The Large numbers of 2-independent sets in extra-free forests

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Abstract

A 2-independent set in a graph G is a subset I of the vertices such that the distance between any two vertices of I in G is at least three. The number of 2-independent sets of a graph G is denoted by $i_2(G)$. For a for a forest F, $i_2(F - e) > i_2(F)$ for each edge e of F. Hence we exclude all forests having isolated vertices. A forest is said to be extra-free if it has no isolated vertex. In this paper, we determine the k-th largest number of 2-independent sets among all extra-free forest of order $n \geq 2$, where $k = 1, 2$ and 3. Extremal graphs achieving these values are also given.

1 Introduction

Throughout this paper, graphs will be finite, simple and loopless. A subset I of $V(G)$ is said to be a *2-independent set* of G such that the distance between any two vertices of I in G is at least three. The set of all 2-independent sets of a graph G is denoted by $\mathscr{I}_2(G)$ and its cardinality by $i_2(G)$. The study of the number of independent sets in a graph has a rich history. The maximum weight k-independent set problem has applications in many practical problems like k-machines job scheduling problem, k-colourable subgraph problem, VLSI design layout and routing problem [3]. Kong and Zhao [4] showed that it is finding a maximum k-independent set of a graph is NP-hard, even when restricted to regular bipartite graphs [5]. W. Duckworth [2] present a simple, yet efficient, heuristic for finding a large 2-independent set of cubic graphs.

For a graph G, we refer to $V(G)$ and $E(G)$ as the vertex set and the edge set, respectively. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. The *(open) neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G, and the *close neighborhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \cup_{x\in A} N_G(x)$ and $N_G[A] = \cup_{x\in A} N_G[x]$. A vertex x is said to be a *leaf* if $|N_G(x)| = 1$. An edge e call *end-edge* if it is incident to a leaf. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ by removing all vertices in A and all edges incident to these vertices. For a subset $F \subseteq E(G)$, the *deletion of F from G* is the graph $G - F$ obtained from G by deleting all edges of F. The *star-product* of two disjoint graphs G_1 and G_2 is the graph $G_1 * G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2\}$, where v_i is a vertex with maximum degree in G_i for $i = 1, 2$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. Denote by P_n a *n-path* with n vertices. For notation and terminology in graphs we follow [1] in general.

In this paper, we determine the k-th largest number of 2-independent sets among all graphs of order n, where $k = 1, 2$ and 3. Extremal graphs achieving these values are also given. The following useful lemmas are needed in this paper.

Lemma 1.1 *If $G_1, G_2, \ldots, G_k$ are components of a graph G, then $i_2(G) = \Pi_{j=1}^{k} i_2(G_j)$.*

Lemma 1.2 *Suppose G is a connected graph and xy is an edge of G, then $i_2(G - xy) \geq i_2(G)$. If G is a tree, then $i_2(G - xy) > i_2(G)$.*

Proof. Note that every 2-independent set of G is also a 2-independent set of $G - xy$. This means that $i_2(G - xy) \geq i_2(G)$. If G is a tree, then $G - xy$ is disconnected. Thus $I = \{x, y\} \in \mathscr{I}_2(G - xy)$ and $I \notin \mathscr{I}_2(G)$. Hence $i_2(G - xy) > i_2(G)$. □

Lemma 1.3 *Suppose T is a tree of order $n \geq 4$ and $e \in E(T)$ is not an end-edge of T, then $T - e$ is an extra-free forest.*

2 Main Results

In this section, we determine the k-th largest number of 2-independent sets among all extra-free forests of order $n \geq 2$, where $k = 1, 2$ and 3. We will prove the following three results.

