On the generalized k-Pell (p, i)-numbers

Kamil Arı*

Karamanoğlu Mehmetbey University,

Faculty of Kamil Özdağ Science,

Department of Mathematics, 70100 Karaman, Turkey

Abstract

The current article focus on the generalized k-Pell (p,i)-numbers for $k = 1, 2, \ldots$ and $0 \leq i \leq p$. It introduces the generalized k-Pell (p,i)-numbers and their generating matrices and generating functions. Some interesting identities are established.

Keywords and phrases: Fibonacci numbers; Pell numbers; Binet's formula; Generating matrix; Generating function.

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1 Introduction

The uses of Fibonacci and Fibonacci-like numbers in many areas of science and engineering are quite remarkable: number theory, combinatorics, special functions, numerical analysis, linear algebra, statistics, etc. The basic properties of Fibonacci and Fibonacci-like numbers are well known and are outlined, for example in [23].

* *E-mail address:* kamilari@kmu.edu.tr

The Fibonacci numbers F_n are the terms of the sequence {0, 1, 1, 2, 3, 5, ...} wherein each term is the sum of the two preceeding terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

In literature, one can find many interesting generalizations of the classic Fibonacci sequence. For example, Horadam's sequence $\{w_n(a, b; p, q)\}$, or briefly $\{w_n\}$ is defined by the recurrence relation $w_0 = a$, $w_1 = b$, $w_n = pw_{n-1} - qw_{n-2}$ $(n \geq 2)$ (see [12]). This sequence is generalized Fibonacci sequence. In [12], Horadam studied the generating function of powers of $\{w_n\}$. Another interesting generalization given by Falcón and Plaza [6] is defined by the follwing equation for any given integer number $k \geq 1$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1,$$

and called k-Fibonacci numbers. This sequence that generalizes, between others, both the classic Fibonacci sequence and the Pell sequence. It is obvious that when $k = 1$, then nth k-Fibonacci number is the nth classic Fibonacci number. In [6], Falcón and Plaza showed the relation between the 4-triangle longest-edge (4TLE) partition and the k-Fibonacci numbers, as another example of the relation between geometry and numbers, and many properties of these numbers are deduced directly from elementary matrix algebra. In [7], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [8], the 3-dimensional k-Fibonacci spirals are studied from a geometric point of view. These curves appear naturally from studying the k-th Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci functions. In [9], the author introduces some sequences obtained from the k-Fibonacci sequences and then some properties of the k-Lucas numbers are proved. In [10], the authors introduce generalized Fibonacci sequences and related identities consisting even and odd terms. Also, in that paper, they present connection formulas for generalized Fibonacci sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence. In [17], the authors introduce the k-Generalized Fibonacci sequence, and then establish some of the interesting properties of k-Generalized Fibonacci numbers. Also, in that paper, they present properties of k-Generalized Fibonacci numbers like Catalan's identity, Cassini's identity and d'ocagnes's identity. In [4], the author consider the k-Fibonacci sequence and many identities are proved for the k-Fibonacci number.

However, another generalization of the classic Fibonacci sequence is given by Stakhov (see [21] and [22]). This generalization is called Fibonacci p-numbers, and defined by the following equation for any given p $(p = 1, 2, ...)$ and $n > p + 1$

$$F_p(n) = F_p(n-1) + F_p(n-p-1),$$

with initial conditions $F_p(1) = ... = F_p(p) = F_p(p+1) = 1$. When $p = 1$, then the sequence of Fibonacci p-numbers, $\{F_p(n)\}$, is reduced to the classic Fibonacci sequence. Also generalizations of Pell numbers can be found in the literature. In [1], P. Catarino consider a generalization of Pell numbers, which the author calls the k-Pell numbers. In [2], the authors give other generalizations which involve other type of Pell numbers, namely the k-Pell-Lucas sequence, in this paper also many properties are proved for the k-Pell-Lucas numbers. In [5],using a diagonal matrix the author get the Binet's formula for k-Pell sequence. Also, in that paper, the n^{th} power of the generating matrix for k-Pell-Lucas sequence is established and basic properties involving the determinant allowed us to obtain its Cassini's identity are given. In [3], the authors consider the k-Pell numbers sequence and present some properties involving the k-Pell numbers. Also, in that paper, using generating matrices the explicit formula for the term of order

n of the k-Pell numbers sequence are given and also using linear algebra the well-known Cassini's identity is obtained. Also in [13], the author consider the fair generalization of the Pell numbers, which he calls the generalized Pell (p,i)-numbers. In that paper, the generalized Pell (p,i)-numbers is defined by the following equation for any given p $(p = 1, 2, ...)$ and $n > p+1$ and $0 \leq i \leq p$

