ON THE TREE DOMINATION NUMBER OF A RANDOM GRAPH

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ABSTRACT. We prove a two-point concentration for the tree domination number of the random graph $G_{n,p}$ provided p is constant or $p \to 0$ sufficiently slow.

1. INTRODUCTION

Let G be a graph with vertex set $[n] = \{1, \ldots, n\}$ and let $\phi \neq S \subseteq [n]$. S is called a dominating set of G iff for every vertex $u \in [n] - S$ there is a vertex $v \in S$ such that $uv \in E(G)$. S is a *tree dominating set* of G iff S is a dominating set and the induced subgraph $G[S]$ is a tree. The domination number $\gamma(G)$ is the smallest integer s such that there exists a dominating set of G of cardinality s. The *tree domination number* $\gamma_T(G)$ is the smallest integer s such that there exists a tree dominating set of G of cardinality s where we set $\gamma_T(G) = 0$ if no such S exists.

Unlike dominating sets there exist graphs without tree dominating sets. Obviously a disconnected graph has no tree dominating set. For each $i \in [3]$, take a connected graph G_i, choose one vertex $v_i \in V(G_i)$, and form the graph G by adding the edges v_1v_2, v_1v_3 and v_2v_3 to $G_1 \cup G_2 \cup G_3$. Then G is a connected graph. Any connected dominating set S of G must contain v_1, v_2, v_3. Then $G[S]$ contains the cycle $v_1v_2v_3$ and is not a tree. This example is readily extendable to further infinite families of connected graphs with no tree dominating sets. Chen et al. [4] provide other examples of graphs with no tree dominating set and ask if it is possible to classify all graphs that do. Rautenbach [10] showed the question is NP-complete even for regular graphs. Chengye et al. [5] found the tree domination number for certain generalized Peterson Graphs. A natural question arises: "With what probability do graphs with tree dominating sets occur?". In this paper we answer this question and prove a much stronger result.

$\mathcal{G}(n,p)$ is the set of all graphs $G_{n,p}$ with vertex set $[n]$ and edges chosen independently with probability $0 \leq p = p(n) \leq 1$. Hence for each $G_{n,p}$, $\Pr(G_{n,p}) = p^{e(G_{n,p})}(1-p)^{\binom{n}{2}-e(G_{n,p})}$. For a given graph property A we say A occurs *asymptotically almost surely* (a.a.s.) if $\Pr(G_{n,p}$ has property $A) \to 1$ as $n \to \infty$.

A graph parameter ξ is said to have a *two-point concentration* iff a.a.s. $\xi(G_{n,p})$ is precisely one of two values (depending on p, n). The first author

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[6] proved a two-point concentration for the strong matching number of the random graph. Weber [11] showed if $p = \frac{1}{2}$ then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 1$ or $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 2$. Godbole and Weiland [8] refined this result showing if p is constant or $p \to 0$ sufficiently slow then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 2$. Bonato and Wang [3] showed for constant p that a.a.s the independent domination number $i(G_{n,p})$ belongs to an interval (depending on p, n) whose width goes to infinity with n. The current authors [7] recently proved if p is constant or $p \to 0$ sufficiently slow then a.a.s. $i(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$. See [2, 9] for further examples.

In this paper we show if p is constant or $p \to 0$ sufficiently slow then a.a.s. $\gamma_T(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ (\ln n + c) + \log_b 2) \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ (\ln n + c) + \log_b 2) \rfloor + 2$, where $c = 2 \ln \left(pe/q^{\frac{3}{2}} \right)$ if $p > (1-p)^{\frac{3}{2}}/e$ and $c = 0$ otherwise. We then briefly discuss how this result relates to other variations of the domination number. Our notation and terminology follow [2, 9].

2. Two-Point Concentration

Throughout this section we will use p as the probability an edge exists in $G = G_{n,p}$ and $q = 1 - p$ as the probability an edge does not exist in G. For convenience, we define $b = \frac{1}{q}$. We will also make extensive use of the inequalities

$$1 - x \leq \exp\{-x\}, \ x \in (-\infty, \infty) \tag{1}$$

$$1 - x \geq \exp\left\{\frac{-x}{1-x}\right\}, \ x \in [0, 1). \tag{2}$$

We begin by defining the random variable X_k as the number of tree dominating sets of cardinality k in G and Y_s as the number of tree dominating sets of size s or less. Clearly $Y_s = \sum_{k=1}^{s} X_k$. Using Cayley's Formula and the fact that a tree of order k has $k-1$ edges,

$$\mathrm{E}(X_k) = \binom{n}{k} k^{k-2} p^{k-1} q^{\binom{k}{2}-k+1} (1 - q^k)^{n-k}$$

and by linearity of expectation

$$\mathrm{E}(Y_s) = \sum_{k=1}^{s} \binom{n}{k} k^{k-2} p^{k-1} q^{\binom{k}{2}-k+1} (1 - q^k)^{n-k}.$$

We now state our first lemma.

