The Singular Chromatic Number of a Graph

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Abstract

A proper coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors. The minimum number of colors in a proper coloring of G is the chromatic number $\chi(G)$ of G. For a graph G and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of G for some positive integer k, the color code of a vertex v of G (with respect to c) is the ordered pair $\text{code}(v) = (c(v), S_v)$, where $S_v = \{c(u) : u \in N(v)\}$. The coloring c is singular if distinct vertices have distinct color codes and the singular chromatic number $\chi_{si}(G)$ of G is the minimum positive integer k for which G has a singular k-coloring. Thus $\chi(G) \le \chi_{si}(G) \le n$ for every graph G of order n. A characterization is established for all triples (a, b, n) of positive integers for which there exists a graph G of order n with $\chi(G) = a$ and $\chi_{si}(G) = b$. It is shown for every vertex v and every edge e in a graph G that $\chi_{si}(G) - 1 \le \chi_{si}(G - v) \le \chi_{si}(G) + \deg v$ and $\chi_{si}(G) - 1 \le \chi_{si}(G - e) \le \chi_{si}(G) + 2$ and that all these four bounds are sharp. We also determine the singular chromatic numbers of cycles and paths.

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1 Introduction

Graph coloring is one of the most popular areas in graph theory. Among the most studied colorings are proper colorings, in which the two vertices in every pair of adjacent vertices are assigned distinct colors. The minimum number of colors in a proper coloring of G is the chromatic number $\chi(G)$ of G. A coloring that provides a method of distinguishing every two adjacent vertices is said to be *neighbor-distinguishing.* A number of neighbor-distinguishing colorings other than proper colorings have been introduced. (See [10, 11, 12, 13], for example.)

It is sometimes desired that not only every two adjacent vertices but every two vertices in a graph can be distinguished from one another in some manner, which is a problem that has received increased attention during the past few decades. Earlier methods used open neighborhoods of vertices in graphs [14, 24], automorphism groups of graphs [3, 15, 16], and distances in graphs [17, 23]. Of course, this goal can be also achieved by assigning colors to the vertices or edges of a graph. For example, we can assign distinct colors to distinct vertices. A coloring by which the vertices in a graph can be distinguished from one another in some way is called *vertex-distinguishing.* Therefore, a vertex-distinguishing coloring is neighbor-distinguishing but the converse is not true in general.

In [18] Harary and Plantholt introduced a way to distinguish the vertices of a graph G by assigning colors to the edges of G in such a way that for every two vertices of G, one of the vertices is incident with an edge assigned one of these colors that the other vertex is not. They referred to the minimum number of colors needed to accomplish this as the *point-distinguishing chromatic index* of G.

There is another edge coloring by which differences among the vertices of a connected graph G can be detected. Let G be a connected graph of order $n \geq 3$ and $c : E(G) \to \{1, 2, \ldots, k\}$ a coloring of the edges of G for some positive integer k. The *(detection) color code* of a vertex v of G with respect to a k-coloring c of the edges of G is the ordered k-tuple $(a_1, a_2, \cdots, a_k)$ where a_i is the number of edges incident with v that are colored i for $1 \leq i \leq k$. The edge coloring c is *detectable* if distinct vertices have distinct color codes. The minimum positive integer k for which G has a detectable k-coloring is the *detection number* of G. The concept of detectable colorings was studied in [1, 2, 5, 6, 7].

In [9] a vertex coloring was introduced to recognize the vertices of a graph. Let G be a graph and $c : V(G) \to \{1, 2, \ldots, k\}$ a coloring of the vertices of G for some positive integer k. The *(recognition) color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $(a_0, a_1, \cdots, a_k)$

where a_0 is the color assigned to v and, for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. The coloring c is called *recognizable* if distinct vertices have distinct color codes and the *recognition number* of G is the minimum positive integer k for which G has a recognizable k-coloring.

Another vertex-distinguishing vertex coloring, introduced in [21], is the *irregular coloring*. For a graph G and a *proper* coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of G for some positive integer k, the (*irregular*) *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $(a_0, a_1, \cdots, a_k)$ where a_0 is the color assigned to v and a_i is the number of vertices adjacent to v that are colored i for $1 \leq i \leq k$. The coloring c is then *irregular* if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring.

In this work, we introduce yet another vertex coloring of graphs that enables us to distinguish every two vertices in a graph G by focusing on the color of each vertex $v \in V(G)$ and the set of colors of the neighbors of v in G. This combines a number of features of the vertex-distinguishing colorings previously considered. We refer to the book [8] for graph theory notation and terminology not described in this paper.

2 Basic Definitions and Preliminary Results

For a graph G, let $c : V(G) \to \mathbb{N}$ be a proper vertex coloring of G. For each vertex v of G, let S_v be the set of colors of the neighbors of v, that is, $S_v = \{c(u) : u \in N(v)\}$ where $N(v)$ is the neighborhood of v. The *color code* code(v) of v is then defined as the ordered pair $(c(v), S_v)$. If code(u) $\neq$ code(v) for every two distinct vertices u and v of G, then c is called a *singular coloring* of G. Therefore, a singular coloring of a graph G is a coloring that uses the color of each vertex together with the set of colors of its neighbors to distinguish all vertices in G. If a singular coloring c uses k colors, then c is a *singular k-coloring*. For each positive integer k, let $\mathbb{N}_k = \{1, 2, \ldots, k\}$. Thus, we assume that every singular k-coloring uses the colors in $\mathbb{N}_k$. A graph G is singularly k-colorable if G has a singular k-coloring. Figure 1 illustrates a singular 4-coloring of the Petersen graph, where each vertex is labeled by its color code and the first coordinate of each color code is the color of that vertex.

The minimum k for which G has a singular k-coloring is called the *singular chromatic number* of G and is denoted by $\chi_{si}(G)$. Since a coloring assigning distinct colors to distinct vertices of a graph G is a singular coloring of G, the singular chromatic number exists for every graph. Since every

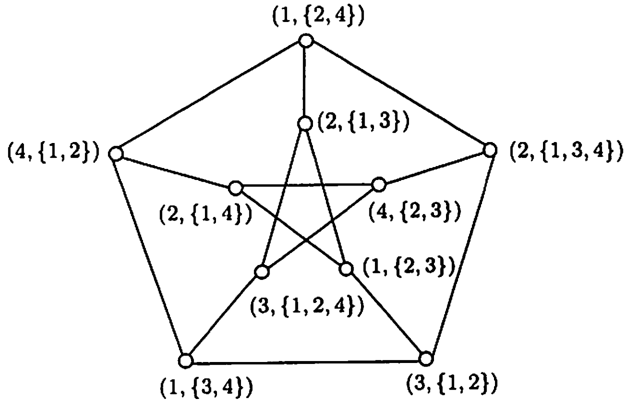

Figure 1: A singular 4-coloring of the Petersen graph

singular coloring is proper, we have the following observation.

