Two types of matchings extend to Hamiltonian cycles in hypercubes *

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Abstract

Ruskey and Savage asked the following question: For $n \geq 2$, does every matching in Q_n extend to a Hamiltonian cycle in Q_n? Fink showed that the answer is yes for every perfect matching, thereby proving Kreweras' conjecture. In this paper, we prove for $n \geq 3$ that every matching in Q_n not covering exactly two vertices at distance 3 extends to a Hamiltonian cycle in Q_n. An edge in Q_n is an *i-edge* if its endpoints differ in the ith position. We show for $n \geq 2$ that every matching in Q_n consisting of edges in at most four types extends to a Hamiltonian cycle in Q_n.

Keywords: Hypercube; Hamiltonian cycle; Matching; Perfect matching

1 Introduction

The n-dimensional hypercube is one of the most popular and efficient interconnection networks. There is a large amount of literature on graph-theoretic properties of hypercubes as well as on their applications in parallel computing; see [7, 9].

The *n-dimensional hypercube*, denoted by Q_n, is a graph whose vertex set consists of all binary strings of length n, i.e., $V(Q_n) = \{u : u = u^1 \cdots u^n$ and $u^i \in \{0,1\}$ for every $i \in \{1,\ldots,n\}\}$, with two vertices being adjacent whenever the corresponding strings differ in just one position; see Figure

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1 for example. An edge in Q_n is an *i-edge* if its endpoints differ in the *i*th position. So all the edges of Q_n can be divided into n types, i.e., 1-edges, ..., n-edges.

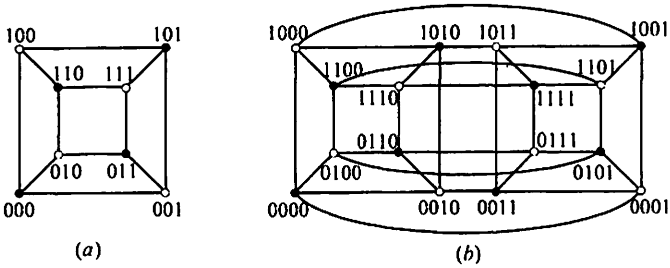


Figure 1. (a) 3-dimensional hypercube; (b) 4-dimensional hypercube.

It is well known that Q_n is Hamiltonian for every $n \geq 2$. This result dates back to 1872 [6]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention.

A set of edges in a graph G is called a *matching* if no two edges have an end-vertex in common. A matching of G is *perfect* if it covers all vertices of G and a matching is *maximal* if no matching with larger size contains it.

Ruskey and Savage [10] asked the following question: For $n \geq 2$, does every matching in Q_n extend to a Hamiltonian cycle in Q_n? Kreweras [8] conjectured for $n \geq 2$ that every perfect matching in Q_n extends to a Hamiltonian cycle in Q_n. Fink [3, 5] solved Kreweras' conjecture by proving the following stronger result. Let K_{Q_n} be the complete graph on the vertices of Q_n. Note that Q_n is a spanning subgraph of K_{Q_n}.

Theorem 1.1. *[3, 5] For every perfect matching M in K_{Q_n}, $n \geq 2$, there exists a perfect matching F in Q_n such that $M \cup F$ forms a Hamiltonian cycle in K_{Q_n}.*

Also, Fink [3] pointed out that the following result is true, and the present authors [11] provided a complete proof.

Lemma 1.2. *[3, 11] Every matching in Q_n extends to a Hamiltonian cycle in Q_n for $n \in \{2, 3, 4\}$.*

A complementary problem of Hamiltonian cycles in Q_n avoiding given matchings has been already settled for arbitrary matchings by Dimitrov et al. [1]. In particular, the authors in [1] proved that Q_n has a Hamiltonian cycle faulting a perfect matching M if and only if $Q_n - M$ is connected.

The *matching graph* $\mathcal{M}(G)$ of a graph G with an even number of vertices has the vertex set consisting of all perfect matchings in G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle in G. Fink [4, 5] proved for $n \geq 4$ that the matching graph $\mathcal{M}(Q_n)$ is bipartite and connected. This also proved Kreweras' conjecture.

Dvořák [2] showed for $n \geq 2$ that every set of at most $2n - 3$ edges in Q_n forming vertex-disjoint paths is contained in a Hamiltonian cycle in Q_n. This result implied that every matching of at most $2n-3$ edges in Q_n extends to a Hamiltonian cycle in Q_n.

The present authors [12] improved Dvořák's result and proved for $n \geq 2$ that every matching of at most $3n-10$ edges in Q_n extends to a Hamiltonian cycle in Q_n.

In this paper, we prove for $n \geq 3$ that every matching in Q_n not covering exactly two vertices at distance 3 extends to a Hamiltonian cycle in Q_n. For the result, however, we now cannot drop the condition "at distance 3". Also, we prove for $n \geq 2$ that every matching in Q_n consisting of edges in at most four types extends to a Hamiltonian cycle in Q_n. The two main results will be proved in the next two sections.

2 A class of maximal matchings in Q_n

The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For an edge $e \in E(G)$, we use $V(e)$ to denote the set of the two endpoints of e. For an edge set $E' \subseteq E(G)$, let $V(E') = \bigcup_{e \in E'} V(e)$.

For a vertex $v \in V(G)$, let $G - v$ denote the resulting graph by deleting from G the vertex v together with all the edges incident with v. For a set $E' \subseteq E(G)$, let $G - E'$ denote the graph with the vertex set $V(G)$ and edge set $E(G) \setminus E'$. Let H and H' be two subgraphs of G. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$.

