/[\alpha])_{q,n} = \sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]] \, \langle [t] \rangle_k \, . \tag{31}$$

Proof. (By induction on n). Clearly, (31) is true for $n = 0$. Now, assume that (31) is also true for $n > 0$. Then by Definition 3.1,

$$\begin{aligned}
&\sum_{k=0}^{n+1} S_q[n+1,k;[\bar{\alpha}]]\,\langle[t]\rangle_k \\
&= \sum_{k=0}^{n} \{S_q[n,k;[\bar{\alpha}]]\,([t]-[k]) + ([k]-[\alpha_n])\,S_q[n,k;[\bar{\alpha}]]\}\,\langle[t]\rangle_k \\
&= ([t]-[\alpha_{n-1}]) \sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]]\,\langle[t]\rangle_k \\
&= ([t]-[\alpha_n])\,([t]/[\alpha])_{q,n} \\
&= ([t]/[\alpha])_{q,n+1}.
\end{aligned}$$

Hence, the proof is finished. □

Taking $q \to 1$, we obtain the corresponding generating function for $S(n,k;\bar{\alpha})$.

Corollary 3.7. *The multiparameter noncentral Stirling numbers of the second kind satisfy the following generating function*

$$(t/\alpha)_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k. \tag{32}$$

Theorem 3.8. *For nonnegative integers n and k, the q-analogue $S_q[n,k;[\bar{\alpha}]]$ satisfy the explicit formula*

$$S_q[n,k;[\bar{\alpha}]] = \frac{1}{\langle[k]\rangle_k} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q ([j]/[\alpha])_{q,n}. \tag{33}$$

Proof. From Theorem 3.6,

$$\begin{aligned}
([k]/[\alpha])_{q,n} &= \sum_{j=0}^{n} S_q[n,j;[\bar{\alpha}]]\,\langle[k]\rangle_j \\
&= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \left\{ \frac{S_q[n,j;[\bar{\alpha}]]\,\langle[k]\rangle_j}{\begin{bmatrix} k \\ j \end{bmatrix}_q} \right\}.
\end{aligned}$$

Applying the q-binomial inversion formula in (11) gives us

$$\frac{S_q[n,k;[\bar{\alpha}]]\,\langle[k]\rangle_k}{\begin{bmatrix} k \\ k \end{bmatrix}_q} = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q ([j]/[\alpha])_{q,n}.$$

Simplifying this equation yields (33). □

Remark 3.9. The explicit formula in Theorem 3.8 is analogous to that in (14).

Remark 3.10. Since the following limits hold:

$$\lim_{q\to 1} \langle [k] \rangle_k = k!, \quad \lim_{q\to 1} \begin{bmatrix} k \\ j \end{bmatrix}_q = \binom{k}{j}, \quad \lim_{q\to 1} ([j]/[\alpha])_{q,n} = (j/\alpha)_n,$$

then, by making use of (9), we have

$$\begin{aligned} \lim_{q\to 1} S_q[n,k;[\bar{\alpha}]] &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j/\alpha)_n \\ &= S(n,k;\bar{\alpha}). \end{aligned}$$

This implies that $S_q[n,k;[\bar{\alpha}]]$ is a proper q-analogue of $S(n,k;\bar{\alpha})$.

Note that the identity from Theorem 2.6 can be written as

$$\langle [t] \rangle_k = \sum_{m=0}^{k} s_q[k,m;[\bar{\alpha}]]([t]/[\alpha])_{q,m}.$$

Combining this with the identity in Theorem 3.6 gives us

$$\begin{aligned} ([t]/[\alpha])_{q,n} &= \sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]] \sum_{m=0}^{k} s_q[k,m;[\bar{\alpha}]]([t]/[\alpha])_{q,m} \\ &= \sum_{m=0}^{n}\left\{\sum_{k=m}^{n} S_q[n,k;[\bar{\alpha}]] s_q[k,m;[\bar{\alpha}]]\right\}([t]/[\alpha])_{q,m}. \end{aligned}$$

