Critical Sets in F-squares

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Abstract

In this paper we address the problem of construction of critical sets in F-squares of the form $F(2n; 2, 2, \ldots\ldots\ldots, 2)$. We point out that the critical set in $F(2n; 2, 2, \ldots\ldots\ldots, 2)$ obtained by Fitina, Seberry and Sarvate (1999) is not correct and prove that in the given context a proper subset is a critical set.

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AMS Subject Classification (1970) :05B15

Keywords and Phrases : Back-circulant latin squares, F-squares, Critical sets.

1 Introduction

In recent years, a number of papers have dealt with the study of critical sets in latin squares which consist of specification of a minimal set of cell entries needed to recreate combinatorial structures uniquely. To name a few, the reader can look into Nelder [12], Smetanuik [15], Curran and van Rees [4], Cooper, Donovan and Seberry [2], Cooper, McDonough and Mavron [3], Donovan, Cooper, Nott and Seberry [5], Donovan and Cooper [6], Fu, Fu and Rodger [9], Donovan and Howse [7] and SahaRay, Adhikari and Seberry [13, 14]. Not much work has been done in regard to construction of critical sets in F-squares and orthogonal F-squares which are natural genaralisations of latin squares and mutually orthogonal latin squares. The papers known to the authors dealing with critical sets in F-squares of a specific form are Fitina, Seberry and Sarvate (1999), to be denoted by FSS (1999) herafter, and Bate and van Rees (2002). A close examination of the critical sets in F-squares obtained in FSS (1999) revealed that in some cases the sets do not satisfy the properties of a critical set. In particular, we deal with F-squares of the type $F(2n; 2, \ldots, 2)$. Before discussing the main results, some background information is needed which is presented in Section 2. In Section 3 we rectify the critical set result stated in Theorem 14 of FSS (1999) and exhibit a correct form of a critical set which is a **proper subset** of the one proposed for $F(2n; 2, \ldots, 2)$.

2 Preliminary Definitions and Notations

A frequency square, or F-square, $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ of order n is an $n \times n$ array with entries chosen from the set $N = \{0, 1, 2, \ldots, v-1\}$ such that each element i occurs α_i times in each row and in each column, where each α_i is a natural number and $\sum_{i=0}^{v-1} \alpha_i = n$. For convenience, an F-square of order n is sometimes represented by a set of ordered triples $F = \{(i, j; k)|$ element k occurs in the position (i, j), $(i, j) \in \{0, 1, 2, \ldots, n-1\}, k \in N\}$. A subset of F will also be called a *partial* F-square.

A *partial* F-square P of order n is an $n \times n$ array with entries chosen from N such that the kth element of N occurs α_{ki} times in the ith row and β_{kj} times in the jth column of P, $k \in N$, $0 \leq \alpha_{ki}, \beta_{kj} \leq \alpha_i$. Then $\alpha'_k = (\alpha_{k1}, \ldots, \alpha_{kn})$ and $\beta'_k = (\beta_{k1}, \ldots, \beta_{kn})$ are said to be frequency vectors for the element k in P along the rows and columns respectively. Then $|P|$ is said to be the size of the *partial* F-square and the set of positions $S_P = \{(i, j)| (i, j; k) \in P, \exists\, k \in N\}$ is said to determine the *shape* of P. Let P and P' be two *partial* F-squares of the same order, with the same size, shape and the same frequency vectors along the rows and columns. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position in P. A F-square *interchange* F_0 is a *partial* F-square for which there exists another *partial* F-square F_0' of the same order, size, shape and the same frequency vector along the rows and columns with the property that F_0 and F_0' are *disjoint* and *mutually balanced.* Thus the rationale behind a F-square *interchange* is that, in a legitimate F-square, F_0 can be replaced by F_0' without altering any property of the original F-square. For example F_0 and F_0' given below are two F-square *interchanges* of order 4.

$F_0 =$

2	2	3	3
3	.	2	.
.	3	.	2
.	.	.	.

$F_0' =$

3	3	2	2
2	.	3	.
.	2	.	3
.	.	.	.

A nonempty subset C of $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ is a *critical set* of F if

1. F is the only F-square of order n which has element k in position (i, j) for each $(i, j; k) \in C$

2.(a) Every proper subset of C is contained in at least two F-squares of the type $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ or

(b) For every $(i, j; k) \in C, l \in N, l \neq k \Rightarrow$ there does not exist any F-square of type $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ which contains $(S \backslash \{(i, j; k)\}) \bigcup \{(i, j; l)\}$.