For a vertex $v \in V(G)$, we call a vertex u a neighbor of v if $uv \in E(G)$. The *distance* between two vertices u and v is the number of edges in a shortest path joining u and v in G, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear.

An *automorphism* of a simple graph G is a permutation π of $V(G)$ which has the property that $uv \in E(G)$ if and only if $\pi(u)\pi(v) \in E(G)$. We say that G is *vertex-transitive* if there is an automorphism π of G such that $\pi(u) = v$ for any two vertices u and v in G. Note that Q_n is vertex-transitive.

Graphs which contain no cycles are usually called *forests*. A forest is *linear* if each component of it is a path.

Lemma 2.1. *Let u, v be two vertices at distance 3 in Q_3 and x be a neighbor of u in Q_3. Let M be a perfect matching in $K_{Q_3} - u - v$. Then there exists*

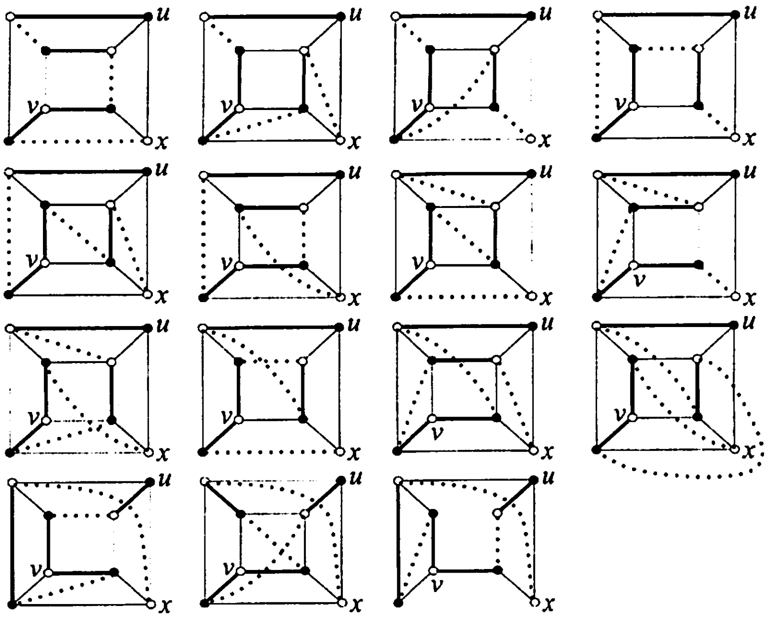


Figure 2. Illustration for the proof of Lemma 2.1 with the edges of M dotted and the edges of F bold.

a linear forest F in Q_3 such that $M \cup E(F)$ forms a Hamiltonian path in K_{Q_3} joining u and x.

Proof. By the vertex-transitivity, we may assume $u = 000$. Then $v = 111$. For any two neighbors x and y of u in Q_3, there exists an automorphism π of Q_3 fixing u and v such that $\pi(y) = x$. Then we may assume $x = 100$. Since M is a perfect matching in $K_{Q_3} - u - v$, there are $5 \times 3 \times 1 = 15$ possibilities of M. By examining all possibilities of M, one can verify that the lemma holds (see Figure 2). □

The set of all i-edges of Q_n is denoted by E_i. Then $E(Q_n) = E_1 \cup \cdots \cup E_n$. Let $[n]$ denote the set $\{1, \ldots, n\}$. For $j \in [n]$ and $\delta \in \{0, 1\}$, let $Q_{n-1}^{\delta,j}$, with the superscripts j being omitted when the context is clear, be the $(n-1)$-dimensional subcube of Q_n induced by the vertex set $\{u \in V(Q_n) : u^j = \delta\}$. Then $Q_n - E_j = Q_{n-1}^0 + Q_{n-1}^1$. We say that Q_n *splits* into two $(n-1)$-dimensional subcubes Q_{n-1}^0 and Q_{n-1}^1 by E_j; see Figure 3 for example.

The *parity* $p(u)$ of a vertex u in Q_n is defined by $p(u) = \sum_{i=1}^{n} u^i \pmod 2$. Then there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity

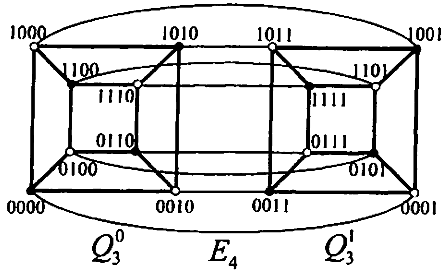


Figure 3. Q_4 splits into two 3-dimensional subcubes Q_3^0 and Q_3^1 by E_4.

1 in Q_n. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively; see Figure 3 for example. We observe that if vertex u is adjacent to vertex v in Q_n, then $p(u) \neq p(v)$. Consequently, $p(u) \neq p(v)$ if and only if $d(u, v)$ is odd. Hence Q_n is bipartite and vertices of each parity form bipartite sets of Q_n.

For $u, v \in V(Q_n)$, let $\Delta(u, v) = \{i \in [n] : u^i \neq v^i\}$. Then $d_{Q_n}(u, v) = |\Delta(u, v)|$. Usually, we use the notations "$u_1, u_2, u_3, \ldots$" to denote a series of vertices, which are distinguish with the encoding sequence $u = u^1 \cdots u^n$.

Theorem 2.2. *For $n \geq 3$, let u, v be two vertices at distance 3 in Q_n and let M be a perfect matching in $K_{Q_n} - u - v$. Then there exists a linear forest F in Q_n such that $M \cup E(F)$ forms a Hamiltonian cycle in K_{Q_n}.*

Proof. We proceed by induction on n. The theorem holds for $n = 3$ by Lemma 2.1. Suppose $n \geq 4$ and the theorem holds for $n - 1$. Since $d_{Q_n}(u, v) = 3$, we may assume $\Delta(u, v) = \{1, 2, 3\}$.