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1),$$

with initial conditions

$$P_p^{(0)}(1) = P_p^{(0)}(2) = ... = P_p^{(0)}(p+1) = 1,$$

$$P_p^{(i)}(1) = P_p^{(i)}(2) = ... = P_p^{(i)}(i) = 0,$$

$$P_p^{(i)}(i+1) = P_p^{(i)}(i+2) = ... = P_p^{(i)}(p+1) = 1,$$

and it is given relationship between the generalized Pell (p,i)-numbers and the generating matrices given for these numbers.

In this paper, we introduce the generalized k-Pell (p,i)-sequence that generalizes the k-Fibonacci sequence given in [6] by Falcón and Plaza, and the Pell (p,i)-sequence given by Kilic in [13]. We present the generating matrices and generating functions for the generalized k-Pell (p,i)-numbers. We show that the characteristic equation of the generalized k-Pell (p,i)-numbers does not have multiple roots for $1 \leq k \in \mathbb{Z}$ and $1 < p \in \mathbb{Z}$.

2 Main Results

We now introduce a new generalization of Fibonacci numbers called as the generalized k-Pell (p,i)-sequence, and then give relationships between these numbers and generating matrices.

Definition 1 *The generalized k-Pell (p,i)-sequence, say $\left\{U_k^{(p,i)}(n)\right\}$, is defined as shown, for any given p, k $(p,k=1,2,...)$ and $n>p+1$ and $0\leq i\leq p$*

$$U_k^{(p,i)}(n)=kU_k^{(p,i)}(n-1)+U_k^{(p,i)}(n-p-1),$$

with initial conditions

$$U_k^{(p,0)}(1)=U_k^{(p,0)}(2)=...=U_k^{(p,0)}(p+1)=1,$$

$$U_k^{(p,i)}(1)=U_k^{(p,i)}(2)=...=U_k^{(p,i)}(i)=0,$$

$$U_k^{(p,i)}(i+1)=U_k^{(p,i)}(i+2)=...=U_k^{(p,i)}(p+1)=1.$$

Particular cases are:

- If $i=p=1$, then the nth generalized k-Pell $(1,1)$-sequence is the $(n+1)$th generalized k-Fibonacci sequence considered in [6],
- If $i=p=1$ and $k=2$, then the nth generalized 2-Pell $(1,1)$-sequence is the $(n+1)$th usual Pell sequence considered in [16],
- If $k=1$, then the nth generalized 1-Pell (p,i)-sequence is the nth generalized Fibonacci-(p,i) sequence considered in [15],
- If $k=2$, then the nth generalized 2-Pell (p,i)-sequence is the nth generalized Pell-(p,i) sequence considered in [13].

Generating matrices are very important tools for obtaining results in number theory. Generating matrices of Fibonacci and Fibonacci-like numbers have been studied in many papers; see for example [11, 13, 14, 19]. We now introduce generating matrix for the generalized k-Pell (p,i)-numbers.

Theorem 2 *For $n, p > 0$ and $k = 1, 2, ...,$ define the $(p+1) \times (p+1)$ matrix H_n as follows:*

$$H_n = \begin{bmatrix} U_k^{(p,0)}(n+p+2) & U_k^{(p,p-1)}(n+p+1) & \cdots & U_k^{(p,0)}(n+p+1) \\ U_k^{(p,0)}(n+p+1) & U_k^{(p,p-1)}(n+p) & \cdots & U_k^{(p,0)}(n+p) \\ U_k^{(p,0)}(n+p) & U_k^{(p,p-1)}(n+p-1) & \cdots & U_k^{(p,0)}(n+p-1) \\ \vdots & \vdots & \cdots & \vdots \\ U_k^{(p,0)}(n+3) & U_k^{(p,p-1)}(n+2) & \cdots & U_k^{(p,0)}(n+2) \\ U_k^{(p,0)}(n+2) & U_k^{(p,p-1)}(n+1) & \cdots & U_k^{(p,0)}(n+1) \end{bmatrix}.$$

Then

$$H_n = A^n E,$$

where the matrices A and E are $(p+1) \times (p+1)$ matrices such that

$$A = \begin{bmatrix} k & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \tag{1}$$

and

$$E = \begin{bmatrix} k+1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

respectively.

