The total number of matchings in triangle graph of a connected graph

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Abstract

For a graph G, let $Z(G)$ be the total number of matchings in G. For a nontrivial graph G of order n with vertex set $V(G) = \{v_1, \dots, v_n\}$, Cvetković *et al.* [2] defined the *triangle graph* of G, denoted by $R(G)$, to be the graph obtained by introducing a new vertex v_{ij} and connecting v_{ij} both to v_i and to v_j for each edge v_iv_j in G. In this short paper, we prove that for a connected graph G, if $Z(R(G)) \geq \left(\frac{1}{2}n - \frac{1}{2} + \frac{5}{2n}\right)^n$, then G is traceable. Moreover, for a connected graph G, we give sharp upper bounds for $Z(R(G))$ in terms of clique number, vertex connectivity and spectral radius of G, respectively.

Keywords: Matching, Hosoya index, degree, traceable graph, clique number, triangle graph.

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1 Introduction

The total number of matchings of a graph is a graphic invariant which is important in structural chemistry. In the chemistry literature this graphic invariant is called the *Hosoya index* of a molecular graph [8]. It was applied to correlations with boiling points, entropies, calculated bond orders, as well as for coding of chemical structures.

Let G be a graph with n vertices. The *Hosoya index* of G, denoted by $Z(G)$, is defined to be the total number of matchings in G, namely,

$$Z(G) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} m(G; s),$$

where $m(G;s)$ is the number of s-matchings of G. An s-matchings of a graph G is a subset M of its edge set with the property that $|M| = s$ and M contains no two edges sharing a common vertex. For convenience, set $m(G;0) = 1$.

During the past decades, numerous results on Hosoya index have been put forward. All these existing results dealt with the extremal problem on Hosoya index. For example, Gutman [6] proved that the linear hexagonal chain is the unique chain with minimal Hosoya index among all hexagonal chains. Zhang [24] showed that zig-zag hexagonal chain is the unique chain with maximal Hosoya index among all hexagonal chains. Hou [11] characterized the trees having minimal and second-minimal Hosoya index among all trees with a given size of matching. Yu and Lv [23] investigated the trees having minimal Hosoya index among all trees with k-pendent vertices. Ou [18] determined the unicyclic graphs with the first and second largest Hosoya index among unicyclic graphs with n vertices. In [16], Li et al. investigated extremal problem for Hosoya index of quasi-tree graphs. For other related results on Hosoya index, see [3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17] and [21] for a survey.

Cvetković *et al.* [2] constructed a composite graph by the following procedure. Suppose that G is connected graph of order n and size m and let $V(G) = \{v_1, \ldots, v_n\}$. Introducing a new vertex v_{ij} and connecting v_{ij} both to v_i and to v_j for each edge v_iv_j. The resulting graph is denoted by $R(G)$, which we call the *triangle graph* of G. The graph $R(G)$ has remarkable properties. For example, Yan and Yeh [22] obtained a remarkable result concerning the Hosoya index of $R(G)$, see Lemma 2.2 in the next section.

A connected graph is said to be *traceable* if it possesses a Hamiltonian path, i.e., a path passing through all vertices of the underlying graph. In this short paper, we prove that for a connected graph G, if $Z(R(G)) \geq \left(\frac{1}{2}n - \frac{1}{2} + \frac{5}{2n}\right)^n$, then G is traceable. Moreover, for a connected graph G, we give sharp upper bounds for $Z(R(G))$ in terms of clique number, vertex connectivity and spectral radius of G, respectively.

We first introduce some basic notation and terminology. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *degree* of each vertex v_i, denoted by $d_G(v_i)$ (or simply d_i), is the number of neighbors of v_i in G. The maximum and minimum vertex degree in G are denoted by Δ and δ, respectively. If $\Delta = \delta$ holds in G, then G is said to be a *regular graph.* The number of vertices of the largest clique in a graph is called its *clique number* and denoted by ω. The *vertex connectivity* of a connected graph G, denoted by ν, is the smallest number of vertices whose removal disconnects G or reduces it to a single vertex. The *index* or *spectral radius* λ_1 of G is the largest eigenvalue of its adjacency matrix. A k-partite graph is said to be complete if any two vertices are adjacent if and only if they belong to different partition classes. Our terminology and notation not defined here will conform to those in [1].