Lemma 2.1. *Let p be constant or going to 0 such that $p \geq e\ln^2 n/n$. If $s = \lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor$, then $\mathrm{E}(Y_s) \to 0$, where $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$ if $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 0$ otherwise.*

Proof. Lemma 2 of [8] states the expected number of dominating sets of size less than or equal to $r = \lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor$ goes to 0 if $p \geq \frac{e\ln^2 n}{n}$. Since every tree dominating set is a dominating set it is clear $E(Y_r) \to 0$ as $n \to \infty$. It remains to show,

$$\sum_{k=r+1}^{s} \mathrm{E}(X_k) \to 0.$$

Using Stirling's inequality and (1),

$$\begin{aligned}
\mathrm{E}(X_k) &= \binom{n}{k} k^{k-2} p^{k-1} q^{\binom{k}{2}-k+1}(1-q^k)^{n-k} \\
&\leq \frac{(ne)^k}{k^2} p^{k-1} q^{\binom{k}{2}-k+1} \exp\left\{-(n-k)q^k\right\} \\
&\leq \exp\left\{k\ln n + k + \frac{k}{2}\ln\left(\frac{1}{q}\right) + (k-1)\ln\left(\frac{p}{q}\right) - nq^k - \frac{k^2}{2}\ln\left(\frac{1}{q}\right)\right\} \\
&= \exp\left\{k\ln n + k\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right) - nq^k - \frac{k^2}{2}\ln\left(\frac{1}{q}\right) - \ln\left(\frac{p}{q}\right)\right\} \\
&:= \exp\{f(k)\}.
\end{aligned}$$

Now

$$f'(k) = \frac{d}{dk}f(k) = \ln n + \ln\left(\frac{pe}{q^{\frac{3}{2}}}\right) + n\ln\left(\frac{1}{q}\right)q^k - k\ln\left(\frac{1}{q}\right).$$

Note $f'(k)$ is decreasing for all positive value of k and $f'(\log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2) \geq 0$ for sufficiently large n. So for sufficiently large n, we have $f(k)$ increasing for all $k \leq \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2$. Hence, setting $k = \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2$ we have

$$\begin{aligned}
\mathrm{E}(Y_s) \leq& (k-r)\exp\{f(k)\} \\
\leq& (k-r)\exp\left\{k\ln n + k\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right) - nq^k - \frac{k^2}{2}\ln\left(\frac{1}{q}\right) - \ln\left(\frac{p}{q}\right)\right\} \\
\leq& \log_b\left(\frac{2\log_b n}{\log_b n + c}\right)\exp\left\{\left(\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right) - \frac{c}{2}\right)\log_b n\right. \\
&+ \left(\ln 2 - \ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)\right)\log_b(\log_b n\ (\ln n + c)) \\
&- \frac{1}{2}\log_b{}^2(\log_b n\ (\ln n + c))\ln\left(\frac{1}{q}\right)
\end{aligned}$$

$$-\left(\frac{\ln 2}{2} - \ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)\right)\log_b 2 - \ln\left(\frac{p}{q}\right)\Bigg\}.$$

If $p \le \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 0$, or, if $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$, then $\mathrm{E}(Y_s) \to 0$ as $n \to \infty$. Here $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ if $p > .242526$ and $p < \frac{(1-p)^{\frac{3}{2}}}{e}$ if $p < .242525$. This calculation can be made arbitrarily precise but the actual value of p such that $p = \frac{(1-p)^{\frac{3}{2}}}{e}$ is irrational. If $p = p(n) \to 0$ then $p < \frac{(1-p)^{\frac{3}{2}}}{e}$ for sufficiently large n. It is now important to note we assumed k is to be positive between r and s, this condition is met so long as $p \ge (e\ln^2 n)/n$. This is the same requirement as Lemma 2 of [8]. □

Lemma 2.2. *Let p be constant or going to 0 such that $\frac{p^2}{96} \ge \frac{\ln(\frac{1}{p})}{\ln n}$. If $s = \lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 2$, then $\mathrm{E}(X_s) \to \infty$, where $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$ if $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 0$ otherwise.*

Proof. Using inequality (2), Stirling's Formula, and that $(n)_k = (1-o(1))n^k$ for $k^2 = o(n)$,

$$\begin{aligned}
\mathrm{E}(X_k) =& \binom{n}{k} k^{k-2}(1-q^{-k})^{n-k} p^{k-1} q^{\binom{k}{2}-k+1} \\
\ge& \binom{n}{k} k^{k-2}(1-q^{-k})^{n} p^{k-1} q^{\binom{k}{2}-k+1} \\
\ge& (1-o(1))\frac{n^k}{k!} k^{k-2}(1-q^{-k})^{n} p^{k-1} q^{\binom{k}{2}-k+1} \text{ (if } k^2 = o(n)) \\
\ge& (1-o(1))\frac{(ne)^k}{k^2\sqrt{2\pi k}}(1-q^{-k})^{n} p^{k-1} q^{\binom{k}{2}-k+1} \text{ (if } k \to \infty) \\
\ge& (1-o(1))\frac{(ne)^k}{k^2\sqrt{2\pi k}} \exp\left\{-\frac{nq^k}{1-q^k} - \binom{k}{2}\ln\left(\frac{1}{q}\right) + (k-1)\ln\left(\frac{p}{q}\right)\right\} \\
\ge& (1-o(1))\exp\Bigg\{k\ln n + \ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)k - \frac{nq^k}{1-q^k} - \frac{k^2}{2}\ln\left(\frac{1}{q}\right) - \ln\left(\frac{p}{q}\right) \\
& -\frac{5}{2}\ln k - \frac{1}{2}\ln(2\pi)\Bigg\} := f(k). \qquad (3)
\end{aligned}$$

