A generalization of a binomial sum for the Stirling numbers of the second kind

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Abstract

A combinatorial sum for the Stirling numbers of the second kind is generalized. This generalization provides us with a new explicit formula for the binomial sum $\sum_{k=0}^{n} k^r a^k b^{n-k} \binom{n}{k}$, where $a, b \in \mathbb{R} - \{0\}$, $n, r \in \mathbb{N}$. As relevant special cases, simple explicit expressions for both the binomial sum $\sum_{k=0}^{n} k^r \binom{n}{k}$ and the raw moment of order r of the binomial distribution $B(n,p)$ are given. All these sums are expressed in terms of generalized r-permutations.

Key words: Stirling numbers of the second kind; Combinatorial identity; Binomial moments; Generalized sampling; Generalized r-permutations

1 Introduction

The Stirling numbers of the second kind, denoted by $\left\{ {r \atop n} \right\}$, count the number of ways of partitioning a set of r (different) elements into n nonempty (indistinguishable) subsets ($n, r \in \mathbb{N}$). Stirling numbers of the second kind (see, e.g., [3, 4, 5, 8, 9, 10, 11, 15, 17]) clearly satisfy

$$\left\{ {r \atop 1} \right\} = \left\{ {r \atop r} \right\} = 1, \ \left\{ {r \atop n} \right\} = 0 \ \text{ for all } \ r < n,$$

obey the recurrence relation

$$\left\{ {r \atop n} \right\} = \left\{ {r-1 \atop n-1} \right\} + n \left\{ {r-1 \atop n} \right\},$$

can be defined by their generating function

$$\frac{x^n}{(1-x)(1-2x)\cdots(1-nx)} = \sum_{r=1}^{\infty} \left\{ {r \atop n} \right\} x^r,$$

or, in terms of the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$, as

$$x^r = \sum_{n=0}^{r} \left\{ {r \atop n} \right\} (x)_n \tag{1.1}$$

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and they can be explicitly computed from the binomial sum

$$\sum_{k=0}^{n}(-1)^{n-k}k^r\binom{n}{k}=n!\left\{{r \atop n}\right\} \tag{1.2}$$

that gives the number of distributions of r different objects into n nonempty distinguishable cells, i.e., the number of surjective applications from a set of r elements onto a set of n elements.

In particular, for $r=n$, Equation (1.2) becomes the well-known additive formula for the factorial [1, 12]

$$\sum_{k=0}^{n}(-1)^{n-k}k^n\binom{n}{k}=n!,$$

while, for all $r<n$, we have

$$\sum_{k=0}^{n}(-1)^{n-k}k^r\binom{n}{k}=0.$$

Note that the binomial sum in the left-hand side of Equation (1.2) is a special case ($a=1$, $b=-1$) of the more general binomial sum

$$S_{a,b}^{n,r}:=\sum_{k=0}^{n}k^r a^k b^{n-k}\binom{n}{k}, \tag{1.3}$$

where we assume that $a,b\in\mathbb{R}-\{0\}$ and n,r are positive integers.

The main goal of this paper is to provide an explicit expression for the binomial sum $S_{a,b}^{n,r}$. Note that for $b=-a$, using Equation (1.2), we have

$$S_{a,-a}^{n,r}=\sum_{k=0}^{n}k^r a^k(-a)^{n-k}\binom{n}{k}=a^n\sum_{k=0}^{n}(-1)^{n-k}k^r\binom{n}{k}=a^n\, n!\left\{{r \atop n}\right\}, \tag{1.4}$$

so that we can restrict to the case $b\neq -a$.

In Section 2, we give a recursion formula for $S_{a,b}^{n,r}$. Using this recurrence relation, in Section 3, we provide an explicit expression for the sum $S_{a,b}^{n,r}$. Next, in Section 4, we evaluate some special binomial sums of the type $S_{a,b}^{n,r}$. The polynomials in n of degree r involved in the explicit expression for $S_{a,b}^{n,r}$ lead us to define, in Section 5, the generalized r-permutations taken from a set of cardinality n. Then all sums $S_{a,b}^{n,r}$ are expressed in terms of such generalized r-permutations. Finally, in Section 6, we present our conclusions.

2 A recursion formula for $S_{a,b}^{n,r}$

The following lemma states a recurrence relation for the binomial sums $S_{a,b}^{n,r}$, that will be used in the next section.