Proof. If $n = 1$, then we write

$$H_1 = \begin{bmatrix} U_k^{(p,0)}(p+3) & U_k^{(p,p-1)}(p+2) & \cdots & U_k^{(p,0)}(p+2) \\ U_k^{(p,0)}(p+2) & U_k^{(p,p-1)}(p+1) & \cdots & U_k^{(p,0)}(p+1) \\ U_k^{(p,0)}(p+1) & U_k^{(p,p-1)}(p) & \cdots & U_k^{(p,0)}(p) \\ \vdots & \vdots & & \vdots \\ U_k^{(p,0)}(4) & U_k^{(p,p-1)}(3) & \cdots & U_k^{(p,0)}(3) \\ U_k^{(p,0)}(3) & U_k^{(p,p-1)}(2) & \cdots & U_k^{(p,0)}(2) \end{bmatrix}.$$

By the definiton of the generalized k-Pell (p,i)-numbers, we have the matrix H_1:

$$H_1 = \begin{bmatrix} k^2+k+1 & k & k & k & \cdots & k & k+1 \\ k+1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Also by a simple calculation, we get

$$H_1 = AE,$$

which completes the proof for $n = 1$. Continuing the proof with induction on n, we suppose that the statement is true for $n-1$ and we prove it for n. Since the matrix A is a companion matrix, we can write

$$H_n = A^n E = AA^{n-1}E = AH_{n-1},$$

from where the proof is completed. ∎

Theorem 3 *For* $n, p > 0$ *and* $k = 1, 2, \ldots,$ *define the* $(p+1) \times (p+1)$

matrix G_n as follows:

$$G_n = \begin{bmatrix} U_k^{(p,p)}(n+p+1) & U_k^{(p,p)}(n+1) & \cdots & U_k^{(p,p)}(n+p) \\ U_k^{(p,p)}(n+p) & U_k^{(p,p)}(n) & \cdots & U_k^{(p,p)}(n+p-1) \\ U_k^{(p,p)}(n+p-1) & U_k^{(p,p)}(n-1) & \cdots & U_k^{(p,p)}(n+p-2) \\ \vdots & \vdots & & \vdots \\ U_k^{(p,p)}(n+2) & U_k^{(p,p)}(n-p+2) & \cdots & U_k^{(p,p)}(n+1) \\ U_k^{(p,p)}(n+1) & U_k^{(p,p)}(n-p+1) & \cdots & U_k^{(p,p)}(n) \end{bmatrix}.$$

Then

$$G_n = A^n,$$

where A is matrix in (1).

Proof. If we consider the definiton of the generalized k-Pell (p,i)-numbers, we can write immediately following matrix-vector relation:

$$\begin{bmatrix} U_k^{(p,p)}(n+p+1) \\ U_k^{(p,p)}(n+p) \\ U_k^{(p,p)}(n+p-1) \\ \vdots \\ U_k^{(p,p)}(n+2) \\ U_k^{(p,p)}(n+1) \end{bmatrix} = \begin{bmatrix} k & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_k^{(p,p)}(n+p) \\ U_k^{(p,p)}(n+p-1) \\ U_k^{(p,p)}(n+p-2) \\ \vdots \\ U_k^{(p,p)}(n+1) \\ U_k^{(p,p)}(n) \end{bmatrix}.$$

Generalizing the above matrix-vector relation to the $(p+1)$ columns, then we have

$$G_n = AG_{n-1}. \tag{2}$$

From the definiton of the generalized k-Pell (p,i)-numbers we obtain $G_1 = A$. Thus, by the inductive argument, from (2) we reach the following result

$$G_n = A^n.$$

This completes the proof. ∎

When $p=2$, then we obtain

$$G_n = \begin{bmatrix} U_k^{(p,2)}(n+3) & U_k^{(p,2)}(n+1) & U_k^{(p,2)}(n+2) \\ U_k^{(p,2)}(n+2) & U_k^{(p,2)}(n) & U_k^{(p,2)}(n+1) \\ U_k^{(p,2)}(n+1) & U_k^{(p,2)}(n-1) & U_k^{(p,2)}(n) \end{bmatrix} = \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n$$

and thus as a consequence of Theorem 3, we can give the Simpson formula of the generalized k-Pell $(2,2)$-numbers as shown:

$$\begin{aligned} 1 &= U_k^{(p,2)}(n+3)\left(U_k^{(p,2)}(n)\right)^2 + U_k^{(p,2)}(n-1)\left(U_k^{(p,2)}(n+2)\right)^2 \\ &+\left(U_k^{(p,2)}(n+1)\right)^3 - 2U_k^{(p,2)}(n)\,U_k^{(p,2)}(n+1)\,U_k^{(p,2)}(n+2) \\ &-U_k^{(p,2)}(n-1)\,U_k^{(p,2)}(n+1)\,U_k^{(p,2)}(n+3). \end{aligned}$$