We note that a latin square is an F-square of type $F = F(n; 1, 1, \ldots, 1)$. A latin square $L = \{(i, j; k)\}$ of order n is called *back circulant* if $k = (i + j)(mod\ n)$ for every triple $(i, j; k) \in L$. Let $I = \{(i, j; i + j : 0 \leq i, j \leq n - 1\}$. Then $\rho_w(I), w \in N$ is the symmetric latin square given by $L = \{(i, j; i + j + w)\}$ with addition reduced modulo n. In particular, $\rho_2(I) = \{(i, j; i + j + 2); 0 \leq i \leq n - 1, 0 \leq j \leq n - 1\}$. Thus, in $\rho_2(I)$, 1 occurs in the **anti diagonal**, i.e., in the cells $(0, n - 1), \ldots, (n - 1, 0)$. Furthermore it is to be noted that i occurs in the cells $\{(0, i - 2), (1, i - 3), \ldots, (i - 2, 0), (i - 1, n - 1), (i, n - 2), \ldots (n - 1, i - 1)\}$ for $i = 2, 3, \ldots, n - 1$ and 0 occurs in the cells $\{(0, n - 2), (1, n - 3), \ldots, (n - 2, 0), (n - 1, n - 1)\}$. In our subsequent discussion, for each i, $i = 0, 2, 3, \ldots, n - 1$ in $\rho_2(I)$, we refer to these collection of cells of occurrence as **reverse transversals** of i.

3 Main Result

In this section, we investigate the general construction of a critical set for the specific type of F-square $F(2n; 2, 2, \ldots, 2)$. We refer to the construction given in FSS (1999) and point out that the critical set mentioned in the proof of Theorem 14 of FSS (1999), while *uniquely completable*, does not satisfy the second condition and thereby cannot be claimed to be a critical set. In order to rectify the Theorem and propose a correct form of the critical set for the specific type of F-square $F(2n; 2, 2, \ldots, 2)$, we refer to the construction given in Section 4.1 of FSS (1999).

Let S be a finite set, say $S = \{0, 1, 2, \ldots, n-1\}$. Let $\Pi = \{C_1, C_2, \ldots C_n\}$ be any ordered collection of n subsets of S, each of size 2, such that each element $k \in S$ occurs in precisely two sets in Π. Let L_i be a 2×2 latin square, formed from the elements of the set C_i, $1 \leq i \leq n$. Then a latin square in the symbols $L_1, L_2, \ldots L_n$ respectively is also an F-square of the type $F(2n; 2, 2 \ldots 2)$ in the elements $0, 1, 2, \ldots n-1$. Let $L_1, L_2, \ldots L_n$ respectively be the latin squares given below:

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \quad \cdots \quad \begin{array}{|c|c|}\hline n-2 & n-1 \\ \hline n-1 & n-2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline n-1 & 0 \\ \hline 0 & n-1 \\ \hline \end{array}$$

Then the F-square

L_1	L_2	L_3	.....	L_{n-1}	L_n
L_2	L_3	L_4	.....	L_n	L_1
L_3	L_4	L_5	....	L_1	L_2
..	..	..	.....	..	..
L_n	L_1	L_2	.....	L_{n-2}	L_{n-1}

is isomorphic to the F-square

$$F_2 : \begin{array}{|c|c|}\hline I & I \\ \hline I & \rho_2(I) \\ \hline \end{array} \tag{3.1}$$

We now quote Theorem 14 from Fitina, Seberry and Sarvate (1999).

Theorem 3.1 *(FSS (1999)) Let $n = 2m$, $m \geq 2$. Let I be of order n. Then the F-square above has a critical set of size $7m^2 - m + 1$. When I is of order $n + 1$, then there is a critical set of size $7m^2 + 6m + 2$.*

In the proof of this theorem the following *partial* F-square given in (3.2) is claimed to be a critical set for (3.1):

$$\hat{F} = \begin{array}{|c|c|}\hline \hat{I} & \hat{I} \\ \hline \hat{I} & \hat{\rho}_2(I) \\ \hline \end{array} \tag{3.2}$$

where

$\hat{I}$ =

0	1	...	$m-1$				
1	...	$m-1$					
...	$m-1$						
$m-1$							
							m
					...	...	...
				m	...	...	$n-2$

,

and

$\hat{\rho}_2(I)$ =

2	3	...	$m-1$		$m+1$	...	1
3	...	$m-1$		$m+1$	...	1	2
...	$m-1$		$m+1$	...	...	...	...
$m-1$		$m+1$	...	...	...	...	...
	$m+1$	...	...	...	...	...	...
...	...	...	...	...	...	...	...
$n-1$	0	1	2	...	...	$n-3$	$n-2$
0	1	2	3	...	...	$n-2$	$n-1$
1	2	3	4	...	...	$n-1$	0

.

We now show that the above mentioned *partial* F-square does not satisfy condition 2 in the definition of a critical set. For example, when $m = 2$ and $n = 2m$ the claimed critical set is as follows.