Observation 2.1 *For every graph G of order n, $\chi(G) \leq \chi_{si}(G) \leq n$.*

We also present some elementary observations.

Observation 2.2 *Let G be a graph.*

(a) $\chi_{si}(G) = 1$ *if and only if* $G = K_1$.

(b) $\chi_{si}(G) = 2$ *if and only if* $G \in \{K_2, K_2 \cup K_1, K_2 \cup 2K_1\}$.

Observation 2.3 *If $\chi_{si}(G) = k$, then each of the following holds.*

(a) *G contains at most k isolated vertices.*

(b) *G contains at most one clique of order k.*

(c) *Each vertex in G is adjacent to at most $k-1$ end-vertices.*

Proposition 2.4 *If G is a singularly k-colorable graph of order n, then*

$$n \leq k \cdot 2^{k-1}. \tag{1}$$

Furthermore, if G is connected, then $n \leq k(2^{k-1} - 1)$.

Proof. Let c be a singular k-coloring of G with color classes $V_1, V_2, \ldots, V_k$. Let $v_i \in V_i$ for $1 \le i \le k$. Since c is a proper coloring, $S_{v_i} \subseteq \mathbb{N}_k - \{i\}$ and so there are at most 2^{k-1} possible codes for v_i. Hence, $|V_i| \le 2^{k-1}$ for $1 \le i \le k$ and so $n = \sum_{i=1}^{k} |V_i| \le k \cdot 2^{k-1}$. If G is connected, then $S_v \ne \emptyset$ for each $v \in V(G)$. Thus $|V_i| \le 2^{k-1} - 1$ for $1 \le i \le k$ and the result now follows. ∎

An immediate consequence of Proposition 2.4 is that if $k(2^{k-1}-1)+1 \le n \le k \cdot 2^{k-1}$, then G contains exactly $n - k(2^{k-1} - 1)$ isolated vertices.

To illustrate Proposition 2.4, we consider the Petersen graph P. Since P is a connected graph of order 10 and $3(2^{3-1} - 1) = 9$, it follows by Proposition 2.4 that $\chi_{si}(P) \ge 4$. Therefore, $\chi_{si}(P) = 4$ as the singular 4-coloring of P in Figure 1 shows.

3 Connected Graphs with Prescribed Order and Singular Chromatic Number

Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, then they are referred to as *true twins*; while if u and v are nonadjacent, then they are *false twins*. Thus if u and v are true twins, then $N[u] = N[v]$; while if u and v are false twins, then $N(u) = N(v)$. The following is a useful observation.

Observation 3.1 *If u and v are twins in a graph G, then $c(u) \ne c(v)$ for every singular coloring c of G.*

We now characterize graphs whose order and singular chromatic number are equal.

Theorem 3.2 *Let G be a graph of order n. Then $\chi_{si}(G) = n$ if and only if G is a complete multipartite graph or an empty graph.*

Proof. Since the result is trivial for $1 \le n \le 3$, let us assume that $n \ge 4$. First suppose that G is a complete multipartite graph. Then every two nonadjacent vertices of G are false twins. Hence, every singular coloring of G must assign distinct colors to distinct vertices by Observation 3.1 and so $\chi_{si}(G) = n$. Also, it is straightforward to see that $\chi_{si}(\overline{K}_n) = n$.

For the converse, suppose that G is neither a complete multipartite graph nor an empty graph. It then follows that there exist three vertices u,

v, and w in G such that $uv \in E(G)$ while $uw, vw \notin E(G)$. Then an $(n-1)$-coloring assigning the color 1 to u and w and the colors $2, 3, \ldots, n-1$ to the remaining $n-2$ vertices is a singular coloring of G. Therefore, $\chi_{si}(G) \le n-1$. ∎

Corollary 3.3 *For every pair a, b of integers with $2 \le a \le b$, there is a connected graph G with $\chi(G) = a$ and $\chi_{si}(G) = b$.*

Proof. While the complete a-partite graph $G = K_{1,1,\ldots,1,b-a+1}$ of order b has chromatic number a, it follows by Proposition 3.2 that $\chi_{si}(G) = b$. ∎

By Observation 2.1 and Proposition 2.4, if G is a nontrivial connected graph of order n with $\chi(G) = a$ and $\chi_{si}(G) = b$, then

$$2 \le a \le b \le n \le b(2^{b-1} - 1). \tag{2}$$

A triple (a, b, n) of positive integers is said to be a *realizable triple* if there is a nontrivial connected graph G of order n such that $\chi(G) = a$ and $\chi_{si}(G) = b$. Thus every realizable triple must satisfy (2). We next show that every triple (a, b, n) satisfying (2) is realizable.

Theorem 3.4 *Every triple (a, b, n) of integers with $2 \le a \le b \le n \le b(2^{b-1} - 1)$ is realizable.*

Proof. We first assume that $b = a$. If $n = a$, then the complete graph of order a has the desired property. Otherwise, we may assume that $a \ge 3$. Let $G_0 \cong K_a$ whose vertex set is $\{v_1, v_2, \ldots, v_a\}$ and obtain another graph G_1 of order $a(2^{a-1}-1)$ from G_0 by adding $a(2^{a-1}-2)$ new vertices in the set $\cup_{i=1}^{a}\{x_{i,R_i} : R_i \subseteq \mathbb{N}_a - \{i\},\ 1 \le |R_i| \le a-2\}$ and joining x_{i,R_i} to v_j if and only if $j \in R_i$. Since $a+1 \le n \le a(2^{a-1}-1)$, there exists a connected graph G of order n that is an induced subgraph of G_1 such that $G_0 \subset G \subseteq G_1$. Observe that the coloring c^* of G_1 given by $c^*(v_i) = c^*(x_{i,R_i}) = i$ for $1 \le i \le a$ is a singular a-coloring of G_1. Furthermore, the coloring c^* restricted to $V(G)$ is a singular coloring of G. Hence, $a = \chi(G_0) \le \chi(G) \le \chi_{si}(G) \le a$, that is, every triple (a, a, n) with $2 \le a \le n \le a(2^{a-1} - 1)$ is realizable.

Next we assume that $b \ge a + 1$. We consider the following two cases.

Case 1. $a = 2$. First suppose that $b \le n \le b^2$. Let $H_0 \cong K_{2,b-2}$ whose partite sets are $\{v_1, v_2\}$ and $\{v_3, v_4, \ldots, v_b\}$. We construct the graphs $H_1, H_2, \ldots, H_b$ as follows. For $1 \le i \le b$, suppose that the graph H_{i-1} has been defined. Then we obtain H_i from H_{i-1} by adding $b-1$ new vertices in the set $U_i = \{u_{j,i} : 1 \le j \le b,\ j \ne i\}$ and joining v_i to every vertex in U_i. Hence, H_b is a connected graph of order b^2.