The condition $k^2 = o(n)$ is satisfied if $p \gg \ln n/n^{\frac{1}{2}}$ and $k = \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 + \epsilon$, where $\epsilon > 0$ and c any constant. Analyzing the derivative $\frac{d}{dk}f(k)$ shows $f(k)$ is increasing as long as k is much smaller than nq^k which is true for large n when assuming the above mentioned condition. Substituting $k = s$ on the left and $k = \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 + \frac{1}{2}$ on the right in (3) it follows that for sufficiently large n

$$\begin{aligned}
&\mathrm{E}(X_s) \\
&\geq (1-o(1))\exp\left\{\frac{1}{2}\log_b n\ \ln n\ \left(1-\frac{q^{\frac{1}{2}}}{1-q^k}\right)+\log_b n\ \left(\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)-\frac{cq^{\frac{1}{2}}}{2(1-q^k)}\right)\right. \\
&\quad -\frac{5}{2}\ln(k)\ -\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)\log_b(\log_b n\ (\ln n+c))-\frac{1}{2}\ln\left(\frac{1}{q}\right)\log_b{}^2(\log_b n\ (\ln n+c)) \\
&\quad \left.+\ln\left(\frac{pe}{q}\right)\log_b 2-\frac{1}{2}\ln\left(\frac{p}{q}\right)-\frac{1}{8}\ln\left(\frac{1}{q}\right)-\frac{1}{2}\ln(2\pi)\right\}
\end{aligned}$$

which clearly goes to infinity with n when p is constant. If $p = p(n) \to 0$ then for n sufficiently large, $p(n) < .242525$ hence $c = 0$, and

$$\begin{aligned}
\mathrm{E}(X_s) \geq& (1-o(1))\exp\left\{\frac{1}{2}\log_b n\ \ln n\ \left(1-\frac{q^{\frac{1}{2}}}{1-q^k}\right)+\log_b n\ \ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)\right. \\
&-\frac{1}{2}\ln\left(\frac{1}{q}\right)\log_b{}^2(\log_b n\ \ln n)+\ln\left(\frac{pe}{q}\right)\log_b 2-\frac{1}{8}\ln\left(\frac{1}{q}\right)-\frac{5}{2}\ln(k) \\
&\left.-\frac{1}{2}\ln(2\pi)\right\} \geq (1-o(1))\exp\{A-B\}
\end{aligned}$$

where

$$\begin{aligned}
A =& \frac{1}{2}\log_b n \ln n\left(1-\frac{q^{\frac{1}{2}}}{1-q^k}\right)+\log_b n \\
B =& \log_b n\ \ln\left(\frac{1}{p}\right)+\frac{1}{2}\ln\left(\frac{1}{q}\right)\log_b{}^2(\log_b n\ \ln n)+\ln\left(\frac{1}{p}\right)\log_b 2 \\
&+\frac{1}{8}\ln\left(\frac{1}{q}\right)+\frac{5}{2}\ln(\log_b n)+\frac{1}{2}\ln(2\pi).
\end{aligned}$$

Note $\log_b n \sim \frac{\ln n}{p}$ and $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$, hence, $p \gg \frac{\log_b n \ln n}{n}$. Thus for n sufficiently large,

$$\begin{aligned}
A &= \frac{1}{2}\log_b n \ln n\left(1-\frac{q^{\frac{1}{2}}}{1-q^s}\right)+\log_b n \\
&= \frac{1}{2}\log_b n \ln n\left(1-\frac{q^{\frac{1}{2}}}{1-\frac{q^{\frac{1}{2}}\log_b n \ln n}{2n}}\right)+\log_b n \\
&\geq \frac{1}{2}\log_b n \ln n\left(1-\frac{q^{\frac{1}{2}}}{1-\frac{pq^{\frac{1}{2}}}{2}}\right)+\log_b n \\
&= \frac{1}{2}\log_b n \ln n\left(\frac{1-\frac{pq^{\frac{1}{2}}}{2}-q^{\frac{1}{2}}}{1-\frac{pq^{\frac{1}{2}}}{2}}\right)+\log_b n.
\end{aligned}$$

Using the inequality $\frac{x}{2} \le 1-(1-x)^{\frac{1}{2}}$ we obtain

$$A \ge \frac{p}{4}\log_b n \ln n \left(\frac{1-(1-p)^{\frac{1}{2}}}{1-\frac{pq^{\frac{1}{2}}}{2}}\right) + \log_b n \ge \frac{p^2}{8}\log_b n \ln n + \log_b n.$$

Define C as:

$$C = \frac{p^2 \log_b n \ln n}{8} + \log_b n. \tag{4}$$

We will now find p such that for n sufficiently large $C/12$ is larger than all terms in B. Hence,

$$\begin{aligned}(1-o(1))\exp\{A-B\} &\ge (1-o(1))\exp\{C-B\} \\ &\ge (1-o(1))\exp\{C/2\} \to \infty.\end{aligned} \tag{5}$$