0	**1**	.	.	**0**	**1**	.	.
1	.	.	.	**1**	.	.	.
.	.	.	.	.	.	.	.
.	.	.	**2**	.	.	.	**2**
0	**1**	.	.	.	**3**	**0**	**1**
1	.	.	.	**3**	**0**	**1**	**2**
.	.	.	.	**0**	**1**	**2**	**3**
.	.	.	**2**	**1**	**2**	**3**	**0**

This set fails to satisfy condition 2 (a) because the proper subset obtained by removing entries (4,7;1) and (7,4;1) is still uniquely completable (the details will be given in the proof of Theorem 3.2). The set also fails condition 2 (b) because there exist entries that can be changed without making it impossible to complete the square; for example, if we change (3,3;2) to (3,3;3) we can obtain the completion

$$
\begin{array}{cccc|cccc}
0 & 1 & 3 & 2 & 0 & 1 & 2 & 3 \\
1 & 3 & 2 & 0 & 1 & 2 & 3 & 0 \\
3 & 2 & 0 & 1 & 2 & 3 & 0 & 1 \\
2 & 0 & 1 & 3 & 3 & 0 & 1 & \mathbf{2} \\
\hline
0 & 1 & 2 & 3 & 2 & \mathbf{3} & 0 & \mathbf{1} \\
1 & 2 & 3 & 0 & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{2} \\
2 & 3 & 0 & 1 & 0 & 1 & \mathbf{2} & \mathbf{3} \\
3 & 0 & 1 & \mathbf{2} & 1 & \mathbf{2} & \mathbf{3} & \mathbf{0}
\end{array}
$$

Thus the claim made in Theorem 14 in FSS (1999) is not correct. We now present the corrected form of a critical set of $F(2n, 2, \ldots, 2)$ which is a proper subset of the set proposed by FSS (1999). To stress this fact, Theorem 3.2 below is stated in the same language as Theorem 14 of FSS (1999).

Theorem 3.2 *Let $n = 2m, m \geq 2$. Let I be of order n. Then the F-square given in (3.1) has a critical set F' of size $7m^2 - 3m + 3$. When I is of order $n = 2m + 1$, then there is a critical set of size $7m^2 + 4m + 4$.*

Proof: The general construction of the *partial* F-square F' is:

$$
F' = \begin{array}{|c|c|}
\hline
\tilde{I} & \tilde{I} \\
\hline
\tilde{I} & \tilde{\rho}_2(I) \\
\hline
\end{array}
\tag{3.3}
$$

where

$\tilde{I}$ =

0	1	2	...	$m-1$					
1	2	...	$m-1$						
2	...	$m-1$							
...	$m-1$								
$m-1$									
									m
								m	...
							...	...	...
						m	...	...	$n-2$

,

$\tilde{\rho}_2(I)$ =

2	...	$m-1$		$m+1$	...	...	...	0	
...	$m-1$		$m+1$	...	...	...	...	1	
$m-1$		$m+1$	...	...	...	...	...	...	
	$m+1$	...	...	...	...	...	...	$m-2$	
$m+1$	...	...	...	...	...	...	...	...	m
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
0	1	...	...	$m-1$	...	...	...	...	$n-1$
				m	...	...	...	$n-1$	0

for $m \geq 3$, and

$\tilde{\rho}_2(I)$ =

	3	0	
3	0	1	2
0	1	2	3
	2	3	0

for $m = 2$.

Thus there are $3(m-1)$ empty cells in $\tilde{\rho}_2(I)$: in the upper left corner, $m-1$ cells in locations $(n, n+m-2), (n+1, n+m-3), .., (n+m-2, n)$; in the last column, $m-1$ cells in locations $(n, 2n-1), (n+1, 2n-1), ..., (n+m-2, 2n-1)$; and in the last row, $m-1$ cells in locations $(2n-1, n), (2n-1, n+1), .., (2n-1, n+m-2)$.

To prove that F' is critical set in F_2 (3.1), we will show below that (i) F' has unique completion to F and (ii) that any proper subset of F' can be completed to at least two F-squares.

(i) Towards unique completion of F' to F we argue as follows:

Step 1: $n-1$ is filled uniquely in cells $(0, 2n-1), (0, n-1), (n-1, 0)$ and $(2n-1, 0)$.

Step 2: The cells in the 0 th column are filled in sequentially in the order $(2n-2, 0), (n-2, 0), (2n-3, 0), (n-3, 0)........(n+m, 0), (m, 0)$.

Step 3: The cells in the 0th row are now filled sequentially in the order $(0, 2n-2), (0, n-2), (0, 2n-3), (0, n-3), .., (0, n+m), (0, m)$.

Step 4: In the nth column, the entries m, $m+1$, $m+2, ..., n-1$ and 1 are placed uniquely in that order.

Step 5: i is placed uniquely in columns 1, $n+1$, 2, $n+2, ...,$ $i-1$, $n+i-1$, and i sequentially in that order for $i = m, ...n-1$.

Step 6: Now $(n, 2n-1) = 1$.