Corollary 4 *For $n,p>0$ and $k=1,2,\ldots,$*

$$U_k^{(p,0)}(n+p+1) = (k+1)\,U_k^{(p,p)}(n+p) + \sum_{j=0}^{p-1} U_k^{(p,p)}(n+j).$$

Proof. Since $H_n = A^nE$, $A^n = G_n$, and the terms a_{21} of the matrices H_n and G_nE are aqual, the formula is obtained. ∎

We now show that the characteristic equation of the generalized k-Pell (p,i)-numbers does not have multiple roots for $1 \le k \in \mathbb{Z}$ and $1 < p \in \mathbb{Z}$.

Lemma 5 *Let $a_p = \frac{k}{p+1}\left(\frac{kp}{p+1}\right)^p$. Then, for $1 < k,p \in \mathbb{Z}$, we have that $a_p < a_{p+1}$.*

Proof. Since $p^2 > p^2-1$ for $1 < p \in \mathbb{Z}$, Kiliç [13] gave for $1 < p \in \mathbb{Z}$

$$\frac{1}{2}\left(\frac{p+1}{p}\right)^2 < \frac{p^2}{p^2-1}.$$

From where, we can easily write that for $1 < k, p \in \mathbb{Z}$

$$\frac{1}{k}\left(\frac{p+1}{p}\right)^2 < \frac{p^2}{p^2-1}.$$

Since also $1 < p \in \mathbb{Z}$, thus we can write for all p

$$\frac{1}{k}\left(\frac{p+1}{p}\right)^2 < \left(\frac{p^2}{p^2-1}\right)^{p-1}.$$

Then we may write from where, we get

$$\begin{aligned}\frac{1}{k}\left(\frac{p+1}{p}\right)^2 &< \left(\frac{p}{k(p-1)} \times \frac{kp}{p+1}\right)^{p-1} \\ &= \left(\frac{p}{k(p-1)}\right)^{p-1} \times \left(\frac{kp}{p+1}\right)^{p-1}.\end{aligned}$$

Therefore we have

$$\frac{1}{p}\left(\frac{k(p-1)}{p}\right)^{p-1} < \frac{1}{p+1}\left(\frac{kp}{p+1}\right)\left(\frac{kp}{p+1}\right)^{p-1},$$

and so

$$\frac{1}{p}\left(\frac{k(p-1)}{p}\right)^{p-1} < \frac{1}{p+1}\left(\frac{kp}{p+1}\right)^{p}.$$

From where, we get for $1 < k, p \in \mathbb{Z}$

$$\frac{k}{p}\left(\frac{k(p-1)}{p}\right)^{p-1} < \frac{k}{p+1}\left(\frac{kp}{p+1}\right)^{p}.$$

Thus, the result is obtained. ■

Lemma 6 *The equation $x^{p+1} - kx^p - 1 = 0$, which is the characteristic equation of the generalized k-Pell (p,i)-numbers, does not have multiple roots for $1 < k, p \in \mathbb{Z}$.*

Proof. Suppose that α is a multiple root of $f(x) = 0$ with $f(x) = x^{p+1} - kx^p - 1$. Since α is a multiple root of $f(x)$, we obtain $f'(\alpha) =$

$\alpha^{p-1}((p+1)\alpha - kp) = 0$. Since $\alpha \neq 0$, and $\alpha \neq 1$, we find $\alpha = \frac{kp}{p+1}$, and hence

$$\begin{aligned} 0 &= -f(\alpha) = -\alpha^{p+1} + k\alpha^p + 1 \\ &= \alpha^p(k-\alpha) + 1 \\ &= \left(\frac{kp}{p+1}\right)^p \left[k - \frac{kp}{p+1}\right] + 1 \\ &= a_p + 1. \end{aligned}$$

Since, by Lemma 5, $a_2 = \frac{4k^3}{27}$ and $a_p < a_{p+1}$ for $1 < k, p \in \mathbb{Z}$, $a_p \neq -1$, which is a contradiction. Thus, we reach the result which is that the characteristic equation of the generalized k-Pell (p,i)-numbers does not have multiple roots for $1 < k, p \in \mathbb{Z}$. ■

Generating functions are one of the most useful and clever tools in mathematics, computer science and statistics, see for example [18, 24] . By using generating functions, we can transform problems about sequences which they generate into problems about real valued functions.We now consider the generating function of the generalized k-Pell (p,p)-numbers.