Case 1. There exists an edge $wt \in M$ such that $\Delta(w, t) \not\subseteq \{1, 2, 3\}$.

Let $j \in \Delta(w, t) \setminus \{1, 2, 3\}$. Split Q_n into two $(n-1)$-dimensional subcubes Q_{n-1}^0 and Q_{n-1}^1 by E_j. Then vertices u and v lie in the same subcube, and vertices w and t lie in different subcubes. By symmetry, we may assume $\{u, v\} \subseteq V(Q_{n-1}^0)$.

Note that $E(K_{Q_n}) = E(K_{Q_{n-1}^0}) \cup E(K_{Q_{n-1}^1}) \cup \{xy : x \in V(Q_{n-1}^0)$ and $y \in V(Q_{n-1}^1)\}$. Let $M_k = M \cap E(K_{Q_{n-1}^k})$ for every $k \in \{0, 1\}$. Let $M^* = M \setminus (M_0 \cup M_1)$; see Figure 4 for example. Then $wt \in M^*$. Since M is a perfect matching in $K_{Q_n} - u - v$, $|M^*|$ is even.

Choose an arbitrary perfect matching S_0 on $V(Q_{n-1}^0) \cap V(M^*)$ in $K_{Q_{n-1}^0}$. Then $M_0 \cup S_0$ is a perfect matching in $K_{Q_{n-1}^0} - u - v$. Since $d_{Q_{n-1}^0}(u, v) = 3$, by the induction hypothesis there exists a linear forest F_0 in Q_{n-1}^0 such that $M_0 \cup S_0 \cup E(F_0)$ forms a Hamiltonian cycle in $K_{Q_{n-1}^0}$; see Figure 5 for example. Note that $M_0 \cup E(F_0) \cup M^*$ forms a linear forest, denoted by F^*.

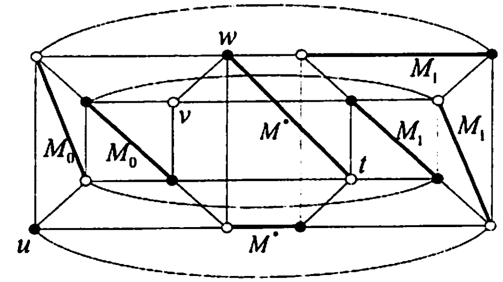
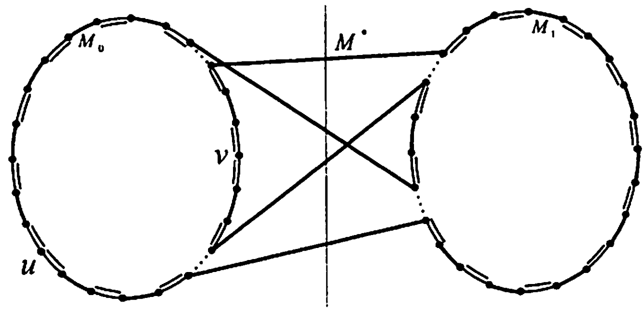


Figure 4. M is divided into M_0, M_1 and M^*, where the edges of M bold.



$K_{Q^0_{n-1}}$ $K_{Q^1_{n-1}}$

Figure 5. Illustration for the construction with the edges of M bold, the edges of F double, and the edges of $S_0 \cup S_1$ dotted.

Let $S_1 = \{xy \in E(K_{Q^1_{n-1}}) : x, y \in V(Q^1_{n-1}) \cap V(M^*)$ and there exists a path joining x and y in $F^*\}$. Note that such a path joining x and y in F^* is a component of F^*. Since S_1 covers all the vertices in $V(Q^1_{n-1}) \cap V(M^*)$, $M_1 \cup S_1$ is a perfect matching in $K_{Q^1_{n-1}}$. By Theorem 1.1 there exists a perfect matching F_1 in Q^1_{n-1} such that $M_1 \cup S_1 \cup F_1$ forms a Hamiltonian cycle, denoted by C_1, in $K_{Q^1_{n-1}}$. By the definition of S_1, one can observe that there is a natural one-to-one correspondence between the edges of S_1 and the components of F^*. In the cycle C_1, replacing every edge $xy \in S_1$ by the corresponding path joining x and y in F^*, we obtain a Hamiltonian cycle formed by edges of $M \cup E(F_0) \cup F_1$ in K_{Q_n}; see Figure 5 for example. Hence the desired linear forest F in Q_n is formed by edges of $E(F_0) \cup F_1$.

Case 2. $\Delta(w, t) \subseteq \{1, 2, 3\}$ for all $wt \in M$.

Let Q_{n-3} be a $(n-3)$-dimensional hypercube. When $n = 4$, $Q_{n-3} = K_2$. Now let $V(Q_{n-3}) = \{x_0, x_1\}$. When $n \geq 5$, since Q_{n-3} is Hamiltonian, we may choose a Hamiltonian cycle $C = x_0, x_1, \ldots, x_{2^{n-3}-1}, x_0$ in Q_{n-3}. Note that for every $k \in \{0, 1, \ldots, 2^{n-3}-1\}$, x_k is a binary string of length $(n-3)$, i.e., $x_k = x_k^1 \cdots x_k^{n-3}$.