It is obvious that the last four terms of B are dominated by the first or second so we will only compare the first and second terms to $C/12$. Comparing the first term,

$$\frac{C}{12} \ge \log_b n \ln\left(\frac{1}{p}\right)$$

if for sufficiently large n

$$\frac{p^2}{96} \ge \frac{\ln\left(\frac{1}{p}\right)}{\ln n}. \tag{6}$$

Comparing the second term,

$$\frac{C}{12} \ge \frac{1}{2}\log_b\left(\log_b n \ln n\right)\ln\left(\log_b n \ln n\right)$$

if for sufficiently large n

$$\frac{p^2}{96} \ge \frac{\ln^2\left(\frac{\ln^2 n}{p}\right)}{2\ln^2 n}. \tag{7}$$

Clearly (6) implies (7) and the assumption $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and the lemma is proved. □

Lemma 2.3. *Let p be constant or going to 0 such that $\frac{p^2}{96} \ge \frac{\ln\left(\frac{1}{p}\right)}{\ln n}$. If $s = \lfloor \log_b n - \log_b\left(\log_b n\ (\ln n + c)\right) + \log_b 2\rfloor + 2$, then*

$$\frac{\mathrm{Var}(X_s)}{\mathrm{E}^2(X_s)} \to 0$$

where $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$ if $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 0$ otherwise.

Proof. Recall $\mathrm{Var}(X_s) = \mathrm{E}(X_s^2) - \mathrm{E}^2(X_s)$ and

$$\mathrm{E}(X_s^2) = \mathrm{E}\left(\left(\sum_{S\subseteq[n],|S|=s} X_S\right)^2\right)$$

where for all $S \subseteq [n]$ and $G = G_{n,p}$

$$X_s(G) = \begin{cases} 1 & S \text{ is a tree dominating set of } G \\ 0 & \text{otherwise.} \end{cases}$$

Expanding the square and using linearity of expectation we have

$$\mathrm{E}(X_s^2) = \mathrm{E}(X_s) + \sum_{\substack{S,T\subseteq[n],|T|=|S|=s \\ S,T \text{ distinct}}} \mathrm{E}(X_S X_T).$$

Note

$$\begin{aligned} &\mathrm{E}(X_S X_T) \\ =&\Pr(S \text{ is a tree dominating set and } T \text{ is a tree dominating set}) \\ =&\Pr(S \text{ and } T \text{ are induced trees and } S \text{ and } T \text{ are dominating sets}) \\ \le&\Pr(S \text{ and } T \text{ are induced trees and } S \text{ and } T \text{ both dominate } (S\cup T)^C) \\ =&\Pr(S \text{ and } T \text{ are induced trees})\Pr(S \text{ and } T \text{ both dominate } (S\cup T)^C). \end{aligned}$$

Let $m = |S \cap T|$ then

$$\Pr(S \text{ and } T \text{ both dominate } (S\cup T)^C) = (1 - 2q^s + q^{2s-m})^{n-2s+m}.$$

Take two trees T_1 and T_2 of order s labeled with the vertices of S and T such that $V(T_1) \cap V(T_2) = S \cap T$, then

$$\Pr(S = T_1 \text{ and } T = T_2) = p^{2(s-1)-|E(T_1)\cap E(T_2)|} q^{2(\binom{s}{2}-s+1)-\binom{m}{2}+|E(T_1)\cap E(T_2)|}.$$

By Cayley's formula there are s^{s-2} ways to choose T_1 and at most s^{s-2} ways to choose T_2. Hence,

$$\Pr(S \text{ and } T \text{ are induced trees}) \le s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)-e(S\cap T)} q^{2\binom{s}{2}-\binom{m}{2}}$$

where $e(S \cap T)$ is the number of edges in $S \cap T$. Since S and T must be trees, $0 \le e(S\cap T) \le m-1$ and $e(S \cap T) = 0$ when $m = 0$. If $p \ge \frac{1}{2}$ then

$$\Pr(S \text{ and } T \text{ are induced trees}) \le s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} q^{2\binom{s}{2}-\binom{m}{2}}.$$

If $p \le \frac{1}{2}$,

$$\Pr(S \text{ and } T \text{ are induced trees}) \le s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} \left(\frac{p}{q}\right)^{-m+1} q^{2\binom{s}{2}-\binom{m}{2}}.$$

If $p \geq \frac{1}{2}$,

$$\operatorname{Var}(X_s) \leq \binom{n}{s}\binom{n-s}{s-0}\binom{s}{0} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} q^{2\binom{s}{2}} (1-2q^s+q^{2s})^{n-2s}$$
$$+ \binom{n}{s} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} \sum_{m=1}^{s-1} f_1(m) + \mathrm{E}(X_s) - \mathrm{E}^2(X_s)$$

where

$$f_1(m) = \binom{s}{m}\binom{n-s}{s-m} \left(1-2q^s+q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2}-\binom{m}{2}}.$$