Step 7: Now i is placed uniquely in columns $i+1,\ n+i+1,\ i+2,\ n+i+2, \ldots,\ n-1,\ 2n-1$ sequentially in that order and $i+2$ is placed in the cells $(2n-1, n+i+1), (n+i+1, 2n-1)$ for $i = 0, 1, 2, \ldots, m-3$.
Step 8: Now i is placed in columns $i+1,\ n+i+1, \ldots\ n-1,\ 2n-1$ sequentially in that order for $i = m-2$ and $m-1$.
Thus the F-square is completed uniquely from F'.

(ii) To prove that any proper subset of the *partial* F-square F' listed above leads to more than one legitimate F-square, we first note that, irrespective of whether n is even or odd, $\tilde{I}$ was proven to be a critical set for I by Curran and van Rees [4]. So deleting any entry from $\tilde{I}$ will lead to a completion of $\tilde{I}$ to a latin square other than I and hence to a F-square different from F_2 in (3.1). So it suffices to show that deleting any entry from $\tilde{\rho}_2(I)$ will also lead to a completion different from F_2 in (3.1). Moreover, it is also clear from the symmetric structure of F_2 in (3.1) that in this context, only entries on or below the diagonal of $\tilde{\rho}_2(I)$ need to be considered as the cases of entries above the diagonal are merely the transpose of the corresponding cases below the diagonal. To this end, we deal with $n = 2m$ and $n = 2m+1$ separately.

Case 1: $n = 2m$

We note that for each triple $(i,\ j;\ x)$ of $\tilde{\rho}_2(I)$, there exists a *partial* F-square $I(x) \subset F_2$ for which $I'(x)$ is a F-square *interchange* and $F' \bigcap I(x) = \{(i,\ j;\ x)\}$. Thus it follows that $F' \setminus \{(i,\ j;\ x)\}$ can be completed to a F-square different from (3.1).
Above the absent m's in $\tilde{\rho}_2(I)$, for each $x = 2, \ldots, m-1$, with $t = \lceil \frac{x-2}{2} \rceil, \ldots, x-2$,

$$\begin{aligned} I(x) \quad = \quad & \{(n+t,\ n-t+x-2;\ x), (n+t,\ m+x-1-t;\ m+x-1), \\ & (m+t+1,\ m+x-1-t;\ x), (m+t+1,\ n-t+x-2;\ m+x-1)\} \end{aligned}$$

has an F-square interchange given by

$$\begin{aligned} I'(x) \quad = \quad & \{(n+t,\ n-t+x-2;\ m+x-1), (n+t,\ m+x-1-t;\ x), \\ & (m+t+1,\ m+x-1-t;\ m+x-1), (m+t+1,\ n-t+x-2;\ x)\}. \end{aligned}$$

Below the absent m's and above the reverse transversal of m's in the lower right corner of $\tilde{\rho}_2(I)$, for each $x = m+1, \ldots, n-1$, with $t = \lceil \frac{x-2}{2} \rceil, \ldots, x-2$,

$$\begin{aligned} I(x) \quad = \quad & \{(n+t,\ n-t+x-2;\ x), (n+t,\ x-1-t;\ x-1), (t+1,\ x-1-t; \\ & x), (t+1,\ n-t+x-2;\ x-1)\} \end{aligned}$$

has an F-square interchange given by

$$\begin{aligned} I'(x) \quad = \quad & \{(n+t,\ n-t+x-2;\ x-1), (n+t,\ x-1-t;\ x), (t+1,\ x-1-t; \\ & x-1), (t+1,\ n-t+x-2;\ x)\}; \end{aligned}$$

for each $x = 1, \ldots, m-1$, with $t = m + \lfloor \frac{x-1}{2} \rfloor, \ldots, n-2$,

$$I(x) = \{(n+t,\ 2n-t+x-2;\ x), (n+t,\ n+x-1-t;\ x-1), (t+1,\ n+x-1-t;\ x), (t+1,\ 2n-t+x-2;\ x-1)\}$$

has an F-square interchange given by

$$I'(x) = \{(n+t,\ 2n-t+x-2;\ x-1), (n+t,\ n+x-1-t;\ x), (t+1,\ n+x-1-t;\ x-1), (t+1,\ 2n-t+x-2;\ x\};$$

and for $x = 0$, with $t = m-1, \ldots, n-2$,

$$I(x) = \{(n+t,\ 2n-t-2;\ 0), (n+t,\ n-1-t;\ n-1), (t+1,\ n-1-t;\ 0), (t+1,\ 2n-t-2;\ n-1)\}$$

has an F-square interchange given by

$$I'(x) = \{(n+t,\ 2n-t-2;\ n-1), (n+t,\ n-1-t;\ 0), (t+1,\ n-1-t;\ n-1), (t+1,\ 2n-t-2;\ 0\}.$$