For every $k \in \{0, 1, \ldots, 2^{n-3}-1\}$, let $Q_3^{x_k}$ be the 3-dimensional subcube of Q_n induced by the vertex set $\{y \in V(Q_n) : y^i = x_k^{i-3}$ for every $i \in \{4, \ldots, n\}\}$. In other words, $Q_3^{x_k}$ is the subcube of Q_n with the positions in $[n] \setminus \{1, 2, 3\}$ fixed by x_k. Then $Q_n - E_4 - \cdots - E_n = Q_3^{x_0} + Q_3^{x_1} + \cdots + Q_3^{x_{2^{n-3}-1}}$; see Figure 6 for example. Recall that x_k is adjacent to x_{k+1} in Q_{n-3}. Then for every vertex $y \in V(Q_3^{x_k})$, there is a unique vertex $y^1y^2y^3x_{k+1}^1 \cdots x_{k+1}^{n-3}$ in $Q_3^{x_{k+1}}$ such that the two vertices are adjacent in Q_n, with subscripts taken modulo 2^{n-3}.

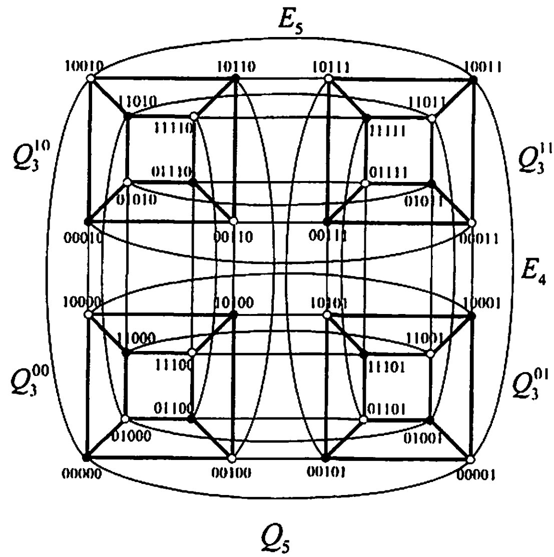


Figure 6. Q_5 splits into four 3-dimensional subcubes Q_3^{00}, Q_3^{01}, Q_3^{10} and Q_3^{11} by E_4 and E_5.

Since $\Delta(u, v) = \{1, 2, 3\}$, we have $\{u, v\} \subseteq V(Q_3^{x_k})$ for some $k \in \{0, 1, \ldots, 2^{n-3} - 1\}$. Without loss of generality we may assume $\{u, v\} \subseteq V(Q_3^{x_0})$. Since $\Delta(w, t) \subseteq \{1, 2, 3\}$ for all $wt \in M$, we have $M \subseteq \bigcup_{k=0}^{2^{n-3}-1} E(K_{Q_3^{x_k}})$. Let $M_k = M \cap E(K_{Q_3^{x_k}})$ for every $k \in \{0, 1, \ldots, 2^{n-3} - 1\}$. Then $M = \bigcup_{k=0}^{2^{n-3}-1} M_k$. Since M is a perfect matching in $K_{Q_n} - u - v$, M_0 is a perfect matching in $K_{Q_3^{x_0}} - u - v$ and M_k is a perfect matching in $K_{Q_3^{x_k}}$ for every $k \geq 1$. By Theorem 1.1 there exists a linear forest F_k in $Q_3^{x_k}$ such that $M_k \cup F_k$ forms a Hamiltonian cycle in $K_{Q_3^{x_k}}$ for every $k \in \{1, \ldots, 2^{n-3} - 1\}$.

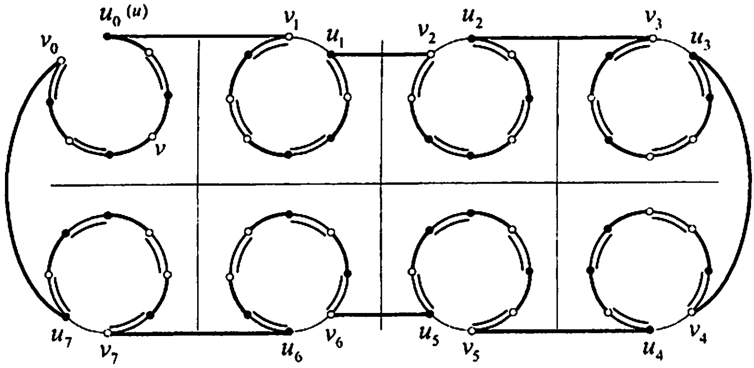


Figure 7. Illustration for the construction with the edges of M double, the edges of F bold.

Let $u_0 = u$ and v_1 be the vertex in $Q_3^{x_1}$ such that $u_0v_1 \in E(Q_n)$. Then $p(u_0) \neq p(v_1)$. From $k = 1$ to $2^{n-3} - 1$, let u_k be the neighbor of v_k in F_k and v_{k+1} be the vertex in $Q_3^{x_{k+1}}$ such that $u_kv_{k+1} \in E(Q_n)$, where the subscripts modulo 2^{n-3}. Then $p(u_k) \neq p(v_k)$ and $p(u_k) \neq p(v_{k+1})$ for every $k \in \{1, \ldots, 2^{n-3} - 1\}$. Hence $p(u_0) \neq p(v_0)$. Since $d_{Q_3^{x_0}}(u_0, v) = 3$, we have $d_{Q_3^{x_0}}(u_0, v_0) = 1$ or $v_0 = v$. Since M_0 is a perfect matching in $K_{Q_3^{x_0}} - u_0 - v$, by Lemma 2.1 in case $d_{Q_3^{x_0}}(u_0, v_0) = 1$ and Theorem 1.1 in case $v_0 = v$, there exists a linear forest F_0 in $Q_3^{x_0}$ such that $M_0 \cup E(F_0)$ forms a Hamiltonian path in $K_{Q_3^{x_0}}$ joining u_0 and v_0. Hence $E(F_0) \cup (\bigcup_{k=1}^{2^{n-3}-1}(E(F_k) \setminus \{u_kv_k\} \cup \{u_{k-1}v_k\})) \cup \{u_{2^{n-3}-1}v_0\}$ forms a linear forest F in Q_n such that $M \cup E(F)$ forms a Hamiltonian cycle in K_{Q_n}; see Figure 7 for example. □