If $p \leq \frac{1}{2}$,

$$\operatorname{Var}(X_s) \leq \binom{n}{s}\binom{n-s}{s-0}\binom{s}{0} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} q^{2\binom{s}{2}} (1-2q^s+q^{2s})^{n-2s}$$
$$+ \binom{n}{s} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} \sum_{m=1}^{s-1} f_2(m) + \mathrm{E}(X_s) - \mathrm{E}^2(X_s)$$

where

$$f_2(m) = \binom{s}{m}\binom{n-s}{s-m} \left(1-2q^s+q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2}-\binom{m}{2}} \left(\frac{p}{q}\right)^{-m+1}.$$

We write $s = \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 + \epsilon$ where $\epsilon = \epsilon(n) = \lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 2 - \log_b n + \log_b(\log_b n\ (\ln n + c)) + \log_b 2$ and observe that $1 < \epsilon \leq 2$.

It is immediately obvious for any s such that $\mathrm{E}(X_s) \to \infty$,

$$\mathrm{E}(X_s) = o(\mathrm{E}^2(X_s)).$$

Further

$$\binom{n}{s}\binom{n-s}{s-0}\binom{s}{0} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} q^{2\binom{s}{2}} (1-2q^s+q^{2s})^{n-2s} - \mathrm{E}^2(X_s)$$
$$\leq \mathrm{E}^2(X_s)\left((1-q^s)^{-4s}-1\right) \leq \mathrm{E}^2(X_s)\left(\exp\left\{\frac{4sq^s}{1-q^s}\right\}-1\right) \text{ (by (2))}$$

and

$$\left(\exp\left\{\frac{4sq^s}{1-q^s}\right\}-1\right) \to 0$$

as $n \to \infty$ provided $p \gg \frac{\ln^{\frac{3}{2}} n}{n^{\frac{1}{2}}}$.

To show

$$\binom{n}{s} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} \sum_{m=1}^{s-1} f_1(m) = o(\mathrm{E}^2(X_s))$$

and

$$\binom{n}{s} s^{2(s-2)} \left(\frac{p}{q}\right)^{2(s-1)} \sum_{m=1}^{s-1} f_2(m) = o(\mathrm{E}^2(X_s)),$$

we first note that for sufficiently large n,

$$\begin{aligned} f_1(m) &\le \binom{s}{m} \frac{n^{s-m}}{(s-m)!} \left(1 - 2q^s + q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2}-\binom{m}{2}} \\ &\le 2\binom{s}{m} \frac{n^{s-m}}{(s-m)!} \left(1 - 2q^s + q^{2s-m}\right)^{n} q^{2\binom{s}{2}-\binom{m}{2}} \\ &\le 2\binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}} \text{ (by (1))}, \end{aligned}$$

where the second inequality holds for $p \gg \frac{\ln^{\frac{3}{2}} n}{n^{\frac{1}{2}}}$. Similarly

$$f_2(m) \le 2\binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}} \left(\frac{p}{q}\right)^{-m+1}.$$

Define

$$\begin{aligned} g_1(m) &:= 2\binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}} \\ g_2(m) &:= 2\binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}} \left(\frac{p}{q}\right)^{-m+1}. \end{aligned}$$

Now consider the ratio of consecutive terms of $g_1(m)$ and $g_2(m)$. Let

$$\begin{aligned} h_1(m) &:= \frac{g_1(m+1)}{g_1(m)} = \frac{q^{-m}(s-m)^2}{n(m+1)} \exp\left\{npq^{2s-m-1}\right\} \\ h_2(m) &:= \frac{g_2(m+1)}{g_2(m)} = \frac{q^{-m}(s-m)^2}{n(m+1)} \left(\frac{p}{q}\right)^{-1} \exp\left\{npq^{2s-m-1}\right\}. \end{aligned}$$

We will show for $i = 1$ or $i = 2$, $h_i(m) \ge 1$ iff $m \ge m_0$ for some $m_0(n) \to \infty$, hence g_i is first decreasing and then increasing. Further we will show $g_i(1) \ge g_i(s-1)$, which implies $\sum_{m=1}^{s-1} f_i(m) \le s g_i(1)$. Observe

$$h_1(1) = \frac{(s-1)^2}{2qn} \exp\left\{npq^{2s-2}\right\} \le \frac{\log_b{}^2 n}{2qn} \exp\left\{\frac{p \log_b{}^2 n \,(\ln n + c)^2}{4nq^{-1+2\epsilon}}\right\} \to 0$$

if $p \gg \ln n / n^{\frac{1}{2}}$, and for sufficiently large n,

$$h_1(s-1) = \frac{q^{-s+1}}{ns} \exp\{npq^s\} \geq \frac{2q^{1-\epsilon}}{\log_b{}^2 n\ (\ln n + c)} \exp\left\{\frac{pq^\epsilon \log_b n\ (\ln n + c)}{2}\right\}$$
$$\geq 1$$

provided $p \neq 1 - o(1)$. By identical calculations for n sufficiently large, $h_2(1) \leq 1$ and $h_2(s-1) \geq 1$.