Since $|V(H_0)| \le n \le |V(H_b)|$, there exists a connected graph G of order n that is an induced subgraph of H_b such that $H_0 \subseteq H_{i-1} \subseteq G \subseteq H_i \subseteq H_b$ for some i with $1 \le i \le b$. Since both H_0 and H_b are bipartite, it follows that $\chi(G) = 2$. If $H_1 \subseteq G \subseteq H_b$, then $\chi_{si}(G) \ge b$ by Observation 2.3(c). If $H_0 \subseteq G \subseteq H_1$, then $\chi_{si}(G) \ge b$ as well since each of the b vertices $v_1, v_2, \ldots, v_b$ must be assigned a distinct color by a singular coloring of G. On the other hand, the coloring c^* of H_b defined by $c^*(v_i) = i$ for $1 \le i \le b$ and $c^*(u_{j,i}) = j$ for $1 \le j \le b$ is a singular b-coloring. Furthermore, the coloring c^* restricted to $V(G)$ is a singular coloring of G. Hence, $\chi_{si}(G) = b$.

Next we assume that $b \ge 4$ and $b^2 + 1 \le n \le b(2^{b-1} - 1)$. First obtain the bipartite graph G_0 of order $b + (b-1)^2$ from $K_{1,b-1}$ whose partite sets are $\{v_1, v_2, \ldots, v_{b-1}\}$ and $\{v_b\}$ by adding $(b-1)^2$ new vertices in the set $\cup_{i=1}^{b-1} U_i$, where $U_i = \{u_{j,i} : 1 \le j \le b-1\}$, and joining $u_{i,i}$ to every vertex in $[U_i - \{u_{i,i}\}] \cup \{v_b\}$. Now for $1 \le i \le b-1$, suppose that the graph G_{i-1} has been defined. We then construct G_i from G_{i-1} by adding $2^{b-1} - b - 1$ new vertices in the set $X_i \cup Y_i$, where

$$X_i = \{x_{i,R_i} : R_i \subseteq \mathbb{N}_{b-1} - \{i\}, 2 \le |R_i| \le b-2\}$$
$$Y_i = \{y_{i,R_i} : R_i \subseteq \mathbb{N}_{b-1} - \{i\}, 1 \le |R_i| \le b-3\}$$

and joining

(a) $x_{i,R_i} \in X_i$ to $u_{j,i} \in U_i$ if and only if $j \in R_i$ and

(b) $y_{i,R_i} \in Y_i$ to (i) v_b and (ii) $u_{j,i} \in U_i$ if and only if $j \in R_i$.

Finally, let G_b be the graph obtained from G_{b-1} by adding $2^{b-1} - 2$ vertices in the set $Z = \{z_R : R \subseteq \mathbb{N}_{b-1}, 1 \le |R| \le b-2\}$ and joining z_R to v_i if and only if $i \in R$. Observe that G_b is a connected graph of order $b(2^{b-1} - 1)$ and furthermore, the coloring c^{**} such that (i) $c^{**}(v) = i$ if $v \in \{v_i\} \cup X_i \cup Y_i$ for $1 \le i \le b-1$, (ii) $c^{**}(u_{j,i}) = j$ for $1 \le j \le b-1$, and (iii) $c^{**}(v) = b$ if $v \in \{v_b\} \cup Z$ is a singular b-coloring of G_b.

Since $|V(G_0)| = b + (b-1)^2 < b^2$, it follows that $|V(G_0)| < n \le |V(G_b)|$ and there exists a connected graph G of order n that is an induced subgraph of G_b such that $G_0 \subset G_{i-1} \subseteq G \subseteq G_i \subseteq G_b$ for some i with $1 \le i \le b$. Since both G_0 and G_b are bipartite, $\chi(G) = 2$. If $G_0 \subset G \subseteq G_{b-1}$, then $\chi_{si}(G) \ge b$ by Observation 2.3(c); while if $G_{b-1} \subseteq G \subseteq G_b$, then $\chi_{si}(G) \ge b$ by Proposition 2.4 since $|V(G)| \ge |V(G_{b-1})| = b(2^{b-1} - 1) - (2^{b-1} - 2) > (b-1)(2^{b-2} - 1)$. On the other hand, in each case, the coloring c^{**} restricted to $V(G)$ is a singular b-coloring of G. Hence, we obtain the desired result for $a = 2$.

Case 2. $a \ge 3$. First suppose that $b \le n \le b(b - a + 1) + (a-1)^2$. Let $H_0 \cong \overline{K}_{b-a+1} + K_{a-1}$ with $V(\overline{K}_{b-a+1}) = \{v_1, v_2, \ldots, v_{b-a+1}\}$ and

$V(K_{a-1}) = \{v_{b-a+2}, v_{b-a+3}, \ldots, v_b\}$. Obtain H_1 of order $b+(a-1)(b-1)$ from H_0 by adding $(a-1)(b-1)$ new vertices in the set $\cup_{i=b-a+2}^{b} U_i$, where $U_i = \{u_{j,i} : 1 \leq j \leq b, j \neq i\}$, and joining v_i to every vertex in U_i for $b-a+2 \leq i \leq b$. Also, obtain H_2 of order $b(b-a+1)+(a-1)^2$ from H_1 by adding $(b-a+1)(b-a)$ new vertices in the set $\cup_{i=1}^{b-a+1} W_i$, where $W_i = \{w_{j,i} : 1 \leq j \leq b-a+1, j \neq i\}$, and joining v_i to every vertex in W_i for $1 \leq i \leq b-a+1$.

Since $|V(H_0)| \leq n \leq |V(H_2)|$, there exists a connected graph G of order n that is an induced subgraph of H_2 such that either $H_0 \subseteq G \subseteq H_1$ or $H_1 \subseteq G \subseteq H_2$. Since H_2 is proper a-colorable and $\chi(H_0) = a$, it follows that $\chi(G) = a$.

If $H_0 \subseteq G \subseteq H_1$, then $\chi_{si}(G) \geq b$ since each of the b vertices $v_1, v_2, \ldots, v_b$ must be assigned a distinct color by a singular coloring of G. If $H_1 \subseteq G \subseteq H_2$, then $\chi_{si}(G) \geq b$ by Observation 2.3(c). To see that $\chi_{si}(G) \leq b$, let c^* be the coloring of H_2 defined by (i) $c^*(v_i) = i$ for $1 \leq i \leq b$, (ii) $c^*(u_{j,i}) = j$ for $1 \leq j \leq b$ and $j \neq i$, and (iii) $c^*(w_{j,i}) = j$ for $1 \leq j \leq b-a+1$ and $j \neq i$. Then c^* is a singular b-coloring of H_2. Furthermore, the coloring c^* restricted to $V(G)$ is a singular b-coloring of G. Therefore, $\chi_{si}(G) = b$, that is, every triple (a, b, n) with $a < b \leq n \leq b(b-a+1)+(a-1)^2$ is realizable.