Comparing the coefficients of $([t]/[\alpha])_{q,m}$ gives us

$$\sum_{k=m}^{n} S_q[n,k;[\bar{\alpha}]] s_q[k,m;[\bar{\alpha}]] = \begin{cases} 0, & if\ m \neq n \\ 1, & if\ m = n \end{cases} = \delta_{mn}.$$

The symbol δ_{mn} is called the *Kronecker delta*. Similarly, we have

$$\sum_{k=m}^{n} s_q[n,k;[\bar{\alpha}]] S_q[k,m;[\bar{\alpha}]] = \delta_{mn}.$$

Hence, the following theorem holds

Theorem 3.11. *For non-negative integers m, n and k, the multiparameter q-noncentral Stirling numbers of the first and second kind satisfy the following orthogonality relation:*

$$\sum_{k=m}^{n} S_q[n,k;[\bar{\alpha}]] s_q[k,m;[\bar{\alpha}]] = \sum_{k=m}^{n} s_q[n,k;[\bar{\alpha}]] S_q[k,m;[\bar{\alpha}]] = \delta_{mn}. \quad (34)$$

When $q \to 1$, we obtain the following orthogonality relation for the multiparameter noncentral Stirling numbers.

Corollary 3.12. *For non-negative integers m, n and k, the multiparameter noncentral Stirling numbers of the first and second kind satisfy the following orthogonality relation:*

$$\sum_{k=m}^{n} S(n,k;\bar{\alpha})s(k,m;\bar{\alpha}) = \sum_{k=m}^{n} s(n,k;\bar{\alpha})S(k,m;\bar{\alpha}) = \delta_{mn}. \qquad (35)$$

Remark 3.13. Note that this orthogonality relation can also be obtained using (24) and (32) by following the same argument above.

Now, using the orthogonality relation in Theorem 3.11, we have the following inverse relation.

Theorem 3.14. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the first and Second kind satisfy the following inverse relation:*

$$f_n = \sum_{k=0}^{n} s_q[n,k;[\bar{\alpha}]]g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]]f_k. \qquad (36)$$

Proof. If the condition

$$f_n = \sum_{k=0}^{n} s_q[n,k;[\bar{\alpha}]]g_k$$

holds, then

$$\begin{aligned}\sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]]f_k &= \sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]] \sum_{m=0}^{k} s_q[k,m;[\bar{\alpha}]]g_m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} S_q[n,k;[\bar{\alpha}]]s_q[k,m;[\bar{\alpha}]] \right\} g_m.\end{aligned}$$

By Theorem 3.11, we have

$$\begin{aligned}\sum_{k=0}^{n} S_q[n,k;[\bar{\alpha}]]f_k &= \sum_{m=0}^{n} \delta_{mn}g_m \\ &= g_n.\end{aligned}$$

The converse can be shown similarly. □

The following theorem can easily be deduced from Theorem 3.14.

Theorem 3.15. *For non-negative integers n and k, the multiparameter q-noncentral Stirling numbers of the first and Second kind satisfy the following inverse relation:*

$$f_k = \sum_{n\geq k} s_q[n,k;[\bar{\alpha}]]g_n \Longleftrightarrow g_k = \sum_{n\geq k} S_q[n,k;[\bar{\alpha}]]f_n. \tag{37}$$

Remark 3.16. When $q \to 1$, we get the corresponding inverse relations for multiparameter noncentral Stirling numbers:

$$f_n = \sum_{k=0}^{n} s(n,k;(\bar{\alpha})g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})f_k. \tag{38}$$

and

$$f_k = \sum_{n\geq k} s(n,k;\bar{\alpha})g_n \Longleftrightarrow g_k = \sum_{n\geq k} S(n,k;\bar{\alpha})f_n. \tag{39}$$

4 Multiparameter q-Noncentral Bell numbers

The classical Bell numbers B_n is defined as the sum of Stirling numbers of the second kind (see [8] for the properties). That is

$$B_n = \sum_{k=0}^{n} S(n,k). \tag{40}$$

The numbers B_n can be expressed exlicitly as

$$B_n = e^{-1}\sum_{i\geq 0}\frac{i^n}{i!}. \tag{41}$$

Similarly, the polynomials

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k \tag{42}$$

is known as the Bell polynomials and is given by the formula

$$B_n(x) = e^{-x}\sum_{i\geq 0}\frac{i^n}{i!}x^i. \tag{43}$$

Parallel to (40) and (42), we establish a multiparameter noncentral version for the numbers B_n and polynomials $B_n(x)$, respectively. For this purpose, we have the following definition.