Below the reverse transversal of $m-1$'s in the lower right corner of $\tilde{\rho}_2(I)$, for $x = m$ with $t = 1, \ldots, \lfloor \frac{n-x}{2} \rfloor$,

$$I(x) = \{(2n-1-t,\ n+m-1+t;\ m), (2n-1-t,\ t;\ n-1), (m-t,\ t;\ m), (m-t,\ n+m-1+t;\ n-1)\}$$

has an F-square interchange given by

$$I'(x) = \{(2n-1-t,\ n+m-1+t;\ n-1), (2n-1-t,\ t;\ m), (m-t,\ t;\ n-1), (m-t,\ n+m-1+t;\ m)\};$$

for each $x = m+1, \ldots, n-1$ with $t = 1, \ldots, \lfloor \frac{n-x}{2} \rfloor$,

$$I(x) = \{(2n-1-t,\ n+x-1+t;\ x), (2n-1-t,\ x-m+t;\ x-m-1), (m-t,\ x-m+t;\ x), (m-t,\ n+x-1+t;\ x-m-1)\}$$

has an F-square interchange given by

$$I'(x) = \{(2n-1-t,\ n+x-1+t;\ x-m-1), (2n-1-t,\ x-m+t;\ x), (m-t,\ x-m+t;\ x-m-1), (m-t,\ n+x-1+t;\ x)\};$$

and for $x = 0$

$$I(x) = \{(2n-1,\ 2n-1;\ 0), (2n-1,\ m;\ m-1), (m,\ m;\ 0), (m,\ 2n-1;\ m-1)\}$$

has an F-square interchange given by

$$\begin{aligned} I'(x) &= \{(2n-1,\ 2n-1;\ m-1),(2n-1,\ m;\ 0),\\ &\quad (m,\ m;\ m-1),(m,\ 2n-1;\ 0)\}. \end{aligned}$$

Case 2: $n = 2m + 1$

This case is a little more difficult than the case where n is even, but we can still show that removal of any of the entries in $\tilde{\rho}_2(I)$ leads to more than one completion. For all elements of $\tilde{\rho}_2(I)$, beginning with the reverse transversal of $(m+1)$'s just below the absent m's and continuing to the reverse transversal of $(m-1)$'s below the anti diagonal, any of the elements can be shown to be necessary in the critical set using a swap with the element that is one lower (considering $n-1$ to be one lower than 0) in the same way as was demonstrated in the case of n even. Below the aforementioned $(m-1)$'s and above the absent m's, however, there is more work to be done. The cases of removal of the triples $(2n-1,\ 2n-1;\ 0)$, $(2n-1,\ 2n-2;\ n-1)$, $(2n-1,\ 2n-3;\ n-2)$, and $(2n-2,\ 2n-2;\ n-2)$ in $\tilde{\rho}_2(I)$ are simpler to deal with. We demonstrate below the required F-square interchanges.

For $(2n-1,\ 2n-1;\ 0)$,

$$\begin{aligned} I(0) &= \{(2n-1,\ 2n-1;\ 0),(2n-1,\ 2;\ 1),(n-2,\ 2;\ 0),(n-2,\ 3;\ 1),\\ &\quad (n-3,\ 3;\ 0),(n-3,\ 4;\ 1),\ldots,(2,\ n-2;\ 0),(2,\ 2n-1;\ 1)\}\\ I'(0) &= \{(2n-1,\ 2n-1;\ 1),(2n-1,\ 2;\ 0),(n-2,\ 2;\ 1),(n-2,\ 3;\ 0),\\ &\quad (n-3,\ 3;\ 1),(n-3,\ 4;\ 0),\ldots,(2,\ n-2;\ 1),(2,\ 2n-1;\ 0)\}. \end{aligned}$$

For $(2n-1,\ 2n-2;\ n-1)$,

$$\begin{aligned} I(n-1) &= \{(2n-1,\ 2n-2;\ n-1),(2n-1,\ 1;\ 0),(n-2,\ 1;\ n-1),\\ &\quad (n-2,\ 2;\ 0),(n-3,\ 2;\ n-1),(n-3,\ 3;\ 0),\ldots,\\ &\quad (2,\ n-3;\ n-1),(2,\ 2n-2;\ 0)\}\\ I'(n-1) &= \{(2n-1,\ 2n-2;\ 0),(2n-1,\ 1;\ n-1),(n-2,\ 1;\ 0),\\ &\quad (n-2,\ 2;\ n-1),(n-3,\ 2;\ 0),(n-3,\ 3;\ n-1),\ldots,\\ &\quad (2,\ n-3;\ 0),(2,\ 2n-2;\ n-1)\}. \end{aligned}$$

For $(2n-1,\ 2n-3;\ n-2)$,

$$\begin{aligned} I(n-2) &= \{(2n-1,\ 2n-3;\ n-2),(2n-1,\ 0;\ n-1),(n-2,\ 0;\ n-2),\\ &\quad (n-2,\ 1;\ n-1),\ldots,(2,\ n-4;\ n-2),(2,\ 2n-3;\ n-1)\}\\ I'(n-2) &= \{(2n-1,\ 2n-3;\ n-1),(2n-1,\ 0;\ n-2),(n-2,\ 0;\ n-1),\\ &\quad (n-2,\ 1;\ n-2),\ldots,(2,\ n-4;\ n-1),(2,\ 2n-3;\ n-2)\}. \end{aligned}$$

