Clique-chromatic Numbers of Line Graphs

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Abstract

The clique-chromatic number of a graph is the least number of colors on the vertices of the graph without a monocolored maximal clique of size at least two. In 2004, Bacsó et al. proved that the family of line graphs has no bounded clique-chromatic number. In particular, the Ramsey numbers provide a sequence of the line graphs of complete graphs with no bounded clique-chromatic number. We complete this result by giving the exact values of the clique-chromatic numbers of the line graphs of complete graphs in terms of Ramsey numbers. Furthermore, the clique-chromatic numbers of the line graphs of triangle-free graphs are characterized.

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1 Introduction

All graphs considered in this paper are simple. We use terminologies from West's textbook [7]. A *triangle* is the complete graph with 3 vertices. A *triangle-free graph* is a graph which contains no triangle as a subgraph. A *hole* in a graph is an induced cycle with at least four vertices. An *odd hole* is a hole with an odd number of vertices. The *neighborhood* of a vertex x in a graph G is the set of vertices adjacent to x, and is denoted by $N_G(x)$. Let $M \subseteq E(G)$ and $e \in E(G)$. We write $G-M$, and $G-e$, for the subgraph of G obtained by deleting all edges of M, and an edge e, respectively. Let $A \subseteq V(G)$ and $v \in V(G)$. We write $G-A$, and $G-v$, for the subgraph of G obtained by deleting all vertices of A, and a vertex v, respectively. The *union* of graphs $G_1, G_2, \ldots, G_k$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$, denoted by $G_1 \cup G_2 \cup \ldots \cup G_k$. A union of graphs $G_1, G_2, \ldots, G_k$ is called *disjoint union* if $G_1, G_2, \ldots, G_k$ have pairwise disjoint vertex sets, and is denoted by $G_1 + G_2 + \cdots + G_k$. For $k \in \mathbb{N}$,

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kG is the disjoint union of k pairwise disjoint copies of a graph G. The *join* of graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{xy \mid x \in V(G)\ ,\ y \in V(H)\}$. For the case $V(H) = \{v\}$, we write $G \vee v$ for $G \vee \{v\}$.

A subset Q of $V(G)$ is a *clique* of G if any two vertices of Q are adjacent. A clique is *maximal* if it is not properly contained in another clique. A *k-coloring* of a graph G is a function $f : V(G) \to X$, where $|X| = k$. A *proper k-coloring* of a graph G is a k-coloring of G such that adjacent vertices have different colors. The *chromatic number* of a graph G is the smallest positive integer k such that G has a proper k-coloring, denoted by $\chi(G)$. Given a k-coloring of a graph G, a clique Q of G is said to be *monocolored* if all vertices of Q are labeled by the same color. A *proper k-clique-coloring* of a graph G is a k-coloring of G without a monocolored maximal clique of G of size at least two. A graph G is *k-clique-colorable* if G has a proper k-clique-coloring. The *clique-chromatic number* of G is the smallest k such that G has a proper k-clique-coloring, denoted by $\chi_c(G)$. Note that $\chi_c(G) = 1$ if and only if G is an edgeless graph. Since any proper k-coloring of G is a proper k-clique-coloring of G, $\chi_c(G) \leq \chi(G)$. If G is a triangle-free graph, then all maximal cliques of G, which is not an isolated vertex, have size two; so $\chi_c(G) = \chi(G)$.

Many families of graphs are 3-clique-colorable, for example, comparability graphs, cocomparability graphs, circular-arc graphs, and the k-power of cycles [2, 3, 4, 5]. In [1], Bacsó et al. proved that almost all perfect graphs are 3-clique-colorable and conjectured that all perfect graphs are 3-clique-colorable. On the other hand, some families of graphs have no bounded clique-chromatic number, for example, triangle-free graphs, and line graphs [1, 6].

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G; and for any edges e and f in G, ef is an edge in $L(G)$ if and only if e and f share a common vertex in G. A graph G is a *line graph* if there is a simple graph H such that $L(H) = G$.