Note that Q_n is a spanning subgraph of K_{Q_n}. Then $Q_n - u - v$ is a spanning subgraph of $K_{Q_n} - u - v$. In Theorem 2.2, when M is a perfect matching in $Q_n - u - v$, $M \cup E(F)$ forms a Hamiltonian cycle in Q_n.

Corollary 2.3. *For $n \geq 3$, let u, v be two vertices at distance 3 in Q_n and M be a perfect matching in $Q_n - u - v$. Then there exists a linear forest F in Q_n such that $M \cup E(F)$ forms a Hamiltonian cycle in Q_n.*

3 Matchings in at most four positions

A *u, v-path* is a path with endpoints u and v, denoted by $P_{u,v}$ when we specify a particular such path. We say that a spanning subgraph of G whose components are k disjoint paths is a *spanning k-path* of G. A spanning 1-path thus is simply a spanning or Hamiltonian path. For a set $E' \subseteq E(G)$, a subgraph H of G *passes through* E' if $E' \subseteq E(H)$.

We say that two matchings M and P of a graph G are *isomorphic* if there exists an automorphism π of G such that $\pi(u)\pi(v) \in P \Leftrightarrow uv \in M$.

In the following Lemmas 3.1, 3.2 and 3.3, by the vertex-transitivity of Q_3, we may assume $u = 000$. Then u is a black vertex.

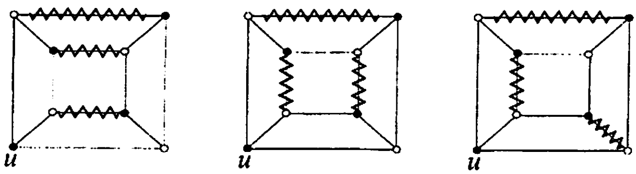


Figure 8. Three non-isomorphic maximal matchings in $Q_3 - u$.

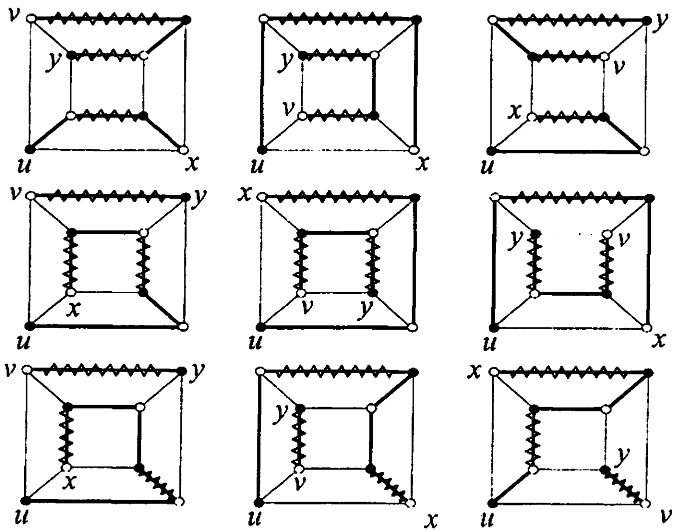


Figure 9. Illustration for the proof of Lemma 3.1 with the edges of M curved and the edges of $P_{u,x} + P_{v,y}$ bold.

Lemma 3.1. *For $u, v \in V(Q_3)$ with $p(u) \neq p(v)$, let M be a matching in $Q_3 - u$ with $v \in V(M)$. Then there exists a spanning 2-path $P_{u,x} + P_{v,y}$ of Q_3 passing through M, where x, y are two vertices at distance 3 in Q_3.*

Proof. Without loss of generality we may assume that M is a maximal matching in $Q_3 - u$. There are three non-isomorphic maximal matchings in $Q_3 - u$ (see Figure 8). Since $p(u) \neq p(v)$, v is a white vertex. Note that $v \in V(M)$. By examining all possibilities of M and v, one can verify that the lemma holds (see Figure 9). □

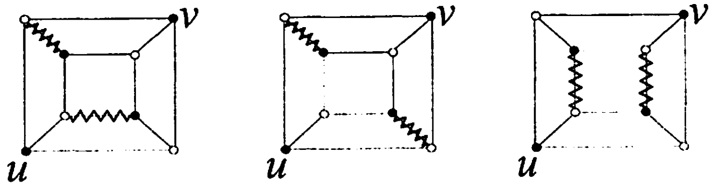


Figure 10. Three non-isomorphic maximal matchings in $Q_3 - u - v$.