Theorem 2.1 *If F is an extra-free forest of order $n \geq 2$, then $i_2(F) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} 3^{\frac{n}{2}}, & \text{if } n \geq 2 \text{ is even;} \\ 4 \cdot 3^{\frac{n-3}{2}}, & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

The equality holds if and only if $F = F_1(n)$, where

$$F_1(n) = \begin{cases} \frac{n}{2} P_2, & \text{if } n \geq 2 \text{ is even;} \\ P_3 \cup \frac{n-3}{2} P_2, & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

Theorem 2.2 *If F is an extra-free forest of order $n \geq 4$ having $F \neq F_1(n)$, then $i_2(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 6 \cdot 3^{\frac{n-4}{2}}, & \text{if } n \geq 4 \text{ is even;} \\ 9 \cdot 3^{\frac{n-5}{2}}, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

The equality holds if and only if $F = F_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even;} \\ P_5 \cup \frac{n-5}{2} P_2, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

Theorem 2.3 *If F is an extra-free forest of order $n \geq 6$ different from $F_1(n)$ and $F_2(n)$, then $i_2(F) \leq f_3(n)$, where*

$$f_3(n) = \begin{cases} 16 \cdot 3^{\frac{n-6}{2}}, & \text{if } n \geq 6 \text{ is even;} \\ 24 \cdot 3^{\frac{n-7}{2}}, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The equality holds if and only if $F = F_3(n)$, where

$$F_3(n) = \begin{cases} 2P_3 \cup \frac{n-6}{2} P_2, & \text{if } n \geq 6 \text{ is even;} \\ P_3 \cup P_4 \cup \frac{n-7}{2} P_2 \text{ or } (P_3 * P_2) \cup \frac{n-5}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

We prove Theorem 2.1 by establishing the following two lemmas.

Lemma 2.4 *If F is an extra-free forest of even order $n \geq 2$, then $i_2(F) \leq 3^{\frac{n}{2}}$, and the equality holding if and only if $F = \frac{n}{2} P_2$.*

Proof. Let F be an extra-free forest of even order $n \geq 2$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 3^{\frac{n}{2}} = i_2(\frac{n}{2} P_2)$. Suppose there exists a component H which is not a star. Let $e \in E(H)$ not an end-edge of H and $F' = F - e$. By Lemma 1.2, $i_2(F') > i_2(F)$. Thus F' is an extra-free forest of even order $n \geq 2$ having $i_2(F') > i_2(F)$. This contradicts the hypothesis of F, so every component of F is a star. Let $F = H_1 \cup H_2 \cup \cdots \cup H_k$. Then $3^{\frac{n}{2}} \leq i_2(F) = \Pi_{i=1}^{k}(|H_i| + 1) \leq 3^{\frac{n}{2}}$. The equalities hold and $|H_i| = 2$ for all i. That is $F = \frac{n}{2} P_2$. □

Lemma 2.5 *If F is an extra-free forest of odd order $n \geq 3$, then $i_2(F) \leq 4 \cdot 3^{\frac{n-3}{2}}$. The equality holds if and only if $F = P_3 \cup \frac{n-3}{2} P_2$.*

Proof. Let F be an extra-free forest of odd order $n \geq 3$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 4 \cdot 3^{\frac{n-3}{2}} = i_2(P_3 \cup \frac{n-3}{2} P_2)$. Since n is odd, there exists an odd component H, say $|H| = m \geq 3$. Then $F - H$ is an extra-free forest of even order $n - m$, by Lemma 2.4, $i_2(F - H) \leq 3^{\frac{n-m}{2}}$. If H is not a star, then $m \geq 5$. Let $e \in E(H)$ be not an end-edge of H. By Lemma

1.2, $F' = F - e$ is an extra-free forest of odd order having $i_2(F') > i_2(F)$. This contradicts the hypothesis of F, so H is a star of odd order $m \geq 3$. Thus $4 \cdot 3^{\frac{n-3}{2}} \leq i_2(F) = i(H) \cdot i(F - H) = (m+1) \cdot 3^{\frac{n-m}{2}} \leq 4 \cdot 3^{\frac{n-3}{2}}$. The equalities hold. Then $m = 3$ and $F - H = \frac{n-3}{2}P_2$. That is $F = P_3 \cup \frac{n-3}{2}P_2$. □

Theorem 2.1 now follow from Lemma 2.4 and Lemma 2.5.

We prove Theorem 2.2 by establishing the following four lemmas.