A *star* is a tree consisting of one vertex adjacent to all the others. A star in a graph G is called *maximal* if it is not properly contained in another star or a triangle in a graph G.

Proposition 1. *Let G be a graph. Then a maximal clique in $L(G)$ corresponds to a triangle or a maximal star in G.*

Proof. A clique in $L(G)$ corresponds to a triangle or a star in G [7, pp.275]. A triangle in G induces a maximal clique of size three in $L(G)$ because no edge of G is incident to all three edges of a triangle in G. Furthermore, a maximal star in G induces a maximal clique in $L(G)$. □

To study a vertex-coloring of $L(G)$, we could study an edge-coloring of G instead. Recall that a *k-edge-coloring* of a graph G is a function $f : E(G) \to X$, where $|X| = k$. Given an edge-coloring of a graph G, a subgraph H of G is said to be *monocolored* if all edges of H are labeled by the same color. Since edges of G correspond to vertices of $L(G)$, by Proposition 1, a k-edge-coloring of G without a monocolored triangle and a monocolored maximal star corresponds to a k-coloring of $L(G)$ without a monocolored maximal clique, which is in fact a proper k-clique-coloring of $L(G)$.

2 Line graphs of the complete graphs

In [1], Bacsó et al. proved that the family of line graphs has no bounded clique-chromatic number. In particular, the family of the line graphs of complete graphs on Ramsey numbers of vertices has no bounded clique-chromatic number. Recall that the *Ramsey number* $R(k_1, k_2, \ldots, k_m)$ is the smallest positive integer such that every m-edge-coloring of $K_{R(k_1,k_2,\ldots,k_m)}$ gives a monocolored complete subgraph on k_i vertices for some $i \in \{1, 2, \ldots, m\}$. We denote the Ramsey number $R(\underbrace{3, 3, \ldots, 3}_{m})$ by $R(m)$.

Bacsó et al. showed that $\chi_c(L(K_{R(m)})) > m$ where $m \in \mathbb{N}$. In this section, we sharpen this bound by showing that $\chi_c(L(K_{R(m)})) = m + 1$. Furthermore, we extend the result to the exact values of the clique-chromatic numbers of the line graphs of all complete graphs.

Lemma 2. *Let $m \in \mathbb{N}$. If a graph G has an m-edge-coloring without a monocolored triangle, then the line graph $L(G \vee v)$ has a proper $(m+1)$-clique-coloring, where v is a vertex that is not in G.*

Proof. For case $m = 1$, let $x \in V(G)$ be fixed. By assumption, G has no triangle. Note that all triangles and maximal stars in $G \vee v$ contain v. Then we define $f : E(G \vee v) \to \{1, 2\}$ by

$$f(ab) = \begin{cases} 2, & \text{if } (a \neq x \text{ and } b = v) \text{ or } (a = x \text{ and } b \neq v) \\ 1, & \text{otherwise.} \end{cases}$$

This function f is a 2-edge-coloring of $G \vee v$ without a monocolored triangle and a monocolored maximal star. Thus f corresponds to a proper 2-clique-coloring of $L(G \vee v)$.

Now, assume $m \geq 2$. Let ϕ be an m-edge-coloring of G without a monocolored triangle. Choose a vertex w in G such that $|N_G(w)| \neq 0$. Let i be a color of an edge incident to w in G. Extend $\phi : E(G) \to \{0, 1, \ldots, m-1\}$ to $\bar{\phi} : E(G \vee v) \to \{0, 1, \ldots, m\}$ by

$$\bar{\phi}(e) = \begin{cases} (i+1) (\text{mod } m), & \text{if } e = wv \\ m, & \text{if } e = uv \text{ for some } u \in V(G) \setminus \{w\} \\ \phi(e), & \text{otherwise.} \end{cases}$$