Lemma 3.2. *Let u, v be two vertices at distance 2 in Q_3 and let x, y be two distinct vertices in Q_3 such that $d(u, x) = d(v, y) = 1$. If M is a matching in $Q_3 - u - v$, then there exists a spanning 2-path $P_{u,x} + P_{v,y}$ of Q_3 passing through M.*

Proof. It suffices to consider the case that M is a maximal matching in $Q_3 - u - v$. For any two vertices v_1 and v_2 in Q_3 satisfying $d(u, v_1) = d(u, v_2) = 2$, there exists an automorphism π of Q_3 fixing u such that $\pi(v_1) = v_2$. Then we may assume $v = 101$. There are three non-isomorphic maximal matchings in $Q_3 - u - v$ (see Figure 10). By examining all possibilities of $\{M, x, y\}$ up to isomorphic, one can verify that the conclusion holds (see Figure 11). □

Lemma 3.3. *Let u, v be vertices in Q_3 with $p(u) = p(v)$. If M is a matching in $Q_3 - u$, then there exists a spanning 2-path $P_{u,v} + P_{x,y}$ of Q_3 passing through M, where x, y are two distinct vertices in Q_3 satisfying $p(x) = p(y) \neq p(u)$.*

Proof. It suffices to consider the case that M is a maximal matching in $Q_3 - u$. There are three non-isomorphic maximal matchings in $Q_3 - u$ (see Figure 8). Since $p(u) = p(v)$, v is a black vertex. By examining all possibilities of M and v up to isomorphic, one can verify that the lemma holds (see Figure 12). □

Lemma 3.4. *[11] Let u, v be two vertices in Q_3 with $p(u) \neq p(v)$. If M is a matching in $Q_3 - u$, then there exists a Hamiltonian path in Q_3 joining u and v passing through M.*

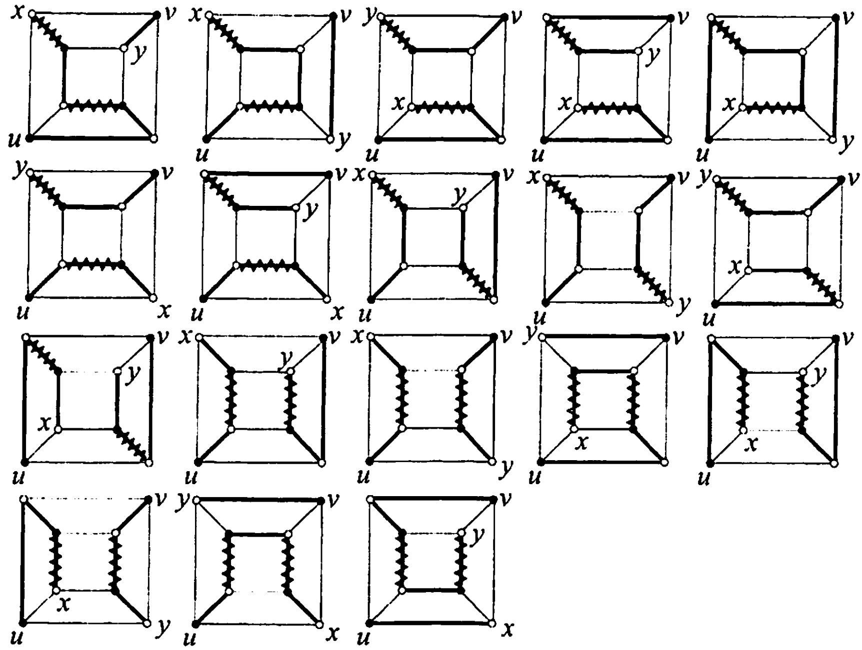


Figure 11. Illustration for the proof of Lemma 3.2 with the edges of M curved and the edges of $P_{u,x} + P_{v,y}$ bold.

Lemma 3.5. *Let u, v be two vertices in Q_4 with $p(u) \neq p(v)$. If M is a matching in $Q_4 - u$, then there exists a Hamiltonian path in Q_4 joining u and v passing through M.*

Proof. It suffices to consider the case that M is a maximal matching in $Q_4 - u$. Since $|M| \leq 7$, there exists $j \in [4]$ such that $|M \cap E_j| \leq 1$. Split Q_4 into subcubes Q_3^0 and Q_3^1 by E_j. By symmetry we may assume $u \in V(Q_3^0)$. Let $M_\delta = M \cap E(Q_3^\delta)$ for every $\delta \in \{0, 1\}$. Note that every vertex $x_\delta \in V(Q_3^\delta)$ has in $Q_3^{1-\delta}$ a unique neighbor, denoted by $x_{1-\delta}$, where $\delta \in \{0, 1\}$.

Case 1. $M \cap E_j = \emptyset$.

If $v \in V(Q_3^1)$, then by Lemma 1.2 there is a Hamiltonian cycle C_1 in Q_3^1 passing through M_1. Let s_1 be a neighbor of v on C_1 such that $vs_1 \notin M$. Then $p(v) \neq p(s_1)$. Since $p(u) \neq p(v)$ and $p(s_1) \neq p(s_0)$, we have $p(u) \neq p(s_0)$. Since $u \notin V(M_0)$, by Lemma 3.4 there exists a Hamiltonian path P_{u,s_0} in Q_3^0 passing through M_0. Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,s_0} + C_1) \cup \{s_0 s_1\} \setminus \{vs_1\}$.

It remains to consider the case $v \in V(Q_3^0)$. Since M is a maximal

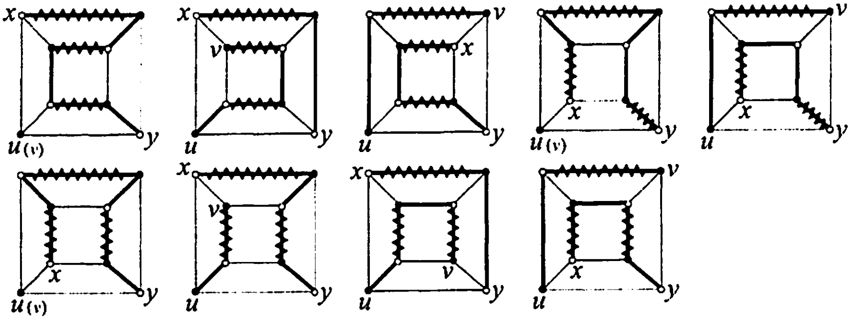


Figure 12. Illustration for the proof of Lemma 3.3 with the edges of M curved and the edges of $P_{u,v} + P_{x,y}$ bold.