Further, $h_1(m) \geq 1$ iff

$$m \geq \log_b\left(\frac{4n}{p}\right) - 2\log_b(\log_b n\,(\ln n + c)) + \log_b\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right) + 2\epsilon - 1$$

and $h_2(m) \geq 1$ iff

$$m \geq \log_b\left(\frac{4n}{p}\right) - 2\log_b(\log_b n\,(\ln n + c)) + \log_b\left(\ln\left(\frac{np(m+1)q^{m-1}}{(s-m)^2}\right)\right) + 2\epsilon - 1.$$

Define

$$x_1(m) = \log_b\left(\frac{4n}{p}\right) + \log_b\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right) - 2\log_b(\log_b n\,(\ln n + c)) + 2\epsilon - 1$$
$$x_2(m) = \log_b\left(\frac{4n}{p}\right) + \log_b\left(\ln\left(\frac{np(m+1)q^{m-1}}{(s-m)^2}\right)\right) - 2\log_b(\log_b n\,(\ln n + c)) + 2\epsilon - 1.$$

Now

$$\frac{d}{dm}x_1(m) = \frac{\left(m^2 - \left(s - 1 - \frac{1}{\ln(\frac{1}{q})}\right)m - \left(1 - \frac{1}{\ln(\frac{1}{q})}\right)s + \frac{2}{\ln(\frac{1}{q})}\right)}{(m+1)(s-m)\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right)}$$

and the zeros of the numerator are:

$$\frac{s - 1 - \frac{1}{\ln(\frac{1}{q})} \pm \sqrt{\left(s - 1 - \frac{1}{\ln(\frac{1}{q})}\right)^2 + 4s\left(1 - \frac{1}{\ln(\frac{1}{q})}\right) - \frac{8}{\ln(\frac{1}{q})}}}{2}$$
$$= \frac{s - 1 - \frac{1}{\ln(\frac{1}{q})} \pm (s+1)\sqrt{\left(1 - \frac{3}{(s+1)\ln(\frac{1}{q})}\right)^2 - \frac{8}{(s+1)^2\ln^2(\frac{1}{q})}}}{2}.$$

Using Taylor Series with remainder about 0 it follows that if $0 \leq z \leq 3 - 2\sqrt{2}$, then for any y such that $|y| \leq z$

$$1 - 3y - \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}} \leq \sqrt{(1 - 3y)^2 - 8y^2} \leq 1 - 3y + \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}}.$$

Letting $y = z = \frac{1}{(s-1)\ln\left(\frac{1}{q}\right)}$, we then have

$$\frac{d}{dm}x_1(m) = \frac{\left(m + 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} - \delta\right)\left(m - s + \frac{2}{\ln\left(\frac{1}{q}\right)} + \delta\right)}{(m+1)(s-m)\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right)}$$

$$\text{where } |\delta| \leq \frac{8}{(s+1)\ln^2\left(\frac{1}{q}\right)\left(1 - \frac{6}{(s+1)\ln\left(\frac{1}{q}\right)} + \frac{1}{(s+1)^2\ln^2\left(\frac{1}{q}\right)}\right)^{\frac{3}{2}}}.$$

Thus $\delta = \Theta\left(\frac{1}{p \ln n}\right) \to 0$ as $n \to \infty$.

Hence on $(-\infty, -1)$ and $\left(\ln^{-1}(\frac{1}{q}) - 1 + \delta, s - 2\ln^{-1}(\frac{1}{q}) - \delta\right)$, $x_1(m)$ is decreasing, and on $\left(-1, \ln^{-1}(\frac{1}{q}) - 1 + \delta\right)$ and $\left(s - 2\ln^{-1}(\frac{1}{q}) - \delta, s\right)$, $x(m)$ is increasing. Thus $m_1 = \ln^{-1}(\frac{1}{q}) - 1 + \delta$ is a relative maximum and $m_2 = s - 2\ln^{-1}(\frac{1}{q}) - \delta$ is a relative minimum of $x_1(m)$.

Note $m_1 \in [1, s-1]$ iff $p \leq 1 - e^{-\frac{1}{2-\delta}}$ and $m_2 \in [1, s-1]$ iff $p \leq 1 - e^{-\frac{2}{1-\delta}}$. Also for n sufficiently large, $x_1(m)$ is continuous on $[1, s-1]$, for every $m \in [1, s-1]$ we have $x_1(m) \in [1, s-1]$, and $s - 1 > x_1(1) > x_1(s-1) > 1$.

If $p > 1 - e^{-\frac{2}{1-\delta}}$, then $x_1(m)$ has an absolute maximum at 1 and an absolute minimum at $s - 1$ on $[1, s-1]$. The above information and the intermediate value theorem imply there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x_1(m_0)$ and $x_1(m_0) > x_1(s-1)$.

If $1 - e^{-\frac{1}{2-\delta}} < p \leq 1 - e^{-\frac{2}{1-\delta}}$, then $x_1(m)$ has an absolute maximum at 1 and an absolute minimum at m_2 on $[1, s-1]$. The above information and the intermediate value theorem imply there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x_1(m_0)$. Further, one can show by iteration that $x_1(m_0) \geq x_1(s-1)$.