We have that $\bar{\phi}$ is an $(m+1)$-edge-coloring of $G \vee v$ without a monocolored triangle and a monocolored maximal star, and hence $\bar{\phi}$ corresponds to a proper $(m+1)$-clique-coloring of $L(G \vee v)$. □

Proposition 3. [1] *Let G be a graph and $m \in \mathbb{N}$. If G contains $K_{R(m)}$ as a subgraph, then $\chi_c(L(G)) \geq m+1$.*

Proof. Suppose that $L(G)$ has a proper m-clique-coloring. Then G has an m-edge-coloring without a monocolored triangle, say f. Thus $f|_{K_{R(m)}}$ is an m-edge-coloring without a monocolored triangle. This contradicts the definition of the Ramsey number $R(m)$. Hence $\chi_c(L(G)) > m$. □

Now we are ready to prove our main result in Theorem 4. Note that $L(K_1)$ is the null graph and $\chi_c(L(K_2)) = \chi_c(K_1) = 1$. Now, let $n \geq 3$. We have that there always exists a positive integer m such that $R(m) \leq n < R(m+1)$ because the Ramsey numbers always exist and $\{R(m)\}_{m=1}^{\infty}$ is a strictly increasing sequence. The main theorem shows the value of the clique-chromatic number of $L(K_n)$ where $n \geq 3$.

Theorem 4. *For $n \geq 3$, $\chi_c(L(K_n)) = m+1$, where the integer m is such that $R(m) \leq n < R(m+1)$.*

Proof. Since $n \geq R(m)$, K_n contains $K_{R(m)}$ as a subgraph. By Proposition 3, $\chi_c(L(K_n)) \geq m+1$.

Case 1. $n = R(m)$. The definition of $R(m)$ implies that $K_{R(m)-1}$ has an m-edge-coloring without a monocolored triangle. By Lemma 2, $L(K_{R(m)})$ has a proper $(m+1)$-clique-coloring. Hence $\chi_c(L(K_{R(m)})) \leq m+1$.

Case 2. $R(m) < n < R(m+1)$. The definition of $R(m+1)$ implies that K_n has an $(m+1)$-edge-coloring without a monocolored triangle, say ϕ. Suppose that ϕ gives a monocolored maximal star S, say labeled all edges in S by color 1. If there is an edge in K_n outside S colored by 1, then K_n contains a monocolored triangle, a contradiction. Thus $E(K_n)\setminus E(S)$ uses m colors, moreover they form a complete graph K_{n-1}. Since $n-1 \geq R(m)$, for every m-edge-coloring of K_{n-1} gives a monocolored triangle, a contradiction. Thus ϕ is an $(m+1)$-edge-coloring of G without a monocolored maximal star. Therefore ϕ corresponds to a proper $(m+1)$-clique-coloring of $L(K_n)$, and hence $\chi_c(L(K_n)) \leq m+1$. □

Given a positive integer m, Theorem 4 provides $R(m+1)-R(m)$ line graphs with clique-chromatic number $m+1$. In fact, there are infinitely many line graphs with the same clique-chromatic number as shown below.

Example 5. *Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$ such that $R(m) \leq n < R(m+1)$. Let $G = kK_{n-t} \vee pK_t$ where $k, p \in \mathbb{N}$ and $1 \leq t \leq n-1$. Then $\chi_c(L(G)) = m+1$.*

Proof. Since K_n is a subgraph of G and $n \geq R(m)$, G contains $K_{R(m)}$ as a subgraph. By Proposition 3, $\chi_c(L(G)) \geq m+1$. Let $kK_{n-t} = K_{n-t}^{(1)} + K_{n-t}^{(2)} + \cdots + K_{n-t}^{(k)}$ and $pK_t = K_t^{(1)} + K_t^{(2)} + \cdots + K_t^{(p)}$ where $K_{n-t}^{(i)}$ and $K_t^{(i)}$ are copies of K_{n-t} and K_t, respectively.