Lemma 2.1 *Let $a,b\in\mathbb{R}-\{0\}$ and let n,r be positive integers. Then*

$$S_{a,b}^{n,r}=n\left(S_{a,b}^{n,r-1}-bS_{a,b}^{n-1,r-1}\right), \tag{2.1}$$

where we adopt the convention that

$$S_{a,b}^{n,0}=(a+b)^n \text{ for all } n\geq 1,\quad S_{a,b}^{0,r}=0 \text{ for all } r\geq 1,\quad\text{and}\quad S_{a,b}^{0,0}=1.$$

Proof. For $n = 1$, we have

$$S_{a,b}^{1,r} = \sum_{k=0}^{1} k^r a^k b^{1-k} \binom{1}{k} = a = \begin{cases} (a+b) - b \cdot 1 = S_{a,b}^{1,0} - bS_{a,b}^{0,0}, & \text{if } r = 1, \\ a - b \cdot 0 = S_{a,b}^{1,r-1} - bS_{a,b}^{0,r-1}, & \text{if } r > 1. \end{cases}$$

For $n > 1$, we have

$$\begin{aligned} S_{a,b}^{n,r} &= \sum_{k=0}^{n} k^r a^k b^{n-k} \binom{n}{k} = \sum_{k=1}^{n} k^r a^k b^{n-k} \binom{n}{k} \\ &= \sum_{k=1}^{n} k^r a^k b^{n-k} \frac{n}{k} \binom{n-1}{k-1} = n \sum_{k=1}^{n} k^{r-1} a^k b^{n-k} \binom{n-1}{k-1} \\ &= n \left[n^{r-1} a^n + \sum_{k=1}^{n-1} k^{r-1} a^k b^{n-k} \left(\binom{n}{k} - \binom{n-1}{k} \right) \right] \\ &= n \left[\sum_{k=1}^{n} k^{r-1} a^k b^{n-k} \binom{n}{k} - b \sum_{k=1}^{n-1} k^{r-1} a^k b^{n-1-k} \binom{n-1}{k} \right] \\ &= \begin{cases} n\left[(a+b)^n - b^n - b(a+b)^{n-1} + b^n\right] = n\left(S_{a,b}^{n,0} - bS_{a,b}^{n-1,0}\right), & \text{if } r = 1, \\ n\left(S_{a,b}^{n,r-1} - bS_{a,b}^{n-1,r-1}\right), & \text{if } r > 1 \end{cases} \end{aligned}$$

and the proof is concluded. □

3 An explicit formula for $S_{a,b}^{n,r}$

The following is the main result of this work.

Theorem 3.1 *Let $a, b \in \mathbb{R} - \{0\}$, $b \neq -a$ and let n, r be positive integers. Then*

$$\begin{aligned} S_{a,b}^{n,r} &= a(a+b)^{n-1} \sum_{C^r} \left(\frac{-b}{a+b}\right)^{i_{r-1}} (n - i_0)(n - i_1) \cdots (n - i_{r-1}), \\ C^r &= \left\{ (i_0, \ldots, i_{r-1}) \in {\mathbb{Z}_0^+}^r \mid i_0 = 0,\ i_k - i_{k-1} \in \{0, 1\}\ \forall k = 1, \ldots, r-1 \right\}. \end{aligned} \tag{3.1}$$

Proof. We proceed by induction on r. For $r = 1$, we have

$$C^1 = \left\{ i_0 \in \mathbb{Z}_0^+ \mid i_0 = 0 \right\} = \{0\},$$

and then

$$S_{a,b}^{n,1} = a(a+b)^{n-1} \left(\frac{-b}{a+b}\right)^0 (n - 0) = a \cdot n \cdot (a+b)^{n-1},$$

so that the result holds for $r = 1$ and for all $n \in \mathbb{N}$. Indeed, using Equation (2.1), we can confirm that, for all $n \in \mathbb{N}$

$$S_{a,b}^{n,1} = n\left(S_{a,b}^{n,0} - bS_{a,b}^{n-1,0}\right) = n\left((a+b)^n - b(a+b)^{n-1}\right) = a \cdot n \cdot (a+b)^{n-1}.$$

Assume that Equation (3.1) is true for a certain $r \geq 1$ and for all $n \in \mathbb{N}$. We must prove that for all $n \in \mathbb{N}$

$$S_{a,b}^{n,r+1} = a(a+b)^{n-1} \sum_{C^{r+1}} \left(\frac{-b}{a+b}\right)^{i_r} (n - i_0)(n - i_1) \cdots (n - i_r),$$

$$C^{r+1} = \left\{ (i_0, \ldots, i_r) \in \mathbb{Z}_0^{+\,r+1} \mid i_0 = 0,\ i_k - i_{k-1} \in \{0,1\}\ \forall k = 1, \ldots, r \right\}.$$