Lemma 2.6 *If F is an extra-free forest of even order $n \geq 6$ having odd components, then $i_2(F) \leq 16 \cdot 3^{\frac{n-6}{2}}$. The equality holds if and only if $F = 2P_3 \cup \frac{n-6}{2}P_2$.*

Proof. Let F be an extra-free forest of even order $n \geq 6$ having odd components such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 16 \cdot 3^{\frac{n-6}{2}} = i_2(2P_3 \cup \frac{n-6}{2}P_2)$. Let H be an odd component of F. Then H and $F - H$ are extra-free forests of odd order, by Lemma 2.5, $16 \cdot 3^{\frac{n-6}{2}} \leq i_2(F) = i(H) \cdot i(F - H) \leq (4 \cdot 3^{\frac{m-3}{2}}) \cdot (4 \cdot 3^{\frac{n-m-3}{2}}) = 16 \cdot 3^{\frac{n-6}{2}}$. The equalities hold and $F = 2P_3 \cup \frac{n-6}{2}P_2$. □

Lemma 2.7 *If F is an extra-free forest of odd order $n \geq 9$ having at least three odd components, then $i_2(F) \leq 64 \cdot 3^{\frac{n-9}{2}}$. The equality holds if and only if $F = 3P_3 \cup \frac{n-9}{2}P_2$.*

Proof. Let F be an extra-free forest of odd order $n \geq 9$ having at least three odd components such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 64 \cdot 3^{\frac{n-9}{2}} = i_2(3P_3 \cup \frac{n-9}{2}P_2)$. If H_1, H_2 and H_3 are three odd components of F, then $|H_1 \cup H_2| = k$ is even and $|F - H_1 - H_2|$ is odd. By Lemma 2.5 and Lemma 2.6, $64 \cdot 3^{\frac{n-9}{2}} \leq i_2(F) = i(H_1 \cup H_2) \cdot i(F - H_1 - H_2) \leq (16 \cdot 3^{\frac{k-6}{2}})(4 \cdot 3^{\frac{n-k-3}{2}}) = 64 \cdot 3^{\frac{n-9}{2}}$. The equalities hold and $H_1 = H_2 = H_3 = P_3$. So $F = 3P_3 \cup \frac{n-9}{2}P_2$. □

Lemma 2.8 *If F is an extra-free forest of even order $n \geq 4$ having $F \neq \frac{n}{2}P_2$, then $i_2(F) \leq 6 \cdot 3^{\frac{n-4}{2}}$. The equality holds if and only if $F = P_4 \cup \frac{n-4}{2}P_2$.*

Proof. Let F be an extra-free forest of even order $n \geq 4$ having $F \neq \frac{n}{2}P_2$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 6 \cdot 3^{\frac{n-4}{2}} = i_2(P_4 \cup \frac{n-4}{2}P_2) > 16 \cdot 3^{\frac{n-6}{2}}$. By Lemma 2.6, F has no odd component. Let H be a largest component of F, say $|H| = m \geq 4$. If F is a star, then $i_2(H) = m + 1$ and, by Lemma 2.4, $6 \cdot 3^{\frac{n-4}{2}} \leq i_2(F) = i(H) \cdot i_2(F - H) \leq (m+1) \cdot 2^{\frac{n-m}{2}} \leq 5 \cdot 3^{\frac{n-4}{2}} < 6 \cdot 3^{\frac{n-4}{2}}$. This is a contradiction, thus H is not a star. Let $e \in E(H)$ be not an end-edge of H and $F' = F - e$. By Lemma 1.2, $i_2(F') > i_2(F)$. Then F' is an extra-free forest of even order n and $i_2(F') > i(F)$. By the hypothesis of F, then $H - e = 2P_2$. That is $F = P_4 \cup \frac{n-4}{2}P_2$. □

Lemma 2.9 *If F is an extra-free forest of odd order $n \geq 5$ having $F \neq P_3 \cup \frac{n-3}{2}P_2$, then $i_2(F) \leq 9 \cdot 3^{\frac{n-5}{2}}$. The equality holds if and only if $F = P_5 \cup \frac{n-5}{2}P_2$.*

Proof. Let F be an extra-free forest of odd order $n \geq 5$ having $F \neq P_3 \cup \frac{n-3}{2}P_2$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 9 \cdot 3^{\frac{n-5}{2}} = i_2(P_5 \cup \frac{n-5}{2}P_2) > 64 \cdot 3^{\frac{n-9}{2}}$. By Lemma 2.7, F has only one odd component H, say $|H| = m$.