Definition 4.1. Let n and k be non-negative integers,

$$\bar{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$$

where $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$ and the numbers α_j, $j = 0, 1, 2, \dots, n-1$, be real numbers. The *multiparameter noncentral Bell numbers*, denoted by $B_n(\bar{\alpha})$, is defined by

$$B_n(\bar{\alpha}) = \sum_{k=0}^{n} S(n, k; \bar{\alpha}) \tag{44}$$

and the multiparameter noncentral Bell polynomials, denoted by $B_n(\bar{\alpha}; x)$, by

$$B_n(\bar{\alpha}; x) = \sum_{k=0}^{n} S(n, k; \bar{\alpha}) x^k. \tag{45}$$

In the following theorem, we present an explicit form for the numbers $B_n(\bar{\alpha})$ and polynomials $B_n(\bar{\alpha}; x)$, respectively.

Theorem 4.2. *For non-negative integers n and k, where $n \geq k$, the polynomials $B_n(\bar{\alpha}; x)$ satisfy the following explicit formulas*

$$B_n(\bar{\alpha}; x) = e^{-x} \sum_{j \geq 0} \frac{(j/\alpha)_n}{j!} x^j. \tag{46}$$

Consequently, when $x = 1$, the numbers $B_n(\bar{\alpha})$ satisfy

$$B_n(\bar{\alpha}; 1) = B_n(\bar{\alpha}) = e^{-1} \sum_{j \geq 0} \frac{(j/\alpha)_n}{j!}. \tag{47}$$

Proof. Combining the explicit formula for $S(n, k; \bar{\alpha})$ in (9) with (45) gives us

$$\begin{aligned} B_n(\bar{\alpha}; x) &= \sum_{k=0}^{n} S(n, k; \bar{\alpha}) x^k \\ &= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j} (j/\alpha)_n x^k}{j!(k-j)!}. \end{aligned}$$

Reindexing the summations and further simplification results to

$$\begin{aligned} B_n(\bar{\alpha}; x) &= \sum_{j \geq 0} \sum_{i \geq 0} \frac{(-1)^i (j/\alpha)_n x^{i+j}}{j! i!} \\ &= \left(\sum_{i \geq 0} \frac{(-x)^i}{i!} \right) \left(\sum_{j \geq 0} \frac{(j/\alpha)_n x^j}{j!} \right) \\ &= e^{-x} \sum_{j \geq 0} \frac{(j/\alpha)_n}{j!} x^j. \end{aligned}$$

Equation (47) is obtained by letting $x = 1$ in the above equation. □

It is known that if X is a Poisson random variable with mean λ, then the n^{th} factorial moment of X, denoted by $E_\lambda[(X)_n]$, is given by

$$E_\lambda[(X)_n] = \lambda^n$$

and the n^{th} moment of X, denoted by $E_\lambda[X^n]$, satisfy

$$E_\lambda[X^n] = B_n(\lambda),$$

where $B_n(\lambda)$ is the Bell polynomials in (42).

Now, considering X to be a Poisson random variable with mean λ, we may write the relation in (5) as

$$(X/\alpha)_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(X)_k.$$

Taking the expected value of both sides of this equation yields

$$\begin{aligned} E_\lambda[(X/\alpha)_n] &= \sum_{k=0}^{n} S(n,k;\bar{\alpha})E_\lambda[(X)_k] \\ &= \sum_{k=0}^{n} S(n,k;\bar{\alpha})\lambda^k \\ &= B_n(\bar{\alpha};\lambda). \end{aligned}$$

This result is embodied in the following theorem.