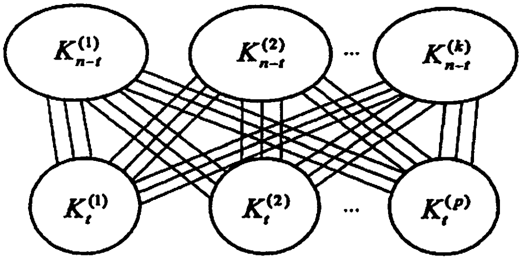

Figure 1: The graph $kK_{n-t} \vee pK_t$

Note that for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$, $K_{n-t}^{(i)} \vee K_t^{(j)} = K_n$. Since $R(m) \leq n < R(m+1)$, we have $\chi_c(L(K_{n-t}^{(1)} \vee K_t^{(1)})) = m + 1$ by Theorem 4. Thus there is an $(m+1)$-edge-coloring ϕ of $K_{n-t}^{(1)} \vee K_t^{(1)}$ without a monocolored triangle and a monocolored maximal star. Now, replicate the colors of the $K_{n-t}^{(1)}$ to $K_{n-t}^{(i)}$ and of $K_t^{(1)}$ to $K_t^{(j)}$ for all i and j, so ϕ can be extended to an $(m+1)$-edge-coloring of G without a monocolored triangle and a monocolored maximal star. Hence $L(G)$ has a proper $(m+1)$-clique-coloring, so $\chi_c(L(G)) \leq m+1$. □

3 Line graphs of triangle-free graphs

In this section, we characterize the clique-chromatic numbers of the line graphs of triangle-free graphs. Given a triangle-free graph G, by Proposition 1, a maximal clique in $L(G)$ corresponds to a maximal star in G. Thus if f is a k-edge-coloring of G without a monocolored maximal star, then f corresponds to a proper k-clique-coloring of $L(G)$.

Theorem 6. *If G is a triangle-free graph, then $\chi_c(L(G)) \leq 3$.*

Proof. Without lost of generality, we may assume that G is connected. Let $x \in V(G)$. Define $A_0 = \{x\}$, $A_1 = N_G(x)$, and $A_i = N_G(A_{i-1}) \setminus (A_{i-1} \cup A_{i-2})$ for all $i \geq 2$. We refer to a vertex having distance i from x as a vertex of distance i. Then A_i contains all vertices with distance i. Each edge in G joins either two vertices of the same distance or vertices of distance $i-1$ and i, for some i. If in the later case, we call such edge *a (distance i)-edge*. We first label all (distance 1)-edges by color 1 or color 2, at least one edge for each color. (If $|N_G(x)| = 1$, label the unique edge by color 1.) Then for each i^{th} step, $i = 2, 3, \ldots$, label a (distance i)-edge by color 1 if it is incident to a (distance $i-1$)-edge of color 2, and by color 2, otherwise. Finally, label all edges joining two vertices of the same distance by color 3. This process garantees that each vertex is incident to edges of at least two colors except the end vertices (vertices incident to a (distance $i-1$)-edge but not to any (distance i)-edge). If an end vertex is incident to all edges of the same color, we can relabel one edge of them by color 3.

Therefore, we have a 3-edge-coloring of G without a monocolored maximal star. So the coloring corresponds to a proper 3-clique-coloring of $L(G)$. Hence $\chi_c(L(G)) \leq 3$. □

The upper bound in Theorem 6 is sharp by the odd cycle C_{2n+1} $(n \geq 2)$ because C_{2n+1} is triangle-free and $\chi_c(L(C_{2n+1})) = 3$. In our purpose, a graph is called *trivial* if it is the complete graph K_1 or K_2. Note that if a graph G has a nontrivial component, then $\chi_c(L(G)) \geq 2$. Recall that a *forest* is a disjoint union of trees. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets. Equivalently, a bipartite graph is a graph which contains no odd cycle.