Using Equation (2.1) and the induction hypothesis, we have

$$\begin{aligned}
S_{a,b}^{n,r+1} &= n\left(S_{a,b}^{n,r} - bS_{a,b}^{n-1,r}\right) = nS_{a,b}^{n,r} - n\cdot b\cdot S_{a,b}^{n-1,r} \\
&= n\cdot a\,(a+b)^{n-1}\sum_{C^r}\left(\frac{-b}{a+b}\right)^{i_{r-1}}(n-i_0)\cdots(n-i_{r-1}) \\
&\quad - n\cdot b\cdot a\,(a+b)^{n-2}\sum_{C^r}\left(\frac{-b}{a+b}\right)^{i_{r-1}}(n-1-i_0)\cdots(n-1-i_{r-1}) \\
&= n\cdot a\,(a+b)^{n-1}\sum_{C^r}\left(\frac{-b}{a+b}\right)^{i_{r-1}}(n-i_0)\cdots(n-i_{r-1}) \\
&\quad + n\cdot a\,(a+b)^{n-1}\sum_{C^r}\left(\frac{-b}{a+b}\right)^{i_{r-1}+1}(n-1-i_0)\cdots(n-1-i_{r-1}) \\
&= a\,(a+b)^{n-1}\sum_{C_0^{r+1}}\left(\frac{-b}{a+b}\right)^{j_r}(n-j_0)\,(n-j_1)\cdots(n-j_r) \\
&\quad + a\,(a+b)^{n-1}\sum_{C_1^{r+1}}\left(\frac{-b}{a+b}\right)^{j'_r}(n-j'_0)\,(n-j'_1)\cdots(n-j'_r) \\
&= a\,(a+b)^{n-1}\sum_{C^{r+1}}\left(\frac{-b}{a+b}\right)^{i_r}(n-i_0)\,(n-i_1)\cdots(n-i_r).
\end{aligned}$$

In the second equality from the bottom, we have made the two following changes of variable

$$\begin{aligned}
j_0 &= 0 \quad \text{and} \quad j_k = i_{k-1} \qquad (1 \le k \le r) \quad \text{for the first sum,} \\
j'_0 &= 0 \quad \text{and} \quad j'_k = i_{k-1} + 1\ (1 \le k \le r) \quad \text{for the second sum.}
\end{aligned}$$

Note that, in particular, for $k = 1$ we have

$$j_1 = i_0 = 0, \qquad j'_1 = i_0 + 1 = 1$$

and this led us to consider the following two equal-sized subsets of C^{r+1}

$$C_0^{r+1} = \{(i_0, \ldots, i_r) \in C^{r+1} \mid i_1 = 0\}, \quad C_1^{r+1} = \{(i_0, \ldots, i_r) \in C^{r+1} \mid i_1 = 1\}$$

and, finally, the last equality follows using the obvious fact that

$$C^{r+1} = C_0^{r+1} \bigcup C_1^{r+1}. \ \square$$

Remark 3.1 Let $a, b \in \mathbb{R} - \{0\}$ and let n, s be positive integers. Making the change of indices $h = n - k$ in the following binomial sum, we get

$$\sum_{k=0}^{n}(n-k)^s\,a^k b^{n-k}\binom{n}{k} = \sum_{h=0}^{n} h^s b^h a^{n-h}\binom{n}{h} = S_{b,a}^{n,s}.$$

Moreover, using the binomial theorem in the following binomial sum, we get

$$\sum_{k=0}^{n} k^r\,(n-k)^s\,a^k b^{n-k}\binom{n}{k} = \sum_{q=0}^{s}(-1)^q\binom{s}{q} n^{s-q} S_{a,b}^{n,r+q},$$

so that the two above binomial sums have been reduced to the type $S_{a,b}^{n,r}$, computed by Equations (1.4) or (3.1), if $b = -a$ or $b \neq -a$, respectively.

4 Some special cases

Now, we highlight the following remarkable applications of Theorem 3.1.