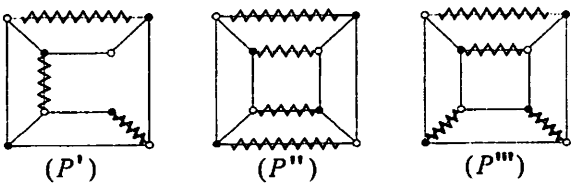


Figure 13. Three non-isomorphic maximal matchings in Q_3.

matching in $Q_4 - u$ and $M \cap E_j = \emptyset$, M_0 is a maximal matching in $Q_3^0 - u$ and M_1 is a maximal matching in Q_3^1. Thus, $|M_0| = 3$ and $3 \leq |M_1| \leq 4$. There are three non-isomorphic maximal matchings, denoted by P', P'' and P''', in Q_3 (see Figure 13). Then M_1 is isomorphic to one of P', P'' and P'''.

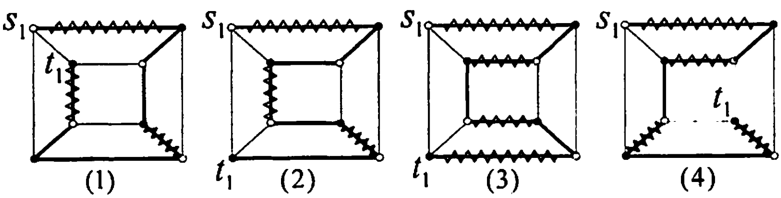


Figure 14. Illustration (up to isomorphic) for the proof of Case 1 in Lemma 3.5 with the edges of M curved and the edges of P_{s_1,t_1} bold.

Since $u \notin V(M_0)$ and $p(u) \neq p(v)$, by Lemma 3.4 there is a Hamiltonian path $P_{u,v}$ in Q_3^0 passing through M_0. If M_1 is isomorphic to P', then since $|E(P_{u,v}) \backslash M_0| - |M_1| = 1$, there exists an edge $s_0t_0 \in E(P_{u,v}) \backslash M_0$ such that

$s_1t_1 \notin M_1$. One can verify that there exists a Hamiltonian path P_{s_1,t_1} in Q_3^1 passing through M_1 (see Figure 14(1)-(2)). Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,v} + P_{s_1,t_1}) \cup \{s_0s_1, t_0t_1\} \setminus \{s_0t_0\}$.

If $v \in V(M_0)$ and M_1 is isomorphic to P'', there exist two edges s_0t_0 and t_0r_0 in $E(P_{u,v}) \setminus M_0$. Since M_1 is a matching, we have $s_1t_1 \notin M_1$ or $t_1r_1 \notin M_1$, say $s_1t_1 \notin M_1$. One can verify that there exists a Hamiltonian path P_{s_1,t_1} in Q_3^1 passing through M_1 (see Figure 14(3)). Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,v}+P_{s_1,t_1}) \cup \{s_0s_1, t_0t_1\} \setminus \{s_0t_0\}$.

If $v \in V(M_0)$ and M_1 is isomorphic to P''', by Lemma 3.1 there exists a spanning 2-path $P_{u,s_0} + P_{v,t_0}$ of Q_3^0 passing through M_0, where s_0, t_0 are two vertices at distance 3 in Q_3^0. Since $d(s_1, t_1) = 3$, one can verify that there exists a Hamiltonian path P_{s_1,t_1} in Q_3^1 passing through M_1 (see Figure 14(4)). Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,s_0} + P_{v,t_0} + P_{s_1,t_1}) \cup \{s_0s_1, t_0t_1\}$.

If $v \notin V(M_0)$ and M_1 is isomorphic to P'' or P''', M is a perfect matching in $Q_4 - u - v$. By Theorem 1.1 there exists a perfect matching F in Q_4 such that $M \cup \{uv\} \cup F$ forms a Hamiltonian cycle in K_{Q_4}. Hence, $M \cup F$ forms a Hamiltonian path in Q_4 joining u and v passing through M.

Case 2. $|M \cap E_j| = 1$.

Let $M \cap E_j = \{w_0w_1\}$, where $w_0 \in V(Q_3^0)$. If $v \in V(Q_3^0)$, then by Lemma 3.4 there is a Hamiltonian path $P_{u,v}$ in Q_3^0 passing through M_0. Let r_0 be a neighbor of w_0 on $P_{u,v}$. Since M is a matching and $w_0w_1 \in M$, we have $w_0r_0 \notin M$. Since $w_1 \notin V(M_1)$ and $p(w_1) \neq p(r_1)$, by Lemma 3.4 there exists a Hamiltonian path P_{w_1,r_1} in Q_3^1 passing through M_1. Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,v}+P_{w_1,r_1}) \cup \{w_0w_1, r_0r_1\} \setminus \{w_0r_0\}$.

So let $v \in V(Q_3^1)$. If $p(u) \neq p(w_0)$, then since $p(u) \neq p(v)$ and $p(w_0) \neq p(w_1)$, we have $p(w_1) \neq p(v)$. Since $u \notin V(M_0)$ and $w_1 \notin V(M_1)$, by Lemma 3.4 there exist Hamiltonian paths P_{u,w_0} in Q_3^0 and $P_{w_1,v}$ in Q_3^1 passing through M_0 and M_1, respectively. Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,w_0} + P_{w_1,v}) \cup \{w_0w_1\}$.