Theorem 4.3. *If X is a Poisson random variable with mean λ, then the following factorial moment holds:*

$$E_\lambda[(X/\alpha)_n] = e^{-\lambda}\sum_{j\geq 0}\frac{(j/\alpha)_n\lambda^j}{j!}. \tag{48}$$

Note that if $\alpha_i = 0$ for $i = 0, 1, \ldots, n-1$, then

$$\begin{aligned} E_\lambda[(X/\alpha)_n] &= e^{-\lambda}\sum_{j\geq 0}\frac{j^n\lambda^j}{j!} \\ &= E_\lambda[X^n]. \end{aligned}$$

This makes our factorial moment $E_\lambda[(X/\alpha)_n]$ a generalization of the ordinary n^{th} moment $E_\lambda[X^n]$. It is also worth mentioning that the factorial moment in (48) is related to the generalized factorial moments of a Poisson random variable X with mean λ given by the pair

$$E_\lambda[(\beta X+\gamma|\ \alpha)_n]=e^{-\lambda}\sum_{i\geq 0}\frac{(i\beta+\gamma|\ \alpha)_n}{i!}\lambda^i \tag{49}$$

$$E_\lambda[(\alpha X-\gamma|\ \beta)_n]=e^{-\lambda}\sum_{i\geq 0}\frac{(i\alpha-\gamma|\ \beta)_n}{i!}\lambda^i, \tag{50}$$

for real or complex numbers α, β and γ. These are the identities established by the authors in [17] where the expression

$$(\beta X+\gamma|\ \alpha)_n=\prod_{j=0}^{n-1}(\beta X+\gamma-j\alpha)$$

is the generalized falling factorial of $\beta X+\gamma$ of increment α. That is, if we let $\beta=1$, $\gamma=0$ and $\bar{\alpha}=(0\cdot\alpha,1\cdot\alpha,\cdots,(n-1)\cdot\alpha)$ in (49), we can verify that

$$E_\lambda[(\beta X+\gamma|\ \alpha)_n]=E_\lambda[(X/\alpha)_n].$$

In the following definition, we present q-analogues for the identities in Definition 4.1, respectively.

Definition 4.4. Let n and k be non-negative integers,

$$[\bar{\alpha}]=([\alpha_0],[\alpha_1],\ldots,[\alpha_{n-1}])$$

where $[\alpha_0]<[\alpha_1]<\ldots<[\alpha_{n-1}]$ and the numbers α_j, $j=0,1,2,\ldots,n-1$, be real numbers. A q-analogue for the numbers $B_n(\bar{\alpha})$, denoted by $B_{q,n}[\bar{\alpha}]$, is defined as

$$B_{n,q}[\bar{\alpha}]=\sum_{k=0}^{n}q^{\binom{k}{2}}S_q[n,k;[\bar{\alpha}]] \tag{51}$$

and for the polynomials $B_n(\bar{\alpha};x)$, denoted by $B_{q,n}[\bar{\alpha};x]$, as

$$B_{n,q}[\bar{\alpha};x]=\sum_{k=0}^{n}q^{\binom{k}{2}}S_q[n,k;[\bar{\alpha}]]x^k. \tag{52}$$

We call $B_{n,q}[\bar{\alpha}]$ and $B_{n,q}[\bar{\alpha};x]$ as *multiparameter q-noncentral Bell numbers* and *polynomials*, respectively.