Next we assume that $b(b-a+1)+(a-1)^2+1 \leq n \leq b(2^{b-1}-1)$. Let $G_0 = H_2$ and we construct $G_1, G_2, \ldots, G_b$ as follows. Let $A = \mathbb{N}_b - \mathbb{N}_{b-a+1}$. For $1 \leq i \leq b-a+1$, suppose that the graph G_{i-1} has been defined. Then we obtain G_i from G_{i-1} by adding $2^{b-1}-b-1$ new vertices in the set $X_i \cup Y_i = X_i \cup [Y_{1,i} \cup Y_{2,i} \cup Y_{3,i}]$, where

$$X_i = \begin{cases} \{x_{i,R_i} : R_i \subseteq \mathbb{N}_{b-a+1} - \{i\},\ 2 \leq |R_i| \leq b-a\} & \text{if } b \geq a+2 \\ \emptyset & \text{if } b = a+1 \end{cases}$$

$$Y_{1,i} = \{y_{i,R_i \cup \{j\}} : R_i \subseteq \mathbb{N}_{b-a+1} - \{i\},\ R_i \neq \emptyset,\ b-a+2 \leq j \leq b\}$$
$$Y_{2,i} = \{y_{i,R_i \cup A} : R_i \subseteq \mathbb{N}_{b-a+1} - \{i\},\ |R_i| \leq b-a-1\}$$

$$Y_{3,i} = \begin{cases} \{y_{i,R_i \cup R} : R_i \subseteq \mathbb{N}_{b-a+1} - \{i\},\ R \subseteq A,\ 2 \leq |R| \leq a-2\} & \text{if } a \geq 4 \\ \emptyset & \text{if } a = 3, \end{cases}$$

and joining

(a) x_{i,R_i} to $w_{j,i}$ if and only if $j \in R_i$ and

(b) $y_{i,R_i \cup R} \in Y_i$, where $R_i \subseteq \mathbb{N}_{b-a+1} - \{i\}$ and $R \subseteq A$, to (i) $w_{j,i}$ if and only if $j \in R_i$ and (ii) v_j if and only if $j \in R$.

Finally, for $b - a + 2 \leq i \leq b$, suppose again that the graph G_{i-1} has been defined. We obtain G_i from G_{i-1} by adding $2^{b-1} - a$ new vertices in the set $Z_i = Z_{1,i} \cup Z_{2,i}$, where

$$Z_{1,i} = \{z_{i,\{j\}} : 1 \leq j \leq b - a + 1\}$$
$$Z_{2,i} = \{z_{i,R_i} : R_i \subseteq \mathbb{N}_b - \{i\},\ 2 \leq |R_i| \leq b - 2\},$$

and joining $z_{i,R} \in Z_i$ to $u_{j,i} \in U_i$ if and only if $j \in R$.

Since $|V(G_0)| < n \leq |V(G_b)|$, there exists a connected graph G of order n that is an induced subgraph of G_b such that $G_0 \subseteq G_{i-1} \subseteq G \subseteq G_i \subseteq G_b$ for some i with $1 \leq i \leq b$. Since G_b is proper a-colorable and $\chi(G_0) = \chi(H_2) = a$, it follows that $\chi(G) = a$. If $G_0 \subset G \subseteq G_{b-1}$, then $\chi_{si}(G) \geq b$ by Observation 2.3(c). If $G_{b-1} \subseteq G \subseteq G_b$, then observe that $|V(G)| \geq |V(G_{b-1})| = b(2^{b-1} - 1) - (2^{b-1} - a) > (b-1)(2^{b-1} - 1) > (b-1)(2^{b-2} - 1)$ and so $\chi_{si}(G) \geq b$ by Proposition 2.4.

To verify that $\chi_{si}(G) \leq b$, let c^{**} be the coloring of G_b such that c^{**} restricted to $V(G_0) = V(H_2)$ equals c^* and $c^{**}(v) = i$ if (i) $1 \leq i \leq b-a+1$ and $v \in X_i \cup Y_i$ or (ii) $b - a + 2 \leq i \leq b$ and $v \in Z_i$ and observe that c^{**} is a singular b-coloring of G_b. Furthermore, the coloring c^{**} restricted to $V(G)$ is a singular b-coloring of G. Hence, $\chi_{si}(G) = b$. ∎

For example, the graphs F_1 and F_2 in Figure 2 show the construction of the graph G described in the proof of Theorem 3.4, verifying that the triples $(2, 4, 24)$ and $(3, 5, 24)$ are realizable, respectively. (Each of the solid vertices belongs to G_0 in each graph.)

4 Vertex or Edge Deletions and the Singular Chromatic Number

It is well known that if v is a vertex in a nontrivial graph G, then either $\chi(G - v) = \chi(G)$ or $\chi(G - v) = \chi(G) - 1$. This is also the case when an edge is deleting from a nonempty graph, that is, either $\chi(G - e) = \chi(G)$ or $\chi(G - e) = \chi(G) - 1$ for every $e \in E(G)$. Therefore, deleting a vertex or an edge from a graph, the chromatic number of the resulting graph never exceeds that of the original graph. For the singular chromatic number, a much different situation can occur.

Theorem 4.1 *For a nontrivial graph G and a vertex v in G,*

$$\chi_{si}(G) - 1 \leq \chi_{si}(G - v) \leq \chi_{si}(G) + \deg v.$$

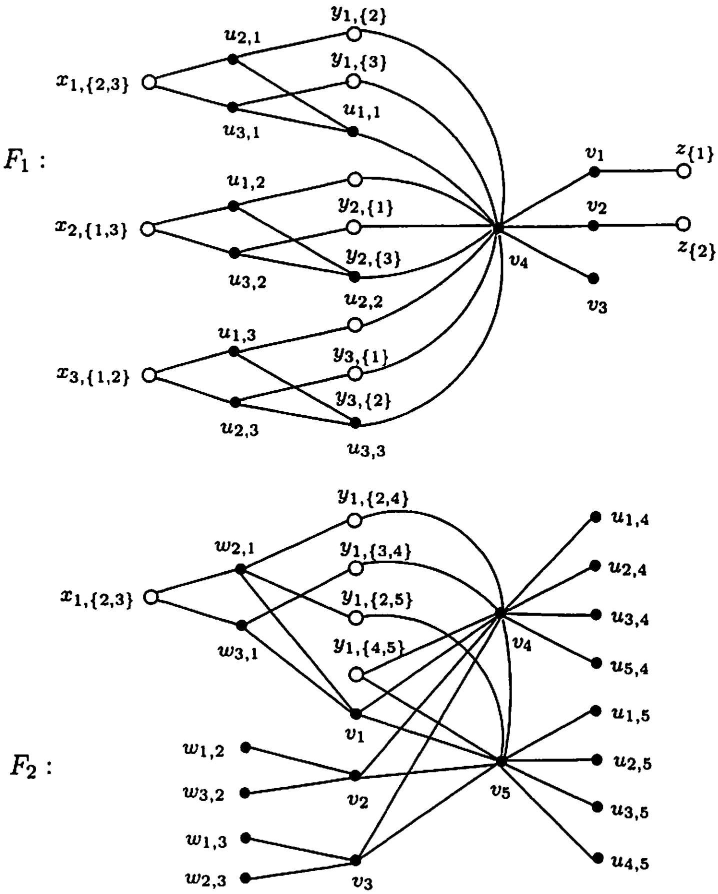

Figure 2: The graphs F_1 and F_2

Proof. We first verify that $\chi_{si}(G) \leq \chi_{si}(G-v)+1$. Suppose that $\chi_{si}(G-v) = k$ and let c be a singular k-coloring of $G - v$. Define a coloring c' of G by $c'(x) = c(x)$ if $x \neq v$ and $c'(v) = k+1$. Then c' is clearly a singular coloring of G using $k+1$ colors and so $\chi_{si}(G) \leq k+1 = \chi_{si}(G-v)+1$.