For $(2n-2,\ 2n-2;\ n-2)$,

$$\begin{aligned} I(n-2) &= \{(2n-2,\ 2n-2;\ n-2),(2n-2,\ 1;\ n-1),(n-3,\ 1;\ n-2),\\ &\quad (n-3,\ 2;\ n-1),\ldots,(1,\ n-3;\ n-2),(1,\ 2n-2;\ n-1)\}\\ I'(n-2) &= \{(2n-2,\ 2n-2;\ n-1),(2n-2,\ 1;\ n-2),(n-3,\ 1;\ n-1),\\ &\quad (n-3,\ 2;\ n-2),\ldots,(1,\ n-3;\ n-1),(1,\ 2n-2;\ n-2)\}. \end{aligned}$$

For the rest of the elements in $\tilde{\rho}_2(I)$ the F-square interchanges are not that straightforward. To facilitate explanation below, we refer to I's in the upper left, upper right and lower left corners of F_2 in (3.1) as I_1, I_2 , and I_3 and respectively.

We first discuss the entries in the lower right corner of $\tilde{\rho}_2(I)$ and exhibit below algorithms to complete the partial F-square $F' \setminus \{(2n-1-t,\ n+x-1+t;\ x))\}$ to a F-square different from (3.1) for $x = m,\ m+1, \ldots, n-3$, with $t = 0, \ldots, \lfloor \frac{n-x}{2} \rfloor$. Algorithm 1 below takes care of completion of $F' \setminus \{(2n-1-t,\ n+x-1+t;\ x))\}$, for the combination of $x = m,\ t = 0$, whereas Algorithm 2 takes care of the other combinations of x and t cited above.

Algorithm 1 Start with F_2 in (3.1).

Step 1: Replace $(2n-1,\ n+m-1;\ m)$ and $(2n-1,\ 0;\ n-1)$ by $(2n-1,\ n+m-1;\ n-1)$ and $(2n-1,\ 0;\ m)$ respectively.

Step 2: Replace $(m+1,\ 0;\ m+1)$ and $(m+1,\ n+m-1;\ n-1)$ by $(m+1,\ 0;\ n-1)$ and $(m+1,,\ n+m-1;\ m+1)$ respectively.

Step 3: Replace $\{(m,\ 0;\ m), (m-1,\ 1;\ m), \ldots, (2,\ m-2;\ m)\}$ by $\{(m,\ 0;\ m+1), (m-1,\ 1;\ m+1), \ldots, (2,\ m-2;\ m+1)\}$ respectively.

Step 4: Replace $\{(m,\ 1;\ m+1), (m-1,\ 2;\ m+1), \ldots, (3,\ m-2;\ m+1)\}$ by $\{(m,\ 1;\ m), (m-1,\ 2;\ m), \ldots, (3,\ m-2;\ m)\}$ respectively.

Step 5: Replace $(2,\ n+m-1;\ m+1)$ by $(2,\ n+m-1\ m)$.

The resultant square is the required completion.

Algorithm 2 Start with F_2 in (3.1).

Step 1: Replace $(2n-1-t,\ n+x+t-1;\ x)$ by $(2n-1-t,\ n+x+t-1;\ n-t-1)$; $(2n-1-t,\ 0;\ n-t-1)$ by $(2n-1-t,\ 0;\ x)$ and $(n-2t-x,\ n+x+t-1;\ n-t-1)$ by $(n-2t-x,\ n+x+t-1;\ x)$.

Step 2: Starting from the cell $(x,\ 0)$ in I_1, locate all the cells along a stair path skipping alternatively $n-t-x-2$ cells in the horizontal direction towards right and then $n-t-x-2$ cells in the vertical direction upwards . Continue this move until the cell $(i_0,\ j_0)$ where $i_0 + j_0 = x$ is reached such that $0 \leq i_0 - (n-2t-x) \leq n-t-x-1$. Swap the element x with $n-t-1$ all throughout in these cells. If $i_0 - (n-2t-x) = 0$, then stop here and the reqiured F- square is obtained. Otherwise, go to Step 3.

Step 3: Replace $\{(n-2t-x,\ 2x+2t-n;\ x), \ldots, (n-2t-x,\ t+x-2;\ n-t-2)\}$ by $\{(n-2t-x,\ 2x+2t-n;\ x+1), \ldots, (n-2t-x,\ t+x-2;\ n-t-1)\}$.

Step 4: Locate the triples $\{(i_0,\ j_0+1;\ x+1), (i_0-1,\ j_0+1;\ x), (i_0-1,\ j_0+2;\ x+1), (i_0-2,\ j_0+2;\ x), \ldots, (n-2t-x+1,\ 2t+2x-n;\ x+1)\}$ and swap the element x with $x+1$.