Let $F' = F - H$. If $F' \neq \frac{n-m}{2}P_2$, by Lemma 2.5 and Lemma 2.8, $9 \cdot 3^{\frac{n-5}{2}} \leq i_2(F) = i(H) \cdot i_2(F-H) \leq (4 \cdot 3^{\frac{m-3}{2}})(6 \cdot 3^{\frac{n-m-4}{2}}) = 24 \cdot 3^{\frac{n-7}{2}} < 9 \cdot 3^{\frac{n-5}{2}}$. This is a contradiction, so $F' = \frac{n-m}{2}P_2$. Since $F \neq P_3 \cup \frac{n-3}{2}P_2$, this imply that $m \geq 5$. If H is a star, then $i_2(H) = m+1$ and, by Lemma 2.4, $9 \cdot 3^{\frac{n-5}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F-H) \leq (m+1) \cdot 3^{\frac{n-m}{2}} \leq 6 \cdot 3^{\frac{n-5}{2}} < 9 \cdot 3^{\frac{n-5}{2}}$. This is a contradiction, so H is not a star. Let $e \in E(H)$ be not an end-edge and $F' = F - e$. By Lemma 1.2, $i_2(F') > i_2(F)$. Then F' is an extra-free forest of odd order n and $i_2(F') > i(F)$. By the hypothesis of F, then $H - e = P_3 \cup P_2$. That is $F = P_5 \cup \frac{n-5}{2}P_2$. □

Theorem 2.2 now follow from Lemma 2.8 and Lemma 2.9.

We prove Theorem 2.3 by establishing the following two lemmas.

Lemma 2.10 *If F is an extra-free forest of even order $n \geq 6$ different from $\frac{n}{2}P_2$ and $P_4 \cup \frac{n-4}{2}P_2$, then $i_2(F) \leq 16 \cdot 3^{\frac{n-6}{2}}$. The equality holds if and only if $F = 2P_3 \cup \frac{n-6}{2}P_2$.*

Proof. Let F be an extra-free forest of even order $n \geq 6$ different from $\frac{n}{2}P_2$ and $P_4 \cup \frac{n-4}{2}P_2$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 16 \cdot 3^{\frac{n-6}{2}} = i_2(2P_3 \cup \frac{n-6}{2}P_2)$.

Claim. F have odd components.

Suppose that F has no odd component. Let H be a largest component of F, say $|H| = m \geq 4$, and $F' = F - H$. If $F' \neq \frac{n-m}{2}P_2$, by Lemma 2.8, then $16 \cdot 3^{\frac{n-6}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') \leq (6 \cdot 3^{\frac{m-4}{2}})(6 \cdot 3^{\frac{n-m-4}{2}}) = 36 \cdot 3^{\frac{n-8}{2}} < 16 \cdot 3^{\frac{n-6}{2}}$. This is a contradiction, so $F = \frac{n-m}{2}P_2$. If H is a star, then $i_2(H) = m+1$ and $16 \cdot 3^{\frac{n-6}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') = (m+1) \cdot 3^{\frac{n-m}{2}} \leq 5 \cdot 3^{\frac{n-4}{2}} < 16 \cdot 3^{\frac{n-6}{2}}$. This is a contradiction, so H is not a star. Let $e \in E(H)$ be not an end-edge and $F^* = F - e = (H-e) \cup \frac{n-m}{2}P_2$. Note that $i_2(F^*) > i_2(F)$. By the hypothesis of F, we can see that $H - e = P_4 \cup P_2$. So $H = P_6$ or $P_4 * P_2$, and $i_2(H) \leq \max\{i_2(P_6), i_2(P_3 * P_2)\} = 13$. Then $16 \cdot 3^{\frac{n-6}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') \leq 13 \cdot 3^{\frac{n-6}{2}} < 16 \cdot 3^{\frac{n-6}{2}}$. This a contradiction, so F have odd components.