To show that $\chi_{si}(G-v) \leq \chi_{si}(G) + \deg v$, suppose that $\chi_{si}(G) = k$ and let c be a singular k-coloring of G. If v is an isolated vertex, then the coloring c restricted to $V(G-v)$ is a singular coloring using at most k colors. Hence, $\chi_{si}(G-v) \leq k+0 = \chi_{si}(G) + \deg v$. Otherwise, let $N(v) = \{v_1, v_2, \ldots, v_d\}$, where $d = \deg v$, and define a coloring c' of $G-v$ by $c'(x) = c(x)$ if $x \notin N(v)$ and $c'(v_i) = k+i$ for $1 \leq i \leq d$. We now verify that c' is a singular coloring of $G-v$. Let x_1 and x_2 be two distinct vertices in $V(G-v)$. Also, let $\text{code}_c(x_i) = (c(x_i), S_{x_i})$ and $\text{code}_{c'}(x_i) = (c'(x_i), S'_{x_i})$ be the codes of x_i with respect to c and c', respectively, for $i = 1,2$. If $c'(x_1) \neq c'(x_2)$, then certainly $\text{code}_{c'}(x_1) \neq \text{code}_{c'}(x_2)$. If $c'(x_1) = c'(x_2)$, then $x_1, x_2 \notin N(v)$ and so $N_{G-v}(x_i) = N_G(x_i)$ for $i = 1,2$. Furthermore, $c(x_1) = c'(x_1) = c'(x_2) = c(x_2)$. Let $A_i = N(x_i) \cap N(v)$ and $B_i = N(x_i) - A_i$ for $i = 1,2$. If $A_1 \neq A_2$, say $v_1 \in A_1 - A_2$, then $k+1 \in S'_{x_1} - S'_{x_2}$ and so $\text{code}_{c'}(x_1) \neq \text{code}_{c'}(x_2)$. If $A_1 = A_2 = A$, then $S_{x_i} = \{c(u) : u \in A\} \cup \{c(u) : u \in B_i\}$ for $i = 1,2$, implying that $\{c(u) : u \in B_1\} \neq \{c(u) : u \in B_2\}$ since $\text{code}_c(x_1) \neq \text{code}_c(x_2)$ and $c(x_1) = c(x_2)$. This in turn implies that $S'_{x_1} \neq S'_{x_2}$, since $S'_{x_i} = \{c'(u) : u \in A\} \cup \{c(u) : u \in B_i\}$ and $\{c'(u) : u \in A\} \cap \{c(u) : u \in B_i\} = \emptyset$ for $i = 1,2$. Hence, c' is a singular coloring of $G - v$ using at most $k + d$ colors, that is, $\chi_{si}(G-v) \leq k + d = k + \deg v$. ∎

Theorem 4.2 *For a nonempty graph G and an edge e in G,*

$$\chi_{si}(G) - 1 \leq \chi_{si}(G-e) \leq \chi_{si}(G) + 2.$$

Proof. Let $e = v_1v_2$. We first show that $\chi_{si}(G) \leq \chi_{si}(G-e)+1$. Assume that $\chi_{si}(G-e) = k$ and let c be a singular k-coloring of $G-e$. Define a coloring c' of G by $c'(x) = c(x)$ if $x \neq v_2$ and $c'(v_2) = k+1$. To verify that c' is a singular coloring of G, let x_1 and x_2 be two distinct vertices in $V(G)$. Also, let $\text{code}_c(x_i) = (c(x_i), S_{x_i})$ and $\text{code}_{c'}(x_i) = (c'(x_i), S'_{x_i})$ be the codes of x_i with respect to c and c', respectively, for $i = 1,2$. If $c'(x_1) \neq c'(x_2)$, then $\text{code}_{c'}(x_1) \neq \text{code}_{c'}(x_2)$. If $c'(x_1) = c'(x_2)$, then $v_2 \notin \{x_1, x_2\}$ and so $c(x_1) = c'(x_1) = c'(x_2) = c(x_2)$. If only one of x_1 and x_2 is adjacent to v_2 in G, say $x_1v_2 \in E(G)$, then $k+1 \in S'_{x_1} - S'_{x_2}$ and so $\text{code}_{c'}(x_1) \neq \text{code}_{c'}(x_2)$. If $v_2 \notin N_G(x_1) \cup N_G(x_2)$, then $\text{code}_{c'}(x_1) = \text{code}_c(x_1) \neq \text{code}_c(x_2) = \text{code}_{c'}(x_2)$. Finally, if $v_2 \in N_G(x_1) \cap N_G(x_2)$, then let $N_i = N_G(x_i) - \{v_2\}$ for $i = 1,2$. Then since $S_{x_i} = \{c(u) : u \in N_i\} \cup \{c(v_2)\}$ for $i = 1,2$ and $S_{x_1} \neq S_{x_2}$, it follows that $\{c(u) : u \in N_1\} \neq \{c(u) : u \in N_2\}$. This implies that

$$S'_{x_1} = \{c'(u) : u \in N_1\} \cup \{k+1\} \neq \{c'(u) : u \in N_2\} \cup \{k+1\} = S'_{x_2},$$

that is, $\text{code}_{c'}(x_1) \neq \text{code}_{c'}(x_2)$. Since c' uses at most $k+1$ colors, $\chi_{si}(G) \leq k+1 = \chi_{si}(G-e)+1$.