Lemma 7. *If G is a forest having a nontrivial component, then $\chi_c(L(G)) = 2$.*

Proof. Use the same coloring in the proof of Theorem 6. Since G has no cycle, all end vertices have degree 1 and there is no edge incident to vertices of the same distance. Thus color 3 is not used in the coloring. Besides G has a nontrivial component, $\chi_c(L(G)) = 2$. □

Lemma 8. *If G is a bipartite graph having a nontrivial component, then $\chi_c(L(G)) = 2$.*

Proof. If G contains no cycle, then G is a forest, it is done by Lemma 7. Now, assume that O_1 is any cycle of G. Since G is bipartite, O_1 is an even cycle. Label edges of O_1 alternately around the cycle by 1,2,1,2,..., then this is a 2-edge-coloring of O_1 without a monocolored maximal star. If $G - E(O_1)$ has a cycle, say O_2, then we color edges of O_2 similarly to O_1. Then similarly consider $G - (E(O_1) \cup E(O_2))$. Continue this process until the resulting graph contains no cycle. Label this resulting graph by the coloring in Lemma 7. Therefore, G has a 2-edge-coloring without a monocolored maximal star, and hence $\chi_c(L(G)) = 2$. □

The next theorem is the main theorem of this section. It contains a characterization of the clique-chromatic numbers of the line graphs of triangle-free graphs.

Theorem 9. *Let G be a triangle-free graph with at least one edge. Then*

$$\chi_c(L(G)) = \begin{cases} 1, & \textit{if all components of } G \textit{ are trivial} \\ 3, & \textit{if } G \textit{ has an odd hole component} \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. If all components of G are trivial, then $\chi_c(L(G)) = 1$. Assume that G has a nontrivial component. If G has an odd hole component, say O, then $\chi_c(L(O)) = 3$. Thus $\chi_c(L(G)) \geq 3$. By Theorem 6, $\chi_c(L(G)) = 3$. Assume that G has no odd hole component. Without lost of generality, assume that G is connected. Let H be the union of all odd holes of G. Then $G - E(H)$ is a bipartite graph. By Lemma 8, $G - E(H)$ has a 2-edge-coloring without a monocolored maximal star, say f. To label edges of H, we can assume that H is connected. Otherwise, consider each component. We can write $H = \bigcup_{i=1}^{n} O_i$ for some $n \in \mathbb{N}$ where each O_i is an odd hole of G, and $V(O_j) \cap V(\bigcup_{i=1}^{j-1} O_i) \neq \emptyset$ for each $2 \leq j \leq n$. Claim that G has a 2-edge-coloring (extend from f) without a monocolored maximal star by induction on n. If $n = 1$, then H is an odd hole of G. Since G is connected, there is a vertex $x \in V(H)$ having an incident edge which is colored by f, say color 1. Label two incident edges of x in H by color 2 and label other edges of H alternately around the cycle by 1,2,1,2,.... Since the number of edges of H is odd, every vertex of H has two incident edges with different colors. Now, assume that $n \geq 2$ and $(G - E(H)) + E(\bigcup_{i=1}^{n-1} O_i)$ has a 2-edge-coloring without a monocolored maximal star, say f'. Thus every vertex of $(G - E(H)) + E(\bigcup_{i=1}^{n-1} O_i)$ has two incident edges with different colors by f'. If O_n and $\bigcup_{i=1}^{n-1} O_i$ have the only one common vertex, say y, then y has two incident edges in $\bigcup_{i=1}^{n-1} O_i$ with different colors. Label edges of O_n alternately around the cycle by 1,2,1,2,.... If $|O_n \cap \bigcup_{i=1}^{n-1} O_i)| \geq 2$, then consider each path in O_n such that each edge of a path is not contained in $\bigcup_{i=1}^{n-1} O_i$. Label edges of each such path alternately by 1,2,1,2,.... Then every vertex of O_n has two incident edges with different colors. So G has a 2-edge-coloring without a monocolored maximal star. Hence $\chi_c(L(G)) = 2$. □

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