2 A sufficient condition in terms of $Z(R(G))$ for a connected graph to be traceable

In this section, we use Hosoya index of the triangle graph of a connected graph to give a sufficient condition for a connected graph to be traceable. We first introduce here a result, which gives a sufficient condition for a connected graph to be traceable.

Lemma 2.1 ([1]). *Let G be a nontrivial graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \ldots \leq d_n$ and $n \geq 4$. Suppose that* *there is no integer $k < \frac{n+1}{2}$ such that $d_k \leq k-1$ and $d_{n-k+1} \leq n-k-1$. Then G is traceable.*

Yan and Yeh [22] obtained a remarkable result concerning the Hosoya index of graph $R(G)$ as follows.

Lemma 2.2 ([22]). *Let G be a nontrivial graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$. Suppose that $R(G)$ is the triangle graph of G. Then*

$$Z(R(G)) = (d_1+1)(d_2+1)\ldots(d_n+1).$$

Now, we use Lemmas 2.1 and 2.2 to prove the following result.

Theorem 2.3. *Let G be a connected graph of order $n \geq 4$ and let $R(G)$ be its triangle graph. If*

$$Z(R(G)) \geq \left(\frac{1}{2}n - \frac{1}{2} + \frac{5}{2n}\right)^n,$$

then G is traceable.

Proof. Suppose to the contrary that G is a non-traceable connected graph with degree sequence $(d_1, d_2, \ldots, d_n)$ such that $d_1 \leq d_2 \leq \ldots \leq d_n$ and $n \geq 4$. By Lemma 2.1, there exists a positive integer $k < \frac{n+1}{2}$ such that $d_k \leq k-1$ and $d_{n-k+1} \leq n-k-1$. Because G is connected and $d_k \leq k-1$, we have $k \geq 2$. By Lemma 2.2, we have

$$\begin{aligned} Z(R(G)) &= (d_1+1)(d_2+1)\ldots(d_n+1) \\ &\leq \left[\frac{(d_1+1)+(d_2+1)+\ldots(d_n+1)}{n}\right]^n \qquad (1) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} d_i + 1\right)^n. \end{aligned}$$

Note that

$$\begin{aligned}\sum_{i=1}^{n} d_i &\le k(k-1)+(n-2k+1)(n-k-1)+ \\ &\quad (k-1)(n-1) \qquad (2)\\ &\quad \text{(because } d_k \le k-1 \text{ and } d_{n-k+1} \le n-k-1) \qquad (3)\\ &= \frac{n(n-1)}{4}+1+\frac{(n-2)(n-3)}{4}-\frac{(k-2)(2n-3k-5)}{4}\\ &\le \frac{n(n-1)}{4}+1+\frac{(n-2)(n-3)}{4} \qquad (4)\\ &= \frac{1}{2}n^2-\frac{3}{2}n+\frac{5}{2}.\end{aligned}$$

So,

$$Z(R(G)) \le \left(\frac{\frac{1}{2}n^2-\frac{3}{2}n+\frac{5}{2}}{n}+1\right)^n = \left(\frac{1}{2}n-\frac{1}{2}+\frac{5}{2n}\right)^n.$$

By our assumption that $Z(R(G)) \ge \left(\frac{1}{2}n-\frac{1}{2}+\frac{5}{2n}\right)^n$, we thus have $Z(R(G)) = \left(\frac{1}{2}n-\frac{1}{2}+\frac{5}{2n}\right)^n$. Therefore, all equalities in (1), (3) and (4) should be attained at the same time. We thus have the following conclusions.

(a). If the equality in (1) is attained, then $d_1 = \ldots = d_n$.

(b). If the equality in (3) is attained, then $d_1 = \ldots = d_k = k-1$, $d_{k+1} = \ldots = d_{n-k+1} = n-k-1$ and $d_{n-k+2} = \ldots = d_n = n-1$.