Corollary 4.1 *Let n, r be positive integers. Then*

$$\sum_{k=0}^{n} k^r \binom{n}{k} = 2^{n-1} \sum_{C^r} \left(-\frac{1}{2}\right)^{i_{r-1}} (n-i_0)\cdots(n-i_{r-1}).$$

Proof. Using Equation (3.1) for $a = b = 1$, the proof is straightforward. □

In particular, using Corollary 4.1 for $r = 1, 2, 3$, we immediately obtain the following well-known combinatorial sums [14, 18]

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}, \ \sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}, \ \sum_{k=0}^{n} k^3 \binom{n}{k} = n^2(n+3)2^{n-3}.$$

Corollary 4.2 *Let n, r be positive integers. Then*

$$\sum_{k=0}^{n} k^r 2^k (-1)^{n-k} \binom{n}{k} = 2\sum_{C^r} (n-i_0)\cdots(n-i_{r-1}).$$

Proof. Using Equation (3.1) for $a = 2,\ b = -1$, the proof is straightforward. □

Corollary 4.3 *Let n, r be positive integers. Then the raw moment (about the origin) $m_{n,p}^r := E[X^r]$ of order r of a binomially distributed random variable $X \sim B(n, p)$ is given by*

$$m_{n,p}^r = p\sum_{C^r} (p-1)^{i_{r-1}} (n-i_0)\cdots(n-i_{r-1}).$$

Proof. Taking into account that (see, e.g., [16])

$$m_{n,p}^r = \sum_{k=0}^{n} k^r p^k (1-p)^{n-k} \binom{n}{k} = S_{p,1-p}^{n,r}$$

and using Equation (3.1) for $a = p,\ b = 1 - p$, the proof is straightforward. □

Remark 4.1 Regarding the raw binomial moments, in [2] the authors derive the following recurrence relation for the moments of the distribution $B(n, 1/2)$

$$m_{n,1/2}^r = n\left(m_{n,1/2}^{r-1} - \frac{1}{2}m_{n-1,1/2}^{r-1}\right).$$

This formula is a special case ($p = 1/2$) of the more general expression

$$m_{n,p}^r = n\left(m_{n,p}^{r-1} - (1-p)\, m_{n-1,p}^{r-1}\right), \quad 0 < p < 1,$$

which can be immediately obtained here using Equation (2.1) for $a = p$ and $b = 1 - p$.

Corollary 4.3 has provided us with a closed expression for the higher-order raw moments $E[X^r]$ of a binomial random variable X. In a recent work [7], the author presents recursive and closed-form expressions for both raw and central moments of the binomial distribution; see [7, Theorem 4.1]. For this purpose, the author first obtains recursive formulae for the raw moments involving the Stirling numbers of the second kind $\left\{ {r \atop n} \right\}$. Then, using the closed expression (1.2) for these numbers and a generalization of the well-known formula (1.1) (see [7, Lemma 3.1]), closed-form expressions for $E[X^r]$ (written as polynomials in p and in $q = 1 - p$) are derived. Regarding the central (binomial) moments $E[(X-\mu)^r]$ ($\mu = np$), they can be expressed in terms of the raw (binomial) moments, using the binomial theorem. More precisely, for an arbitrary real parameter μ, the r-th general moment of the random variable X about the point μ (and, in particular, the r-th central moment of X about the mean $E[X] = \mu$) is given by [7, Theorem 2.2]

$$E[(X-\mu)^r] = \sum_{i=0}^{r} \binom{r}{i} (-\mu)^{r-i} E\left[X^i\right]. \tag{4.1}$$

Indeed, our Remark 3.1 and Corollaries 4.1 & 4.3 are closely related to Theorems 2.2 & 4.1 and Lemma 3.1 in [7].

Closed formulae for the higher-order binomial moments are of relevance in the analysis of the storage capacity and retrieval error probabilities in neural associative memory networks -which have been widely applied, for example, in artificial intelligence and for modeling the brain [7]. Recent approaches to this problem can be found, e.g. in [6]. In synthesis, one of the most efficient models of neural associative memory for biological modeling and applications is the so-called *Willshaw model with binary neurons and synapses.* For an exact analysis of this model, one needs to compute the *Willshaw-Palm probability distribution of the neurons dendritic potentials*; see [6] for more details. Since the Willshaw-Palm distribution is more difficult to formulate, the analysis of binary neural associative networks of Willshaw type often relies on a binomial approximation [6, 7]. For this reason, in [6, Theorem 4.2] the raw and central moments of the Willshaw-Palm distribution are computed from the raw and central moments of the binomial distribution (given in [7, Theorem 4.1]). Further, in [6] the author analyzes the convergence of the Willshaw-Palm distribution and the corresponding binomial approximation, by determining asymptotic conditions when their moments become identical.