Step 5: Adjust the elements in the cells $(i,\ j)$, where $i = n-2t-x+1, \ldots, i_0$ and
$j = x-i+2, \ldots, n-t-1-i$, accordingly so that the resultant square is a F-square.

We now consider the entries in the upper left corner of $\tilde{\rho}_2(I)$ and proceed to demonstrate alternate completions for the partial F-square $F'\backslash\{n+x-t-2, n+t; x)$ for $x = 2, \ldots, m-1$ with $t = 0, \ldots, \lfloor \frac{x-2}{2} \rfloor$.

In the one case where m is odd, $x = m-1$, and $t = \lfloor \frac{x-2}{2} \rfloor = \frac{m-3}{2}$, we proceed as follows.

Algorithm 3 Start with F_2 in (3.1).

Step 1: Replace $(n+\frac{m-3}{2}, n+\frac{m-3}{2}; m-1)$ by $(n+\frac{m-3}{2}, n+\frac{m-3}{2}; \frac{m-1}{2})$.

Step 2: Replace $(2n-1, n+\frac{m-3}{2}; \frac{m-1}{2})$ by $(2n-1, n+\frac{m-3}{2}; m-1)$ and $(n+\frac{m-3}{2}, 2n-1; \frac{m-1}{2})$ by $(n+\frac{m-3}{2}, 2n-1; m-1)$.

Step 3: Replace $(2n-1, m; m-1)$ by $(2n-1, m; \frac{m-1}{2})$ and $(m, 2n-1; m-1)$ by $(m, 2n-1; \frac{m-1}{2})$.

Step 4: Starting from $(n-\frac{m+1}{2}, m; \frac{m-1}{2})$, proceed similarly as in Step 2 of Algorithm 2, skipping $\frac{m-3}{2}$ cells. Stop at $(m+1, \frac{3m-1}{2}; \frac{m-1}{2})$ and exchange the entry $\frac{m-1}{2}$ with $m-1$.

Step 5: Replace $(m, \frac{3m+1}{2}; \frac{m-1}{2}), \ldots,$
$(m, n-2; m-2)$ by $(m, \frac{3m+1}{2}; \frac{m+1}{2}), \ldots, (m, n-2; m-1)$.

Step 6: Replace $(m+1, \frac{3m+1}{2}; \frac{m+1}{2}), \ldots, (m+1, n-2; m-1)$ by $(m+1, \frac{3m+1}{2}; \frac{m-1}{2}), \ldots, (m+1, n-2; m-2)$.

For all other x and t in the range under consideration, we proceed as described below.

Algorithm 4 Start with F_2 in (3.1).

Step 1: Replace $\{(n+x-t-2, n+t; x), (n+m-t-2, n+t; m), (2n-1, n+t; t+1), (2n-1, x+1; x)\}$ by $\{(n+x-t-2, n+t; m), (n+m-t-2, n+t; t+1), (2n-1, n+t; x), (2n-1, x+1; t+1)\}$.

Step 2: Starting in location $(2n-1, x+1)$, proceed as in Step 2 of Algorithm 2, skipping $x-t-2$ cells in each step vertically upwards and horizontally to the right until (i_0, j_0), where $i_0 + j_0 - 2n = x$ and $0 \leq i_0 - (n+m) < x-t-1$, switching entries x and $t+1$.

Step 3: Replace $\{(n+x-t-1, m+t-x+2; m+1), \ldots, (n+m-t-2, t+3; m+1)\}$ by $\{(n+x-t-1, m+t-x+2; m), \ldots, (n+m-t-2, t+3; m)\}$.

Step 4: Replace $\{(n+x-t-2, m+t-x+2; m), \ldots, (n+m-t-3, t+3; m)\}$ by $\{(n+x-t-2, m+t-x+2; m+1), \ldots, (n+m-t-3, t+3; m+1)\}$.

Step 5: Fill in the rest of row $n+x-t-2$ of I_3 except the last entry, analogously to Step 3 of Algorithm 2. The last entry will be $(n+x-t-2, n-1; x)$.

If $t+1$ originally appeared in one of the locations $(n+x-t-2, j_0)$, $(n+x-t-1, j_0), \ldots, (i_0-1, j_0)$,

Step 6A: As in Step 5 of Algorithm 2, adjust rows $n+x-t-1$ through $n+m-t-2$ from column $t+4$ to column j_0-1, then adjust columns j_0 through $n-1$ as necessary.

If $t+1$ is originally above location $n+x-2-t$ in column j_0,

Step 6B: Adjust rows $n+x-t-1$ from left to right until the column where $t+1$ would be needed in row $n+m-t-2$. Call this column j_1.

Temporarily change the entry in location (i_0, j_0) back to its original value and change the entry in location (i_0, j_1) to $t+1$. Now adjust columns j_1 through $n-1$ as necessary to complete the F-square.