Lemma 7 *Let $U_k^{(p,p)}(n)$, $n > p+1$ and $p > 1$, be the nth generalized k-Pell (p,p) number. Then,*

$$x^n = U_k^{(p,p)}(n+1)\, x^p + \sum_{j=0}^{p-1} U_k^{(p,p)}(n-j)\, x^j.$$

Proof. Since $n > p+1$ and $p > 1$, we first suppose that $p = 2$ and so $n = 4$. Since $U_k^{(p,2)}(5) = k^2$, $U_k^{(p,2)}(3) = 1$ and $U_k^{(p,2)}(4) = k$, and the characteristic equation of the generalized k-Pell $(p,2)$-numbers is $x^3 - kx^2 - 1$, we obtain

$$\begin{aligned} x^4 &= x.x^3 = x\left(kx^2+1\right) = kx^3 + x = k\left(kx^2+1\right) + x = k^2x^2 + x + k \\ &= U_k^{(p,2)}(5)\, x^2 + U_k^{(p,2)}(3)\, x + U_k^{(p,2)}(4). \end{aligned}$$

Thus the proof is complete for the first case. Continuing the proof with induction on n, we suppose that the statement is true for n and we prove it for $n+1$. From where and since the characteristic equation of the generalized k-Pell (p,p)-numbers is $x^{p+1}=kx^p+1$, we have

$$\begin{aligned}
x^{n+1} &= x^n.x=\left(U_k^{(p,p)}(n+1)\,x^p+\sum_{j=0}^{p-1}U_k^{(p,p)}(n-j)\,x^j\right)x \\
&= U_k^{(p,p)}(n+1)\,x^{p+1}+\sum_{j=0}^{p-1}U_k^{(p,p)}(n-j)\,x^{j+1} \\
&= U_k^{(p,p)}(n+1)\,(kx^p+1)+\sum_{j=0}^{p-1}U_k^{(p,p)}(n-j)\,x^{j+1} \\
&= kU_k^{(p,p)}(n+1)\,x^p+U_k^{(p,p)}(n-p+1)\,x^p \\
&\quad +U_k^{(p,p)}(n-p+2)\,x^{p-1}+...+U_k^{(p,p)}(n)\,x+U_k^{(p,p)}(n+1) \\
&= \left(kU_k^{(p,p)}(n+1)+U_k^{(p,p)}(n-p+1)\right)x^p+U_k^{(p,p)}(n-p+2)\,x^{p-1} \\
&\quad +...+U_k^{(p,p)}(n)\,x+U_k^{(p,p)}(n+1) \\
&= U_k^{(p,p)}(n+2)\,x^p+U_k^{(p,p)}(n-p+2)\,x^{p-1} \\
&\quad +...+U_k^{(p,p)}(n)\,x+U_k^{(p,p)}(n+1) \\
&= U_k^{(p,p)}(n+2)\,x^p+\sum_{j=0}^{p-1}U_k^{(p,p)}(n+1-j)\,x^j,
\end{aligned}$$

which completes the proof. ■

We now derive the generating function of the generalized k-Pell (p,p)-numbers, and then give exponential representation for these numbers. Let

$$\begin{aligned}
g_p(x) &= U_k^{(p,p)}(p+1)+U_k^{(p,p)}(p+2)\,x+U_k^{(p,p)}(p+3)\,x^2 \\
&\quad +...+U_k^{(p,p)}(n+p+1)\,x^n+....
\end{aligned}$$

So we obtain

$$g_p(x)-kxg_p(x)-x^{p+1}g_p(x)=\left(1-kx-x^{p+1}\right)g_p(x).$$

By the definition of the generalized k-Pell (p,p)-numbers, it can be written $\left(1-kx-x^{p+1}\right) g_p(x) = U_k^{(p,p)}(p+1) = 1$. Thus,

$$g_p(x) = \frac{1}{1-kx-x^{p+1}},$$

for $0 \leq kx + x^{p+1} < 1$.