If $p \leq 1 - e^{-\frac{1}{2-\delta}}$, then $x_1(m)$ has an absolute maximum at m_1 and an absolute minimum at m_2 on $[1, s-1]$. The above information and the intermediate value theorem imply there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x_1(m_0)$. Further, one can show by iteration that $x_1(m_0) \geq x_1(s-1)$.

Thus, in any of the three cases, there exists a unique $m_0 \in [1, s-1]$ such that for each $m \geq m_0 = x_1(m_0)$ we have $m \geq x_1(m)$.

Also

$$\frac{d}{dm}x_2(m) = \frac{\left(m^2 - \left(s - 1 - \frac{1}{\ln(\frac{1}{q})}\right)m - \left(1 - \frac{1}{\ln(\frac{1}{q})}\right)s + \frac{2}{\ln(\frac{1}{q})}\right)}{(m+1)(s-m)\left(\ln\left(\frac{np(m+1)q^{m-1}}{(s-m)^2}\right)\right)},$$

and by identical calculations we see $x_2(m)$ is decreasing and increasing on the same intervals as $x_1(m)$. Thus the absolute extrema of $x_2(m)$ in $[1, s-1]$ are located at the same locations as the relative extrema of $x_1(m)$. Further, $x_2(1) > x_2(s-1)$. Hence there exists $m_0 \in [1, s-1]$ such that for all $m \geq m_0$ we have $m \geq x_2(m)$.

Now, for n sufficiently large

$$\ln\left(\frac{n(m_0+1)q^{m_0}}{(s-m_0)^2}\right) \geq \ln\left(nsq^{s-1}\right) \geq \ln\left(\frac{\log_b n\ (\ln n + c)s}{4q^{1-\epsilon}}\right)$$

which goes to infinity as n goes to infinity. Also, $\log_b\left(\frac{4n}{p}\right) \gg 2\log_b(\log_b n\ (\ln n + c))$ and $2\epsilon - 1$ is bounded, thus $m_0 \to \infty$. Therefore, if $i = 1$ or 2 $h_i(m) \geq 1$ iff $m \geq m_0 \to \infty$ as $n \to \infty$. So $m \geq m_0$ iff $h_i(m) \geq 1$.

It remains to show $g_i(1) \geq g_i(s-1)$ for $i = 1$ and 2. First note $g_1(1) = g_2(1)$ and $g_1(s-1) \leq g_2(s-1)$, so we only need show that $g_2(1) \geq g_2(s-1)$.

This last inequality is true iff

$$\frac{n^{s-1}}{(s-1)!}\exp\left\{nq^{2s-1}\right\} \geq n\exp\left\{nq^{s+1}\right\}q^{-\binom{s-1}{2}}\left(\frac{p}{q}\right)^{-s+2}$$

iff

$$\frac{n^{s-2}}{(s-1)!}p^{s-2}q^{\binom{s-1}{2}-s+2}\exp\left\{nq^{2s-1} - nq^{s+1}\right\} \geq 1$$

and for n sufficiently large

$$\begin{aligned}
&\frac{n^{s-2}}{(s-1)!}p^{s-2}q^{\binom{s-1}{2}-s+2}\exp\left\{nq^{2s-1} - nq^{s+1}\right\}\\
&\geq \frac{n^{s-2}}{s!}p^s q^{\frac{s^2}{2}}\exp\{-nq^{s+1}\}\\
&\geq \frac{n^{s-2}}{s^s}p^s q^{\frac{s^2}{2}}\exp\{-nq^{s+1}\}\\
&\geq \exp\left\{(s-2)\ln n - s\ln(s) - s\ln\left(\frac{1}{p}\right) - \frac{s^2}{2}\ln\left(\frac{1}{q}\right) - nq^{s+1}\right\}\\
&\geq \exp\left\{\frac{(1-q^{\epsilon+1})}{2}\log_b n\ \ln n - \left(2 + \frac{\epsilon}{2}\right)\ln n - \frac{1}{2}\log_b{}^2\left(\log_b n(\ln n + c)\right)\ln\left(\frac{1}{q}\right)\right.\\
&\qquad \left. - \left(\ln s + \ln\left(\frac{1}{p}\right) + \frac{cq^{\epsilon+1}}{2}\right)\log_b n\right\}\\
&\geq \exp\left\{\frac{p}{2}\log_b n\ \ln n - 3\ln n - \frac{1}{2}\log_b{}^2\left(\log_b n\ (\ln n + c)\right)\ln\left(\frac{1}{q}\right)\right.\\
&\qquad \left. - \left(\ln s + \ln\left(\frac{1}{p}\right) + \frac{cq^{\epsilon+1}}{2}\right)\log_b n\right\},
\end{aligned}$$

which goes to infinity as $n \to \infty$ so long as $p \gg \frac{\ln s}{\ln n}$ and $p \gg \frac{\ln^2\left(\frac{\ln^2 n}{p}\right)}{\ln^2 n}$. These conditions are clearly satisfied by the hypothesis.