If $p(u) = p(w_0)$, then $d(u, w_0) = 2$ and $p(w_1) = p(v)$. Since M_1 is a matching in $Q_3^1 - w_1$, by Lemma 3.3 there exists a spanning 2-path $P_{w_1,v} + P_{s_1,t_1}$ of Q_3^1 passing through M_1, where s_1, t_1 are two distinct vertices in Q_3^1 such that $p(s_1) = p(t_1) \neq p(w_1)$. Then $p(u) = p(w_0) \neq p(s_0) = p(t_0)$. In Q_3^0, since $d(u, w_0) = 2$ and $s_0 \neq t_0$, we have $d(u, s_0) = d(w_0, t_0) = 1$ or $d(u, t_0) = d(w_0, s_0) = 1$. Without loss of generality, we may assume $d(u, s_0) = d(w_0, t_0) = 1$. Since M_0 is a matching in $Q_3^0 - u - w_0$, by Lemma 3.2 there exists a spanning 2-path $P_{u,s_0} + P_{w_0,t_0}$ of Q_3^0 passing through M_0. Then the desired Hamiltonian path in Q_4 is formed by edges of $E(P_{u,s_0} + P_{w_0,t_0} + P_{s_1,t_1} + P_{w_1,v}) \cup \{s_0s_1, t_0t_1, w_0w_1\}$. □

Theorem 3.6. *For $n \geq 2$, let M be a matching in Q_n such that $|\{i \in [n] : M \cap E_i \neq \emptyset\}| \leq 4$. Then there exists a Hamiltonian cycle in Q_n passing through M.*

Proof. If $n \in \{2, 3, 4\}$ or M is a perfect matching in Q_n, then by Theorem 1.1 or Lemma 1.2 the theorem holds. So in what follows we may assume that $n \geq 5$ and M is a matching in Q_n which is not perfect. Since $|\{i \in [n] : M \cap E_i \neq \emptyset\}| \leq 4$, without loss of generality we may assume $\{i \in [n] : M \cap E_i \neq \emptyset\} \subseteq \{1, 2, 3, 4\}$. Then $M \subseteq E_1 \cup E_2 \cup E_3 \cup E_4$.

Let Q_{n-4} be a $(n-4)$-dimensional hypercube. When $n = 5$, let $V(Q_{n-4}) = \{x_0, x_1\}$. When $n \geq 6$, choose a Hamiltonian cycle $C = x_0, x_1, \ldots, x_{2^{n-4}-1}, x_0$ in Q_{n-4}. Note that for every $k \in \{0, 1, \ldots, 2^{n-4}-1\}$, x_k is a binary string of length $(n-4)$.

For every $k \in \{0, 1, \ldots, 2^{n-4}-1\}$, let $Q_4^{x_k}$ be the 4-dimensional subcube of Q_n induced by the vertex set $\{y \in V(Q_n) : y^i = x_k^{i-4}$ for every $i \in \{5, \ldots, n\}\}$. Then $Q_n - E_5 - \cdots - E_n = Q_4^{x_0} + Q_4^{x_1} + \cdots + Q_4^{x_{2^{n-4}-1}}$ and $\bigcup_{k=0}^{2^{n-4}-1} E(Q_4^{x_k}) = E_1 \cup E_2 \cup E_3 \cup E_4$. Hence $M \subseteq \bigcup_{k=0}^{2^{n-4}-1} E(Q_4^{x_k})$. Let $M_k = M \cap E(Q_4^{x_k})$ for every $k \geq 0$. Then $M = \bigcup_{k=0}^{2^{n-4}-1} M_k$.

Since M is a matching in Q_n which is not perfect, without loss of generality we may assume M_0 is not perfect in $Q_4^{x_0}$. First apply Lemma 1.2 to obtain a Hamiltonian cycle C_k in $Q_4^{x_k}$ passing through M_k for every $k \in \{1, \ldots, 2^{n-4}-1\}$.

For every $k \in \{0, 1, \ldots, 2^{n-4}-1\}$, since x_k is adjacent to x_{k+1} in Q_{n-4}, every vertex $y \in V(Q_4^{x_k})$ has in $Q_4^{x_{k+1}}$ a unique neighbor $y^1 y^2 y^3 y^4 x_{k+1}^0 \cdots x_{k+1}^{n-4}$, with subscripts taken modulo 2^{n-4}. Let $u_0 \in V(Q_4^{x_0}) \setminus V(M_0)$ and v_1 be the neighbor of u_0 in $Q_4^{x_1}$. Then $p(u_0) \neq p(v_1)$. From $k = 1$ to $2^{n-4}-1$, let u_k be a neighbor of v_k on C_k such that $u_k v_k \notin M$ and let v_{k+1} be the neighbor of u_k in $Q_4^{x_{k+1}}$, where the subscripts modulo 2^{n-4}. Then $p(u_k) \neq p(v_k)$ and $p(u_k) \neq p(v_{k+1})$ for every $k \in \{1, \ldots, 2^{n-4}-1\}$. Hence $p(u_0) \neq p(v_0)$. Since M_0 is a matching in $Q_4^{x_0} - u_0$, by Lemma 3.5 there exists a Hamiltonian path P_{u_0,v_0} in $Q_4^{x_0}$ passing through M_0. Then the desired Hamiltonian cycle in Q_n is formed by edges of $E(P_{u_0,v_0}) \cup (\bigcup_{k=1}^{2^{n-4}-1}(E(C_k) \cup \{u_{k-1}v_k\} \setminus \{u_k v_k\})) \cup \{u_{2^{n-4}-1}v_0\}$. □

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