Applying the explicit formula in Theorem 3.8 to (52) yields

$$\begin{aligned}B_{n,q}[\bar{\alpha};x] &= \sum_{k=0}^{n}q^{\binom{k}{2}}S_q[n,k;[\bar{\alpha}]]x^k\\ &= \sum_{k=0}^{n}\frac{q^{\binom{k}{2}}}{\langle[k]\rangle_k}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q([j]/[\alpha])_{q,n}x^k.\end{aligned}$$

Reindexing the summations and since

$$\langle [k] \rangle_k = [k]_q! q^{\binom{k}{2}},$$

we have

$$\begin{aligned} B_{n,q}[\bar{\alpha}; x] &= \sum_{j\geq 0}\sum_{i\geq 0} \frac{(-1)^i q^{\binom{i}{2}} ([j]/[\alpha])_{q,n} x^{i+j}}{[j]_q! [i]_q!} \\ &= \left(\sum_{j\geq 0} \frac{([j]/[\alpha])_{q,n}}{[j]_q!} x^j \right) \left(\sum_{i\geq 0} q^{\binom{i}{2}} \frac{(-x)^i}{[i]_q!} \right). \end{aligned}$$

Applying the known q-exponential function

$$\hat{e}_q(t) = \sum_{i\geq 0} q^{\binom{i}{2}} \frac{t^i}{[i]_q!}$$

and we obtain the following explicit formula

$$B_{n,q}[\bar{\alpha}; x] = \hat{e}_q(-x) \sum_{j\geq 0} \frac{([j]/[\alpha])_{q,n}}{[j]_q!} x^j.$$

Moreover, when $x = 1$, we have

$$B_{n,q}[\bar{\alpha}] = \hat{e}_q(-1) \sum_{j\geq 0} \frac{([j]/[\alpha])_{q,n}}{[j]_q!}.$$

These results are embodied in the following theorem.

Theorem 4.5. *The explicit forms for the q-analogues $B_{n,q}[\bar{\alpha}; x]$ and $B_{n,q}[\bar{\alpha}]$ are given respectively by*

$$B_{n,q}[\bar{\alpha}; x] = \hat{e}_q(-x) \sum_{j\geq 0} \frac{([j]/[\alpha])_{q,n}}{[j]_q!} x^j \tag{53}$$

and

$$B_{n,q}[\bar{\alpha}] = \hat{e}_q(-1) \sum_{j\geq 0} \frac{([j]/[\alpha])_{q,n}}{[j]_q!}. \tag{54}$$

The following is easily observed.

Remark 4.6.

$$\begin{aligned} \lim_{q\to 1} B_{n,q}[\bar{\alpha}; x] &= B_n(\bar{\alpha}; x), \\ \lim_{q\to 1} B_{n,q}[\bar{\alpha}] &= B_n(\bar{\alpha}). \end{aligned}$$

5 Summary and Recommendations

This study defined q-analogues of both kinds of multiparameter noncentral Stirling numbers by means of triangular recurrence relations. Horizontal and vertical recurrence relations, explicit formula, generating functions and the orthogonality and inverse relations of the q-analogues have been established. The horizontal and vertical recurrence relations and the generating functions were derived by proper application of the triangular recurrence relations, while the explicit formula and the orthogonality and inverse relations were obtained using the generating functions and the q-binomial inversion formula. Moreover, a multiparameter noncentral Bell numbers were defined in terms of the second kind multiparameter noncentral Stirling numbers and certain Dobinski-type formula has been obtained. Consequently, a factorial moment of Poisson random variable was established using the Dobinski-type formula. Furthermore, a kind of q-analogue of multiparameter noncentral Bell numbers is defined as the sum of multiparameter q-noncentral Stirling numbers of the second kind.

As a continuation to this research work, the authors recommend the following interesting problems:

1. To define the multiparameter q-noncentral Stirling numbers in terms of q-differential operators which is parallel to the one being done in [4].

2. To establish a q-analogue of multiparameter noncentral Stirling numbers in line with relation (13) satisfying

$$UV - qVU = hV^s.$$

   This can possibly lead us to a more general combinatorial interpretation of the generalized Stirling numbers in terms of rook theory.

3. To establish a q-factorial moment of Poisson random variable by means of the multiparameter q-noncentral Bell numbers.

4. To obtain the Hankel transform or determinant of the sequence of multiparameter noncentral Bell numbers as well as the sequence of multiparameter q-noncentral Bell numbers.

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