In a completely different context, the binomial sums associated to the higher-order moments of other probability distributions, can also be applied to the portfolio choice problem in the field of financial economics. For instance, in [13], a recursive formula for the central moments of the univariate lognormal distribution is presented, and then this recursive scheme is extended to the multivariate case. For these purposes, as mentioned above, the author uses the binomial sum (4.1) to reduce the computation of the central moments to the computation of the raw moments. Finally, the so-obtained recursive formula for the computation of *all* central moments of the multivariate lognormal distribution -up to a given maximum total order- is used to provide an approximate solution to a multi-asset portfolio choice problem.

5 Generalized r-permutations

The 2^{r-1} polynomials in n of degree r involved in Equation (3.1), lead us to define the generalized or "mixed" sampling scheme. Here, the term "mixed" means that the selection of each one of the r elements taken from an n-set can be done either with or without replacement.

Definition 5.1 *Let S be a set of n elements and r a positive integer ($r \geq 2$). Let $(u_1, \ldots, u_{r-1}) \in \{0,1\}^{r-1}$ s.t. $\sum_{i=1}^{r-1} u_i < n$. Then a generalized sample of the type*

$(u_1, \ldots, u_{r-1})$ of r elements taken from S is an ordered sequence $(x_1, \ldots, x_r)$ of r elements of S, selected according to the following criterion: For all $i = 1, \ldots, r-1$, if $u_i = 0$ ($u_i = 1$, respectively) then the i-th selected element x_i is (is not, respectively) replaced to the population S. The so performed sampling scheme is called generalized or mixed sampling of the type $(u_1, \ldots, u_{r-1})$.

Definition 5.1 can be reformulated in terms of r-permutations (arrangements) of the n-set S, as follows.

Definition 5.2 *Let S be a set of n elements and r a positive integer ($r \geq 2$). Let $(u_1, \ldots, u_{r-1}) \in \{0,1\}^{r-1}$ s.t. $\sum_{i=1}^{r-1} u_i < n$. Then a generalized r-permutation of the type $(u_1, \ldots, u_{r-1})$ of the n-set S is an ordered sequence $(x_1, \ldots, x_r)$ of r elements of S, selected according to the following restriction: If $u_i = 1$ for some $i = 1, \ldots, r-1$, then the j-th selected element x_j is required to be different from the i-th selected element x_i for all $j = i+1, \ldots, r$.*

The number of mixed samples of size r (or r-permutations) of the type $(u_1, \ldots, u_{r-1})$ taken from an n-set S is denoted by ${}_nP_r^{u_1, \ldots, u_{r-1}}$, and it is given by

$$\begin{gathered} {}_nP_r^{u_1, \ldots, u_{r-1}} = (n - i_0)(n - i_1) \cdots (n - i_{r-1}), \\ i_0 = 0 \ \text{ and } \ i_k = u_1 + \cdots + u_k, \quad 1 \leq k \leq r-1, \end{gathered} \tag{5.1}$$

since the first element x_1 of the sample can be always selected in n different ways from the n-set S and after selecting the first k elements $x_1, \ldots, x_k$ ($1 \leq k \leq r-1$) from the population S, its cardinality will be decreased by the sum $i_k = u_1 + \cdots + u_k$, and thus the number of remaining elements in the n-set S will be $n - i_k$.

Obviously, the number of generalized r-permutations, obtained via this mixed sampling scheme, is the same irrespective of whether the r-th (last) element is replaced or not, and that is why the type of the samples of size r is defined by a binary $(r-1)$-tuple and not by an r-tuple.

Remark 5.1 The sampling scheme of the type $(u_1, \ldots, u_{r-1})$ generalizes the usual sampling both with and without replacement. In other words, the mixed r-permutations of the type $(u_1, \ldots, u_{r-1})$ generalize the usual r-permutations both with and without repetitions allowed. That is, the r-permutations of the type $(0, \ldots, 0)$ (of the type $(1, \ldots, 1)$, respectively) are just the usual r-permutations with (without, respectively) repetitions, and using Equation (5.1), we have

$${}_nP_r^{0, \ldots, 0} = n^r, \quad {}_nP_r^{1, \ldots, 1} = n(n-1) \cdots (n-(r-1))$$

Now, we can express the sums $S_{a,b}^{n,r}$, evaluated in Theorem 3.1, in terms of generalized r-permutations, as follows.