By Claim and Lemma 2.6, we obtain that $i_2(F) = 16 \cdot 3^{\frac{n-6}{2}}$ and $F = 2P_3 \cup \frac{n-6}{2}P_2$. □

Lemma 2.11 *If F is an extra-free forest of odd order $n \geq 7$ different from $P_3 \cup \frac{n-3}{2}P_2$ and $P_5 \cup \frac{n-5}{2}P_2$, then $i_2(F) \leq 24 \cdot 3^{\frac{n-7}{2}}$. The equality holds if and only if $F = P_3 \cup P_4 \cup \frac{n-7}{2}P_2$ or $(P_3 * P_2) \cup \frac{n-5}{2}P_2$.*

Proof. Let F be an extra-free forest of odd order $n \geq 7$ different from $P_3 \cup \frac{n-3}{2}P_2$ and $P_5 \cup \frac{n-5}{2}P_2$ such that $i_2(F)$ is as large as possible. Then $i_2(F) \geq 24 \cdot 3^{\frac{n-7}{2}} = i_2(P_3 \cup P_4 \cup \frac{n-7}{2}P_2) = i_2((P_3 * P_2) \cup \frac{n-5}{2}P_2) > 64 \cdot 3^{\frac{n-9}{2}}$. By Lemma 2.7, so F has the only one odd component H, say $|H| = m \geq 3$. Let $F' = F - H$. If $F' \neq \frac{n-m}{2}P_2$ and $F' \neq P_4 \cup \frac{n-m-4}{2}P_2$, by Lemma 2.10, then $i_2(F') \leq 16 \cdot 3^{\frac{n-m-6}{2}}$. Thus, by Lemma 2.5, $24 \cdot 3^{\frac{n-7}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') \leq (4 \cdot 3^{\frac{m-3}{2}})(16 \cdot 3^{\frac{n-m-6}{2}}) = 64 \cdot 3^{\frac{n-9}{2}} < 24 \cdot 3^{\frac{n-7}{2}}$. This is a contradiction, so $F' = \frac{n-m}{2}P_2$ or $F' = P_4 \cup \frac{n-m-4}{2}P_2$.
Case 1. $F' = P_4 \cup \frac{n-m-4}{2}P_2$. By Lemma 2.5, $24 \cdot 3^{\frac{n-7}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') \leq (4 \cdot 3^{\frac{m-3}{2}})(6 \cdot 3^{\frac{n-m-4}{2}}) = 24 \cdot 3^{\frac{n-7}{2}}$. So the equalities hold and $F = P_3 \cup P_4 \cup \frac{n-7}{2}P_2$.
Case 2. $F' = \frac{n-m}{2}P_2$. If H is a star, then $m \geq 5$ and $i_2(H) = m + 1$ and $24 \cdot 3^{\frac{n-7}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') \leq (m+1)(3^{\frac{n-m}{2}}) \leq 6 \cdot 3^{\frac{n-5}{2}} < 24 \cdot 3^{\frac{n-7}{2}}$. This a contradiction, so H is not a star. Let $e \in E(H)$ be not an end-edge of H and $F^* = F - e$. Then F^* is an extra-free forest of odd order $n \geq 7$ and $i_2(F^*) > i_2(F)$. By the hypothesis of F, we can obtain that $H - e = P_3 \cup P_2$. Note that $F \neq P_5 \cup \frac{n-5}{2}P_2$. This implies that $H = P_3 * P_2$ and $i_2(H) = 8$. Thus $24 \cdot 3^{\frac{n-7}{2}} \leq i_2(F) = i_2(H) \cdot i_2(F') = 24 \cdot 3^{\frac{n-7}{2}}$. So the equalities hold and $F = (P_3 * P_2) \cup \frac{n-5}{2}P_2$. □

Theorem 2.3 now follow from Lemmas 2.10 and 2.11.

References

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