By our assumption that $k < \frac{n+1}{2}$, from (b), we know that $d_1 = \ldots = d_k < n-1 = d_n$. It is a contradiction to (a). So, G is traceable, as desired. □

3 Sharp upper bounds for $Z(R(G))$

In this section, we give sharp upper bounds for $R(G)$ in terms of clique number, vertex connectivity and spectral radius, respectively. First, we recall the well-known Turán's theorem, which is stated as follows.

Theorem 3.1 ([20]). *Let G be a connected K_{q+1}-free graph of order n and size m. Then*

$$m \leq \left(1 - \frac{1}{q}\right) \cdot \frac{n^2}{2}$$

with equality if and only if G is a complete q-partite graph in which all classes are of equal cardinality.

Now, for a connected graph G, we give a sharp upper bound for $Z(R(G))$ in terms of the clique number of G.

Theorem 3.2. *Let G be a connected graph of order n and clique number ω, and let $R(G)$ be its triangle graph. Then*

$$Z(R(G)) \leq \left[n\left(1 - \frac{1}{\omega}\right) + 1\right]^n \tag{5}$$

with equality if and only if G is a complete ω-partite graph in which all classes are of equal cardinality.

Proof. Suppose that G has degree sequence $(d_1, d_2, \ldots, d_n)$. Since G has clique number ω, then G is a $K_{\omega+1}$-free graph. Let $m(G)$ denote the number of edges in G.

By Theorem 3.1, we have

$$m(G) \leq \left(1 - \frac{1}{\omega}\right) \cdot \frac{n^2}{2}. \tag{6}$$

According to (6) and Lemma 2.2, we have

$$\begin{aligned} Z(R(G)) &= (d_1+1)(d_2+1)\ldots(d_n+1) \\ &\leq \left[\frac{(d_1+1)+(d_2+1)+\ldots(d_n+1)}{n}\right]^n & (7) \\ &= \left(\frac{2m(G)}{n}+1\right)^n \\ &\leq \left[\frac{2}{n}\cdot\left(1-\frac{1}{\omega}\right)\cdot\frac{n^2}{2}+1\right]^n & (8) \\ &= \left[n\left(1-\frac{1}{\omega}\right)+1\right]^n. \end{aligned}$$

Now, we check the equality condition in (5). Suppose that $Z(R(G)) = \left[n\left(1-\frac{1}{\omega}\right)+1\right]^n$. Then both equalities in (7) and (8) must be attained simultaneously. When the equality in (7) is attained, G must be a regular graph. When the equality in (8) is attained, G is a complete ω-partite graph in which all classes are of equal cardinality. So, if $Z(R(G)) = \left[n\left(1-\frac{1}{\omega}\right)+1\right]^n$, then G is a complete ω-partite graph in which all classes are of equal cardinality. Conversely, if G is a complete ω-partite graph in which all classes are of equal cardinality, then both equalities in (7) and (8) are attained. So, $Z(R(G)) = \left[n\left(1-\frac{1}{\omega}\right)+1\right]^n$.

This completes the proof. □

Lemma 3.3 ([19]). *Let G be a connected graph on $n \geq 3$ vertices, with clique number ω and vertex connectivity ν. Then*

$$\omega - \nu \leq n - 2$$

with equality if and only if G is the short kite $KT_{n,n-1}$ or K_3.

Lemma 3.4 ([19]). *Let G be a connected graph on $n \geq 3$ vertices, with clique number ω and index λ_1. Then*

$$\omega - \lambda_1 \leq 1$$

with equality if and only if G is the complete graph K_n.

By Theorem 3.2, Lemmas 3.3 and 3.4, we immediately have the following

Corollary 3.5. *Let G be a connected graph on $n \geq 3$ vertices with clique number ω, vertex connectivity ν. Then*

$$Z(R(G)) \leq \left[n\left(1-\frac{1}{\nu+n-2}\right)+1\right]^n$$

with equality if and only if $G \cong K_3$.

Corollary 3.6. *Let G be a connected graph on $n \geq 3$ vertices with clique number ω, index λ_1. Then*

$$Z(R(G)) \leq \left[n\left(1 - \frac{1}{1+\lambda_1}\right) + 1\right]^n$$

with equality if and only if $G \cong K_n$.

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