Corollary 5.1 *Let $a, b \in \mathbb{R} - \{0\}$, $b \neq -a$ and let n, r be positive integers. Then*

$$S_{a,b}^{n,r} = a(a+b)^{n-1} \sum_{(u_1, \ldots, u_{r-1}) \in \{0,1\}^{r-1}} \left(\frac{-b}{a+b}\right)^{u_1 + \cdots + u_{r-1}} {}_nP_r^{u_1, \ldots, u_{r-1}},$$

$${}_nP_r^{u_1, \ldots, u_{r-1}} = n(n-u_1)(n-(u_1+u_2)) \cdots (n-(u_1+u_2+\cdots+u_{r-1})).$$

Proof. Using Equations (3.1) and (5.1), the proof is straightforward. □

Remark 5.2 According to Corollary 5.1 and Equation (1.3), Corollary 4.2 can be rewritten as

$$\sum_{(u_1,\ldots,u_{r-1})\in\{0,1\}^{r-1}} {}_nP_r^{u_1,\ldots,u_{r-1}} = \frac{1}{2}S_{2,-1}^{n,r},$$

while it can be proved by induction on r that for all $n, r \in \mathbb{N}$

$$\sum_{(u_1,\ldots,u_{r-1})\in\{0,1\}^{r-1}} (-1)^{u_1+\cdots+u_{r-1}} \ {}_nP_r^{u_1,\ldots,u_{r-1}} = n.$$

6 Summary & Conclusions

We have derived both a recurrence relation and an explicit formula for the binomial sum $S_{a,b}^{n,r} = \sum_{k=0}^{n} k^r a^k b^{n-k}\binom{n}{k}$ $(a, b \in \mathbb{R}-\{0\}, n, r \in \mathbb{N})$. That sum has been expressed in terms of the generalized r-permutations of the type $(u_1, \ldots, u_{r-1}) \in \{0,1\}^{r-1}$ from the n-set S, associated to the mixed sampling scheme of the same type (the i-th selected element is replaced or not replaced iff $u_i = 0$ or 1, respectively). The most relevant special case is the so-obtained explicit formula for the binomial sum $S_{1,1}^{n,r} = \sum_{k=0}^{n} k^r\binom{n}{k}$.

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References

[1] R. Anglani, M. Barile, *Two very short proofs of a combinatorial identity*, Integers 5 (2005), #A18.

[2] A. Benyi, S.M. Manago, *A recursive formula for moments of a binomial distribution*, College Math. J. 36 (2005), 68-72.

[3] R.A. Brualdi, *Introductory Combinatorics*, second ed., Elsevier, Amsterdam, 1991.

[4] C.A. Charalambides, *Enumerative Combinatorics*, CRC Press, Boca Raton, FL, 2002.

[5] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, second ed., Addison-Wesley, Reading, MA, 1994.

[6] A. Knoblauch, *Neural associative memory and the Willshaw-Palm probability distribution*, SIAM J. Appl. Math. 69 (2008), no.1, 169-196.

[7] A. Knoblauch, *Closed-form expressions for the moments of the binomial probability distribution*, SIAM J. Appl. Math. 69 (2008), no.1, 197-204.

[8] D.E. Knuth, *The Art of Computer Programming: Fundamental Algorithms, Vol. 1: Fundamental Algorithms*, third ed., Addison-Wesley, Reading, MA, 1997.

[9] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1980.

[10] S. Roman, *The Umbral Calculus*, Dover, New York, 2005.

[11] K.H. Rosen (Ed.), *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Boca Raton, FL, 2000.

[12] S.M. Ruiz, *An algebraic identity leading to Wilson's theorem*, Math. Gaz. 80 (1996), 579-582.

[13] G. Skoulakis, *A recursive formula for computing central moments of a multivariate lognormal distribution*, The American Statistician 62 (2008), no.2, 147-150.

[14] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2009, published electronically at http://www.research.att.com/~njas/sequences/A102573.

[15] R.P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, Cambridge, 1997.

[16] A. Stuart, J.K. Ord, *Kendall's Advanced Theory of Statistics*, Vol. 1, Oxford University Press, Oxford, 1998.

[17] J.H. Van Lint, R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

[18] E.W. Weisstein, *Binomial Sums*, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BinomialSums.html.