We illustrate below the alternate completions demostrated above for some specific choices of n and $(i,\ j;\ x)$. The usual Roman fonts denote the entries in $F' \setminus \{(i,\ j;\ x))\}$, while the bold fonts denote the entries which are different from those in the corresponding cells of F_2 in (3.1) and the "." s are the usual entries of F_2 in (3.1).

Example 1 $n = 10, \quad (i,\ j;\ x) = (15,\ 13;\ 0).$

0	1	2	3	4	.	.	.	.	.	0	1	2	3	4	.	.	.	.	.
1	2	3	4	.	.	.	.	.	.	1	2	3	4	.	.	.	.	.	.
2	3	4	.	.	.	.	.	.	.	2	3	4	.	.	.	.	.	.	.
3	4	.	.	.	.	.	.	.	.	3	4	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.	4	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	**9**	.	.	.	.	5	.	.	.	**0**	.	.	.	.	.	5
.	.	.	.	.	.	.	.	5	6	.	.	.	.	.	.	.	.	5	6
.	.	.	.	.	.	.	5	6	7	.	.	.	.	.	.	.	5	6	7
.	.	.	.	.	.	5	6	7	8	.	.	.	.	.	.	5	6	7	8
0	1	2	3	4	.	.	.	.	.	2	3	4	.	6	7	8	9	0	.
1	2	3	4	.	.	.	.	.	.	3	4	.	6	7	8	9	0	1	.
2	3	4	.	.	.	.	.	.	.	4	.	6	7	8	9	0	1	2	.
3	4	.	.	.	.	.	.	.	.	.	6	7	8	9	0	1	2	3	.
4	.	.	.	.	.	.	.	.	.	6	7	8	9	0	1	2	3	4	5
.	.	.	.	**0**	.	.	.	.	.	7	8	9	**9**	1	2	3	4	5	6
.	.	.	.	.	.	.	.	.	5	8	9	0	1	2	3	4	5	6	7
.	.	.	.	.	.	.	.	5	6	9	0	1	2	3	4	5	6	7	8
.	.	.	.	.	.	.	5	6	7	0	1	2	3	4	5	6	7	8	9
.	.	.	.	.	.	5	6	7	8	.	.	.	.	5	6	7	8	9	0

Also observe that the case $n = 10$, $(i,\ j;\ x) = (13,\ 15;\ 0)$ can be addressed by taking the transpose of Example 1.

Example 2 $n = 11, (i, j; x) = (12, 11; 3)$

0	1	2	3	4	.	.	.	.	.	.	0	1	2	3	4	.	.	.	.	.	.
1	2	3	4	.	.	.	.	.	.	.	1	2	3	4	.	.	.	.	.	.	.
2	3	4	.	.	.	.	.	.	.	.	2	3	4	.	.	.	.	.	.	.	.
3	4	.	.	.	.	.	.	.	.	.	3	4	.	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.	.	4	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	5	.	.	.	.	.	.	.	.	.	.	5
.	.	.	.	.	.	.	.	.	5	6	.	.	.	.	.	.	.	.	.	5	6
.	.	.	.	.	.	.	.	5	6	7	.	.	.	.	.	.	.	.	5	6	7
.	.	.	.	.	.	.	5	6	7	8	.	.	.	.	.	.	.	5	6	7	8
.	.	.	.	.	.	5	6	7	8	9	.	.	.	.	.	.	5	6	7	8	9
0	1	2	3	4	.	.	.	.	.	.	2	3	4	.	6	7	8	9	10	0	.
1	2	3	4	**6**	**7**	**8**	**9**	**10**	**0**	**3**	**5**	4	.	6	7	8	9	10	0	1	.
2	3	4	**6**	**5**	**8**	**7**	**10**	**9**	**1**	**0**	4	.	6	7	8	9	10	0	1	2	.
3	4	.	**5**	**7**	**6**	.	**8**	.	**10**	.	**1**	6	7	8	9	10	0	1	2	3	.
4	.	.	.	.	.	.	.	**3**	.	**1**	6	7	8	9	10	0	1	2	3	4	5
.	.	.	.	.	.	.	.	.	.	.	7	8	9	10	0	1	2	3	4	5	6
.	.	.	.	.	.	**3**	.	**1**	.	5	8	9	10	0	1	2	3	4	5	6	7
.	.	.	.	.	.	.	.	.	5	6	9	10	0	1	2	3	4	5	6	7	8
.	.	.	.	**3**	.	**1**	.	5	6	7	10	0	1	2	3	4	5	**8**	7	8	9
.	.	.	.	.	.	.	5	6	7	8	0	1	2	3	4	5	6	7	8	9	10
.	.	.	.	**1**	.	5	6	7	8	9	**3**	.	.	.	5	6	7	8	9	10	0

Acknowledgement: We would like to thank the referee for his insightful comments.

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