Therefore, we get

$$\begin{aligned}
\ln g_p(x) &= \ln\left[1-kx-x^{p+1}\right]^{-1} \\
&= \left(kx+x^{p+1}\right) + \frac{1}{2}\left(kx+x^{p+1}\right)^2 + \ldots + \frac{1}{n}\left(kx+x^{p+1}\right)^n + \ldots \\
&= x\left[(k+x^p) + \frac{x}{2}(k+x^p)^2 + \ldots + \frac{x^{n-1}}{n}(k+x^p)^n + \ldots\right] \\
&= x\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(k+x^p)^n.
\end{aligned}$$

From where, we have

$$g_p(x) = \exp\left[x\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(k+x^p)^n\right].$$

References

[1] Catarino P., On some identities and generating functions for k-Pell numbers, *International Journal of Mathematical Analysis*, **7**(38) (2013), 1877-1884.

[2] Catarino P., Vasco P., On some Identities and Generating Functions for k- Pell-Lucas sequence, *Applied Mathematical Sciences*, **7**(98) (2013), 4867-4873.

[3] Catarino P., Vasco P., Some Basic Properties and a Two-by-Two Matrix involving the k-Pell numbers, *International Journal of Mathematical Analysis*, **7**(45) (2013), 2209-2215.

[4] Catarino P., On some identities for k-Fibonacci sequence, *Int. J. Contemp. Math. Sciences*, **9**(1) (2014), 37-42.

[5] Catarino P., A Note involving Two-by-Two Matrices of the k-Pell and k-Pell-Lucas sequences, *International Mathematical Forum*, **8**(32) (2013), 1561-1568.

[6] Falcón S., Plaza A., On the Fibonacci k-numbers. *Chaos, Solitons & Fractals*, **32**(5) (2007), 1615–24.

[7] Falcón S., Plaza A., The k-Fibonacci sequence and the Pascal 2-triangle. *Chaos, Solitons & Fractals*, **33**(1) (2007), 38–49.

[8] Falcón S., Plaza A., On the 3-dimensional k-Fibonacci spirals. *Chaos, Solitons & Fractals*, **38** (2008), 993-1003.

[9] Falcón S., On the k-Lucas Numbers, *Int. J. Contemp. Math. Sciences*, **6**(21) (2011), 1039-1050.

[10] Gupta V. K., Panwar Y. K., Sikhwal O., Generalized Fibonacci sequences, *Theoretical Mathematics & Applications*, **2**(2) (2012), 115-124.

[11] Hoggatt V.E., *Fibonacci and Lucas Numbers*, Houghton-Mifflin, 1969.

[12] Horadam A. F., Generating functions for powers of a certain generalized sequences of numbers, *Duke Math. J.*, **32** (1965), 437-446.

[13] Kılıç E., The generalized Pell (p, i)-numbers and their Binet formulas, combinatorial representations, sums. *Chaos, Solitons & Fractals*, **40**(4) (2009), 2047-2063.

[14] Kılıç E., The generalized order-k-Fibonacci–Pell sequence by matrix methods, *Journal of Computational and Applied Mathematics*, **209** (2007), 133–145.

[15] Kılıç E., Stakhov A. P., On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices, *Chaos, Solitons & Fractals*, **40**(5) (2009), 2210-2221.

[16] Koshy T., *Pell and Pell-Lucas Numbers with Applications*, Springer, 2014.

[17] Panwar Y. K., Singh M., k-Generalized Fibonacci Numbers, *Applied Mathematics and Physics*, **2**(1) (2014), 10-12.

[18] Rosen K.H., *Handbook of discrete and combinatorial mathematics*, CRC Press, Boca Raton, FL, 2000

[19] Stakhov AP., A generalization of the Fibonacci Q-matrix. *Rep Natl Acad Sci Ukraine*, **9** (1999), 46–9.

[20] Stakhov AP, Sluchenkova AA, Massingua V. *Introduction into Fibonacci coding and cryptography*, Kharkov, Osnova, 1999.

[21] Stakhov A., Rozin B. Theory of Binet formulas for Fibonacci and Lucas p-numbers. *Chaos, Solitons & Fractals*, **27**(5) (2005), 1163–77.

[22] Stakhov A., Rozin B. The continuous functions for the Fibonacci and Lucas p-numbers. *Chaos, Solitons & Fractals*, **28**(4) (2006), 1014–25.

[23] Vajda S., *Fibonacci and Lucas Numbers, and the Golden Section, Theory and applications*, John Wiley and Sons, New York, 1989.

[24] Wilf H.S., *Generatingfunctionology*, A K Peters 1994, second/ third ed., Second edition available from <http://www.math.upenn.edu/wilf/DownldGF.html>.