We next show that $\chi_{si}(G-e) \leq \chi_{si}(G)+2$. Assume that $\chi_{si}(G) = k$ and let c be a singular k-coloring of G. We then obtain a coloring c' of $G-e$ defined by $c'(x) = c(x)$ if $x \neq v_1, v_2$ and $c'(v_i) = k+i$ for $i = 1,2$. Then an argument similar to the one used above will show that c' is a singular coloring of $G-e$. Since c' uses at most $k+2$ colors, it follows that $\chi_{si}(G-e) \leq k+2 = \chi_{si}(G)+2$. ∎

All upper and lower bounds described in Theorems 4.1 and 4.2 are sharp. For vertex deletion, let d be a nonnegative integer and i an integer with $-1 \leq i \leq d$. Then there exists a graph G containing a vertex v whose degree is d and such that $\chi_{si}(G-v) = \chi_{si}(G) + i$. If $d = 0$, then observe that $\chi_{si}((n-1)K_1) = \chi_{si}(nK_1) - 1$ and $\chi_{si}(K_n) = \chi_{si}(K_n \cup K_1)$ for $n \geq 2$. If $d \geq 1$, then let $G_{-1} = K_{d+1}$. For $0 \leq i \leq d$, let G_i be the graph of order $i + d + 3$ obtained from a complete bipartite graph with partite sets $\{v_1, v_2\}$ and $\{v_3, v_4, \ldots, v_{d+2}\}$ by adding $i + 1$ pendant edges at v_1. Then $\chi_{si}(G_{-1} - v) = \chi_{si}(G_{-1}) - 1$; while for $0 \leq i \leq d$, observe that $\chi_{si}(G_i - v_2) = \chi_{si}(K_{1,i+d+1}) = (d+2)+i = \chi_{si}(G_i)+i$. For edge deletion, observe for each positive integer n that $\chi_{si}(K_{1,n} \cup K_1) = \chi_{si}(K_{1,n+1}) - 1$, $\chi_{si}(K_{2,n}) = \chi_{si}(K_2 + nK_1)$, $\chi_{si}(K_{1,n} \cup (n+3)K_1) = \chi_{si}(K_{1,n+1} \cup (n+2)K_1) + 1$, and $\chi_{si}(K_{1,n+3}) = \chi_{si}(K_{1,n+3} + e) + 2$.

5 The Singular Chromatic Numbers of Cycles and Paths

Recall that, for a graph G and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of G for some positive integer k, the *irregular color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\text{code}_{ir}(v) = (a_0, a_1, \ldots, a_k)$ where a_0 is the color assigned to v and a_i is the number of vertices adjacent to v that are colored i for $1 \leq i \leq k$. The coloring c is then *irregular* if distinct vertices have distinct irregular color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. This concept was first introduced by Radcliffe and Zhang [21, 22] and further studied by Okamoto, Radcliffe, and Zhang [20] and Anderson, Barrientos, Bringham, Carrington, Kronman, Vitray, and Yellen [4]. In particular, the irregular chromatic numbers of cycles and paths have been determined. We state the result on cycles as follows.

Theorem 5.1 [4] *For each integer $n \geq 3$, let k be the unique positive integer such that $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$. Then*

$$\chi_{ir}(C_n) = \begin{cases} k & \text{if } n \neq k\binom{k}{2} - 1 \\ k+1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

For each vertex v of G, let M_v be the *multiset* of colors of the neighbors of v. Then $\text{code}_{ir}(u) \neq \text{code}_{ir}(v)$ if and only if $(c(u), M_u) \neq (c(v), M_v)$. If c is a singular coloring of a graph G, then for every two distinct vertices u and v, either (i) $c(u) \neq c(v)$ or (ii) $S_u \neq S_v$. If $S_u \neq S_v$, then certainly $M_u \neq M_v$. Hence, every singular coloring of a graph G is an irregular coloring of G. The following is an immediate consequence of this.

Observation 5.2 *For every graph G, $\chi(G) \leq \chi_{ir}(G) \leq \chi_{si}(G)$.*

Observe also that an irregular coloring of a graph G is not necessarily a singular coloring of G. For example, the proper 2-coloring of P_4 is irregular but not singular and, in fact, $\chi_{ir}(P_4) = 2$ while $\chi_{si}(P_4) = 3$. For cycles, however, the two concepts are actually interchangeable, that is, $\chi_{ir}(G) = \chi_{si}(G)$ if G is a cycle. The following is therefore an immediate consequence of Theorem 5.1.

Corollary 5.3 *For each integer $n \geq 3$, let k be the unique positive integer such that $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$. Then*

$$\chi_{si}(C_n) = \begin{cases} k & \text{if } n \neq k\binom{k}{2} - 1 \\ k+1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

Now let us consider paths in more detail. As we saw earlier, there are paths P for which $\chi_{ir}(P) < \chi_{si}(P)$. We first present some useful lemmas and observations. If c is a coloring of the vertices of an n-cycle $C : v_1, v_2, \ldots, v_n, v_1$, then the *color sequence* of C (with respect to c) is $c(v_1), c(v_2), \ldots, c(v_n), c(v_1)$.

Lemma 5.4 *If c is a singular k-coloring of an n-cycle such that there are two vertices u and v with* $\text{code}(u) = (c(u), \{c(v)\})$ *and* $\text{code}(v) = (c(v), \{c(u)\})$, *then there exists a singular k-coloring of an n-cycle C with respect to which the color sequence of C contains a subsequence of the form $c(u), c(v), c(u), c(v)$ (or $c(v), c(u), c(v), c(u)$).*

Proof. Suppose that c is a singular k-coloring of an n-cycle $C : v_1, v_2, \ldots, v_n, v_1$. Without loss of generality, assume that $\text{code}(v_2) = (2, \{1\})$ and $\text{code}(v_\ell) = (1, \{2\})$, where ℓ is an integer with $1 \leq \ell \leq n$ and $\ell \neq 2$. If $\ell \in \{1, 3\}$, then the result immediately follows. Otherwise, $6 \leq \ell \leq n - 2$. Let c' be the coloring of C such that the color sequence of C (with respect to c') is

$$s : c(v_1), c(v_2), c(v_\ell), c(v_{\ell-1}), \ldots, c(v_3), c(v_{\ell+1}), c(v_{\ell+2}), \ldots, c(v_n), c(v_1).$$

Then c' is a singular k-coloring of C. Furthermore, the first four terms of s are $1, 2, 1, 2$. ■

Suppose that c is a singular k-coloring of P_n. Since the number of possible codes is $k\binom{k-1}{2} + k(k-1) = k\binom{k}{2}$, it follows that $n \leq k\binom{k}{2}$. We state this fact as follows.

Observation 5.5 *If $\chi_{si}(P_n) = k$, then $n \le k\binom{k}{2}$.*

If c is a coloring of the vertices of a path $P: v_1, v_2, \ldots, v_n$, then the *color sequence* of P (with respect to c) is $c(v_1), c(v_2), \ldots, c(v_n)$. For $4 \le n \le 9$, let s_n be the sequence given by

$s_4: 1,2,1,3$	$s_6: 1,2,1,3,2,3$	$s_8: 1,2,1,3,1,3,2,3$
$s_5: 1,2,1,3,2$	$s_7: 1,2,3,2,3,1,3$	$s_9: 1,2,3,2,3,1,3,1,2$

and observe that the coloring of P_n whose color sequence is s_n is a singular 3-coloring of P_n. Hence, $\chi_{si}(P_1) = 1$, $\chi_{si}(P_2) = 2$, and $\chi_{si}(P_n) = 3$ for $3 \le n \le 9$. We next consider paths of order $n \ge 10$.