We have thus shown for, $i = 1$ or $i = 2$,

$$\sum_{m=1}^{s-1} f_i(m) \leq s g_1(1).$$

Finally, we show $\binom{n}{s}\left(\frac{p}{q}\right)^{2(s-1)} s^{2(s-1)+1} g_i(1) = o(\mathrm{E}^2(X_s))$. Since $g_1(1) = g_2(1)$ we need only perform the estimate once. Now

$$\begin{aligned}
&\frac{\binom{n}{s}\left(\frac{p}{q}\right)^{2(s-1)} s^{2(s-1)+1} g(1)}{\mathrm{E}^2(X_s)} \\
&= \frac{2s^2 n^{s-1}\exp\{n(-2q^s + q^{2s-1})\}}{\binom{n}{s}(1-q^s)^{2(n-s)}(s-1)!} \\
&\leq \frac{2s^3\exp\{n(-2q^s + q^{2s-1})\}}{(1-o(1))n(1-q^s)^{2n}} \quad (s^2 = o(n)) \\
&\leq \frac{2s^3\exp\{n(-2q^s + q^{2s-1})\}}{(1-o(1))n\exp\left\{\frac{-2nq^s}{1-q^s}\right\}} \text{(by (2))} \\
&\leq \frac{2s^3}{(1-o(1))n}\exp\left\{n\left(-2q^s + \frac{2q^s}{1-q^s} + q^{2s-1}\right)\right\} \\
&\leq \frac{2s^3}{(1-o(1))n}\exp\left\{\frac{n(2q+1)q^{2s-1}}{1-q^s}\right\} \\
&\leq \frac{2\log_b{}^3 n}{(1-o(1))n}\exp\left\{\frac{3\log_b{}^2 n\ (\ln n + c)^2 q^{2\epsilon-1}}{4n(1-q^s)}\right\} \\
&\to 0
\end{aligned}$$

if $p \gg \frac{\ln n}{n^{\frac{1}{3}}}$. We have thus shown if $s = \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 + \epsilon = \lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 2$ then $\mathrm{Var}(X_s) = o(\mathrm{E}^2(X_s))$ and the lemma is proved. □

We now can state our main result.

Theorem 2.4. *If p is constant or going to 0 such that $\frac{p^2}{96} \geq \frac{\ln(\frac{1}{p})}{\ln n}$, then a.a.s. $\gamma_T(G_{n,p})$ is equal to $\lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 2$, where $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$ if $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ and $c = 0$ otherwise.*

Proof. Let $s = \lfloor \log_b n - \log_b (\log_b n\ (\ln n + c) + \log_b 2 \rfloor$. By Markov's Inequality and Lemma 2.1

$$\Pr(Y_s \geq 1) \leq \mathrm{E}(Y_s) \to 0.$$

Thus a.a.s. $\gamma_T(G) \geq s+1$ or $\gamma_T(G) = 0$. By Chebyshev's Inequality, Lemma 2.2, and Lemma 2.3

$$\Pr(X_{s+2} = 0) \leq \Pr(|X_{s+2} - \mathrm{E}(X_{s+2})| \geq \mathrm{E}(X_{s+2})) \leq \frac{\mathrm{Var}(X_{s+2})}{\mathrm{E}^2(X_{s+2})} \to 0.$$

Thus a.a.s. there exists a tree dominating set of cardinality $s+2$, hence $\gamma_T(G) \leq s+2$ and $\gamma_T(G) \neq 0$. Thus $\gamma_T(G)$ is either $s+1$ or $s+2$. □

3. Relationship to Other Domination Parameters

Let G be a graph with vertex set $[n]$ and let $\phi \neq S \subseteq [n]$. S is a *connected dominating set* of G iff S is a dominating set of G and the induced subgraph $G[S]$ is connected. S is a *total dominating set* of G iff for every vertex $u \in [n]$ there is a vertex $v \in S$ such that $uv \in E(G)$. The *connected domination number* $\gamma_c(G)$ is the smallest integer s such that there exists a connected dominating set of G of cardinality s. The *total domination number* $\gamma_t(G)$ is the smallest integer s such that there exists a total dominating set of G of cardinality s.

The tree domination number is closely related to both the connected and total domination numbers. In fact a non-trivial tree dominating set of a graph G is both a connected dominating set and a total dominating set of G. Thus by determining the (non-trivial) tree domination number of a random graph we have also bounded its connected and total domination numbers above. Godbole's result [8] on the domination number of a random graph provides a lower bound.

Let $a(n,p) = \lfloor \log_b n - \log_b (\log_b n\ \ln n) \rfloor + 1$ and $b(n,p,c) = \lfloor \log_b n - \log_b (\log_b n\ (\ln n + c)) + \log_b 2 \rfloor + 2$ where $b = \frac{1}{1-p}$. Hence, we have shown:

Theorem 3.1. *If p is constant, then a.a.s.*

$$a(n,p) \leq \gamma_c(G_{n,p}), \gamma_t(G_{n,p}) \leq b(n,p,c).$$

where $c = 2\ln\left(\frac{pe}{q^{\frac{3}{2}}}\right)$ *if* $p > \frac{(1-p)^{\frac{3}{2}}}{e}$ *and* $c = 0$ *otherwise.* □

It should be noted Bonato and Wang [3] proved $\gamma_t(G_{n,p}) \leq \lfloor \log_b n - \log_b (\log_b n\ \ln n) \rfloor + 2$. We conjecture a similar two-point concentration for $\gamma_c(G_{n,p})$.

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