Lemma 5.6 *If $n \ge 10$ and k is a positive integer such that $(k-1)\binom{k}{2}+1 \le n \le k\binom{k}{2}$, then $\chi_{si}(P_n) = k$.*

Proof. Note that $k \ge 4$. Since $n > (k-1)\binom{k-1}{2}$, it follows that $\chi_{si}(P_n) \ge k$ by Observation 5.5. First, assume that $n \ne k\binom{k}{2} - 1$ and let $C: v_1, v_2, \ldots, v_n, v_1$ be an n-cycle. By Corollary 5.3, there exists a singular k-coloring c of C. Furthermore, since $n > (k-1)\binom{k}{2}$, there exist two vertices u and v such that code$(u) = (c(u), \{c(v)\})$ and code$(v) = (c(v), \{c(u)\})$, say code$(u) = (1, \{2\})$ and code$(v) = (2, \{1\})$. By Lemma 5.4, we may assume that $c(v_1) = c(v_3) = 1$ and $c(v_2) = c(v_4) = 2$. Then the coloring of a path of order n whose color sequence is $c(v_3), c(v_4), \ldots, c(v_n), c(v_1), c(v_2)$ is a singular k-coloring.

If $n = k\binom{k}{2} - 1$, then let c be a singular k-coloring of $C \cong C_{n^*}$, where $n^* = k\binom{k}{2}$. Furthermore, we may assume that the color sequence s of C contains $1,2,1,2$ as a subsequence. Let $C: v_1, v_2, \ldots, v_{n^*}, v_1$ and suppose that $c(v_1) = c(v_3) = 1$ and $c(v_2) = c(v_4) = 2$. Furthermore, suppose that $c(v_5) = 3$. Since c is a singular k-coloring of a cycle of order $k\binom{k}{2}$, every possible code must be used. Hence, we may also assume that s contains either (i) $2,3,2,3$ or (ii) $3,2,3,2$ as a subsequence as well.

If (i) occurs, then we may assume that $c(v_\ell) = c(v_{\ell+2}) = 2$ and $c(v_{\ell+1}) = c(v_{\ell+3}) = 3$ for some integer ℓ with $4 \le \ell \le n^* - 3$ and $\ell \ne 5$. Let

$$s: \ c(v_{\ell+1}), c(v_\ell), \ldots, c(v_5), c(v_{\ell+2}), c(v_{\ell+3}), \ldots, c(v_{n^*}), c(v_1), c(v_2), c(v_3).$$

If (ii) occurs, then assume that $c(v_\ell) = c(v_{\ell+2}) = 3$ and $c(v_{\ell+1}) = c(v_{\ell+3}) = 2$ for some integer ℓ with $7 \le \ell \le n^* - 4$ and let

$$s: \ c(v_{\ell+2}), c(v_{\ell+1}), c(v_\ell), \ldots, c(v_5), c(v_{\ell+3}), c(v_{\ell+4}), \ldots, c(v_{n^*}), \\ c(v_1), c(v_2), c(v_3).$$

In each case, the coloring of a path of order $n^* - 1$ having s as the color sequence is a singular k-coloring. ■

Now we are prepared to present the complete result on paths.

Theorem 5.7 *For every integer $n \geq 2$, $\chi_{si}(P_n) = k$, where k is the unique positive integer such that $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$.*

Proof. Since the result has been verified for $2 \leq n \leq 9$, assume that $n \geq 10$. Hence, $k \geq 4$. By Observation 5.5, it suffices to show that $\chi_{si}(P_n) \leq k$ by providing a singular k-coloring of P_n.

For $k = 4$, we may assume that $10 \leq n \leq 18$ by Lemma 5.6. Let $A_1 : v_1, v_2, \ldots, v_9$ be a path of order 9 and let c_1 be a 4-coloring of A_1 such that the color sequence of A_1 is $4, 1, 2, 4, 2, 3, 4, 3, 4$. Let $A_2 : u_1, u_2, \ldots, u_{n-9}$ be a path of order $n-9$. Since $1 \leq n-9 \leq 9$, there exists a singular 3-coloring c_2 of A_2 and we may assume that $c_2(u_1) = 1$ and $c_2(u_2) = 3$ (if $n \geq 11$). Let A be a path of order n obtained from A_1 and A_2 by adding the edge v_9u_1 and consider the coloring c such that $c(v) = c_i(v)$ if $v \in V(A_i)$ for $i = 1, 2$. Then c is a singular 4-coloring of A. Hence, $\chi_{si}(P_n) = 4$ for $10 \leq n \leq 18$. Consequently, $\chi_{si}(P_n) = 4$ for $10 \leq n \leq 24$.

For $k = 5$, assume that $25 \leq n \leq 40$ by Lemma 5.6. Let $A_1 : v_1, v_2, \ldots, v_{16}$ be a path of order 16 and let c_1 be a 5-coloring of A_1 such that the color sequence of A_1 is $5, 1, 2, 5, 2, 3, 5, 3, 4, 5, 4, 5, 1, 5, 2, 5$. Let $A_2 : u_1, u_2, \ldots, u_{n-16}$ be a path of order $n-16$. Since $9 \leq n-16 \leq 24$, there exists a singular 4-coloring c_2 of A_2 and we may assume that $c_2(u_1) = 3$ and $c_2(u_2) = 1$. Let A be a path of order n obtained from A_1 and A_2 by adding the edge $v_{16}u_1$ and consider the coloring c such that $c(v) = c_i(v)$ if $v \in V(A_i)$ for $i = 1, 2$. Then c is a singular 5-coloring of A. Hence, $\chi_{si}(P_n) = 5$ for $25 \leq n \leq 50$.

Now suppose that $\chi_{si}(P_n) = k$ for every n with $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$ for some $k \geq 5$. We show that $\chi_{si}(P_n) = k+1$ for every n with $k\binom{k}{2} + 1 \leq n \leq (k+1)\binom{k+1}{2}$. By Lemma 5.6, we may further assume that $k\binom{k}{2} + 1 \leq n \leq k\binom{k+1}{2}$.

First suppose that k is odd, say $k = 2\ell + 1$ for some integer $\ell \geq 2$. Consider a complete graph H of order $2\ell + 1$ whose vertex set is $\mathbb{N}_{2\ell+1}$. Then H can be factored into ℓ Hamiltonian cycles $H_1, H_2, \ldots, H_\ell$. Let $H_i : a_{1,i}, a_{2,i}, \ldots, a_{2\ell+1,i}, a_{1,i}$ for $1 \leq i \leq \ell$. Consider ℓ copies $B_1, B_2, \ldots, B_\ell$ of a path of order $6\ell+3$, where $B_i : v_{1,i}, v_{2,i}, \ldots, v_{6\ell+3,i}$. Let c_1 be a $(2\ell+2)$-

coloring of the graph $\cup_{i=1}^{\ell} B_i$ such that the color sequence s_i of B_i is

$$s_i : 2\ell+2, a_{1,i}, a_{2,i}, 2\ell+2, a_{3,i}, a_{4,i}, \ldots, 2\ell+2, a_{2\ell-1,i}, a_{2\ell,i}, 2\ell+2, a_{2\ell+1,i}, a_{1,i},$$
$$2\ell+2, a_{2,i}, a_{3,i}, 2\ell+2, a_{4,i}, a_{5,i}, \ldots, 2\ell+2, a_{2\ell,i}, a_{2\ell+1,i}.$$

Let A_1 be the graph obtained from the paths $B_1, B_2, \ldots, B_\ell$ by (i) adding the edge $v_{6\ell+3,i}v_{1,i+1}$ for $1 \le i \le \ell-1$ and (ii) deleting the two vertices $v_{6\ell+2,\ell}$ and $v_{6\ell+3,\ell}$. Hence A_1 is a path of order $n_1 = \ell(6\ell+3)-2$, where

$$k^2 = (2\ell+1)^2 < n_1 < (2\ell+1)\binom{2\ell+1}{2} + 1 = k\binom{k}{2} + 1.$$

Let $A_2 : u_1, u_2, \ldots, u_{n_2}$ be a path of order $n_2 = n - n_1$. Since $k\binom{k}{2} + 1 \le n \le k\binom{k+1}{2}$, it follows that $0 < n_2 < k\binom{k}{2}$. Hence, there exists a singular k-coloring of A_2 using the colors in $\mathbb{N}_k$. In particular, let c_2 be a singular k-coloring of A_2 such that $c_2(u_1) = a_{2\ell,\ell}$ and $c_2(u_2) = a_{2\ell+1,\ell}$. Then obtain $A \cong P_n$ from A_1 and A_2 by adding the edge $v_{6\ell+1,\ell}u_1$ and observe that the coloring c such that $c(v) = c_i(v)$ if $v \in V(A_i)$ for $i = 1, 2$ is a singular $(2\ell+2)$-coloring of A.

If k is even, then write $k = 2\ell$, where ℓ is an integer with $\ell \ge 3$. Consider a complete graph H of order 2ℓ whose vertex set is $\mathbb{N}_{2\ell}$. Then H can be factored into $\ell-1$ Hamiltonian cycles $H_1, H_2, \ldots, H_{\ell-1}$ and a 1-factor H_0. Let $H_i : a_{1,i}, a_{2,i}, \ldots, a_{2\ell,i}, a_{1,i}$ for $1 \le i \le \ell-1$. Consider $\ell-1$ copies $B_1, B_2, \ldots, B_{\ell-1}$ of a path of order 6ℓ, where $B_i : v_{1,i}, v_{2,i}, \ldots, v_{6\ell,i}$. Furthermore, let $B_0 : v_{1,0}, v_{2,0}$ be a path of order 2. Let c_1 be a $(2\ell+1)$-coloring of the graph $\cup_{i=0}^{\ell-1} B_i$ such that the color sequence s_i of B_i is

$$s_i : 2\ell+1, a_{1,i}, a_{2,i}, 2\ell+1, a_{3,i}, a_{4,i}, \ldots, 2\ell+1, a_{2\ell-1,i}, a_{2\ell,i},$$
$$2\ell+1, a_{1,i}, a_{2\ell,i}, 2\ell+1, a_{2\ell-1,i}, a_{2\ell-2,i}, \ldots, 2\ell+1, a_{3,i}, a_{2,i}$$

for $1 \le i \le \ell-1$, while $s_0 : 2\ell+1, a$, where $a \in \mathbb{N}_{2\ell}$ and $a_{1,1}a \in E(H_0)$. Let A_1 be the graph obtained from the paths $B_0, B_1, \ldots, B_{\ell-1}$ by (i) adding the edge $v_{2,0}v_{1,1}$, (ii) adding the edge $v_{6\ell,i}v_{1,i+1}$ for $1 \le i \le \ell-2$, and (iii) deleting the two vertices $v_{6\ell-1,\ell-1}$ and $v_{6\ell,\ell-1}$. Hence A_1 is a path of order $n_1 = (\ell-1)(6\ell)$, where

$$k^2 = (2\ell)^2 \le n_1 < (2\ell)\binom{2\ell}{2} + 1 = k\binom{k}{2} + 1.$$

Let $A_2 : u_1, u_2, \ldots, u_{n_2}$ be a path of order $n_2 = n - n_1$. Since $k\binom{k}{2} + 1 \le n \le k\binom{k+1}{2}$, it follows that $0 < n_2 \le k\binom{k}{2}$. Hence, there exists a singular k-coloring of A_2 using the colors in $\mathbb{N}_k$. Let c_2 be a singular k-coloring of A_2 such that $c_2(u_1) = a_{3,\ell-1}$ and $c_2(u_2) = a_{2,\ell-1}$. Then obtain $A \cong P_n$ from A_1 and A_2 by adding the edge $v_{6\ell,\ell-1}u_1$ and observe that the coloring c

defined by $c(v) = c_i(v)$ if $v \in V(A_i)$ for $i = 1, 2$ is a singular $(2\ell+1)$-coloring of A. This completes the proof. ■

For example, consider a path of order 60. Since $5\binom{5}{2} + 1 \le 60 \le 6\binom{6}{2}$, we first consider the graph $H \cong K_5$ with $V(H) = \mathbb{N}_5$. Observe that H can be factored into two Hamiltonian cycles H_1 and H_2, say $H_1 : 1, 2, 3, 4, 5, 1$ and $H_2 : 1, 3, 5, 2, 4, 1$. Then let $A_1 \cong P_{28}$ and consider a 6-coloring of A_1 whose color sequence is

$$s_{A_1} :\ 6, 1, 2, 6, 3, 4, 6, 5, 1, 6, 2, 3, 6, 4, 5, 6, 1, 3, 6, 5, 2, 6, 4, 1, 6, 3, 5, 6.$$

Let $A_2 \cong P_{32}$. Since $4\binom{4}{2} + 1 \le 32 \le 5\binom{5}{2}$, there exists a singular 5-coloring of A_2 the first two terms of whose color sequence s_{A_2} are $2, 4$. Then the coloring of P_{60} whose color sequence is s_{A_1} followed by s_{A_2} is a singular 6-coloring.

We summarize the results on cycles and paths as follows.

Theorem 5.8 *For each $n \ge 3$, let k be the unique positive integer such that $(k-1)\binom{k-1}{2} + 1 \le n \le k\binom{k}{2}$. Then*

$$\chi_{si}(P_n) = k$$

$$\chi_{si}(C_n) = \begin{cases} k & \text{if } n \ne k\binom{k}{2} - 1 \\ k+1 & \text{if } n = k\binom{k}{2} - 1. \end{cases}$$

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