SOME IDENTITIES OF BOOLE AND EULER POLYNOMIALS

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ABSTRACT. In this paper, we give a new and interesting identities of Boole and Euler polynomials which are derived from the symmetry properties of the p-adic fermionic integrals on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote, respectively, the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the p-adic fermionic integral on $\mathbb{Z}_p$ is defeind by Kim to be

$$(1) \qquad I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [5])}$$

Let $f_1(x) = f(x+1)$. Then, by (1), we get

$$(2) \qquad I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \quad \text{(see [7, 5, 10, 8, 9])}.$$

From (2), we can derive the following integral equation :

$$(3) \qquad I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l).$$

As is well known, the ordinary Euler polynomials are defined by the generating function to be

$$(4) \qquad \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

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with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1, 9, 12, 11, 13, 14, 17, 15, 16, 2, 3, 4, 6, 7, 5, 10, 8]).

When $x = 0$, $E_n = E_n(0)$ is called the n-th Euler number.

The Stirling number of the first kind is defined by

$$(5) \qquad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l$$

where $n \in \mathbb{N} \cup \{0\}$ (see [6, 7, 5]).

The Boole polynomials are defined by the generating function to be

$$(6) \qquad \sum_{n=0}^{\infty} Bl_n(x|\lambda) \frac{t^n}{n!} = \frac{1}{1+(1+t)^{\lambda}} (1+t)^x,$$

(see [5, 14]).

When $\lambda = 1$, $2Bl_n(x|1) = Ch_n(x)$ are the Changhee polynomials which are defined by

$$\sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} = \frac{2}{t+2}(1+t)^x, \quad \text{(see [7, 5])}.$$

Let us take $f(x) = e^{tx}$. Then, by (2), we get

$$(7) \qquad \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

From (7), we have

$$(8) \qquad \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Thus, by comparing the coefficients on the both sides of (8), we get

$$(9) \qquad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \text{where } n \in \mathbb{N} \cup \{0\}.$$

The purpose of this paper is to give identities of symmetry for the Boole and Euler polynomials which are derived from the symmetric properties of the p-adic fermionic integrals on $\mathbb{Z}_p$.

2. Identities of Symmetry for Boole and Euler Polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, For $\lambda \in \mathbb{Z}_p$, let us take $f(x) = (1+t)^{\lambda x}$. Then, by (2), we get

$$\int_{\mathbb{Z}_p} (1+t)^{\lambda x} d\mu_{-1}(x) = \frac{2}{1+(1+t)^{\lambda}} = 2\sum_{n=0}^{\infty} Bl_n(\lambda) \frac{t^n}{n!}, \tag{10}$$

where $Bl_n(0|\lambda) = Bl_n(\lambda)$ are called the Boole numbers.

By (10), we easily get

$$\int_{\mathbb{Z}_p} (1+t)^{x+\lambda y} d\mu_{-1}(y) = \frac{2}{1+(1+t)^{\lambda}} (1+t)^{x}. \tag{11}$$

By (6) and (11), we get

$$\int_{\mathbb{Z}_p} (x+\lambda y)_n d\mu_{-1}(y) = 2Bl_n(x|\lambda), \quad (n \in \mathbb{Z}_{\geq 0}),$$

and

$$\sum_{n=0}^{\infty} 2Bl_n(x|\lambda) \frac{(e^t-1)^n}{n!} = \frac{2}{e^{\lambda t}+1} e^{xt} = \sum_{n=0}^{\infty} E_n\left(\frac{x}{\lambda}\right) \lambda^n \frac{t^n}{n!}. \tag{12}$$

The Stirling number of the second kind is defined by the generating function to be

$$(e^t-1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}, \quad \text{(see [7, 5, 14])}. \tag{13}$$

By (12) and (13), we get

$$\sum_{n=0}^{m} Bl_n(\lambda|x) S_2(m,n) = \frac{1}{2} E_m\left(\frac{x}{\lambda}\right) \lambda^m, \tag{14}$$

where $m \in \mathbb{Z}_{\geq 0}$.

Let $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1$ (mod 2). Then, by (1), we see that

$$\begin{aligned}(15)\qquad &\int_{\mathbb{Z}_p} (1+t)^{w_1w_2x+w_2j+w_1y}\, d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (1+t)^{w_1w_2x+w_2j+w_1y} (-1)^y \\ &= \lim_{N\to\infty} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} (1+t)^{w_1w_2x+w_2j+w_1(i+w_2y)} (-1)^{i+w_2y}.\end{aligned}$$

From (15), we have

$$\begin{aligned}(16)\qquad &\sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1w_2x+w_2j+w_1y}\, d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} (-1)^{i+j+y} (1+t)^{w_1w_2(x+y)+w_2j+w_1i}.\end{aligned}$$

By the same method as (16), we get

$$\begin{aligned}(17)\qquad &\sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1w_2x+w_1j+w_2y}\, d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{p^N-1} (-1)^{i+j+y} (1+t)^{w_1w_2(x+y)+w_1j+w_2i}.\end{aligned}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, we have*

$$\begin{aligned}&\sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1w_2x+w_2j+w_1y}\, d\mu_{-1}(y) \\ &= \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1w_2x+w_1j+w_2y}\, d\mu_{-1}(y).\end{aligned}$$

Corollary 2.2. *For $n \geq 0$, $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, we have*

$$\sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} (w_1 w_2 x + w_2 j + w_1 y)_n \, d\mu_{-1}(y)$$
$$= \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (w_1 w_2 x + w_1 j + w_2 y)_n \, d\mu_{-1}(y).$$

Now, we observe that

$$(18) \quad \int_{\mathbb{Z}_p} (1+t)^{w_1 w_2 x + w_1 j + w_2 y} \, d\mu_{-1}(y)$$
$$= \frac{2}{1+(1+t)^{w_2}} (1+t)^{w_1 w_2 x + w_1 j}$$
$$= \sum_{n=0}^{\infty} 2 Bl_n (w_1 w_2 x + w_1 j | \, w_2) \frac{t^n}{n!}.$$

Thus, by (18), we get

$$(19) \quad \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1 w_2 x + w_1 j + w_2 y} \, d\mu_{-1}(y)$$
$$= \sum_{n=0}^{\infty} \left(2 \sum_{j=0}^{w_2-1} (-1)^j Bl_n (w_1 w_2 x + w_1 j | \, w_2) \right) \frac{t^n}{n!},$$

and

$$(20) \quad \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (1+t)^{w_1 w_2 x + w_1 j + w_2 y} \, d\mu_{-1}(y)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (w_1 w_2 x + w_1 j + w_2 y)_n \, d\mu_{-1}(y) \right) \frac{t^n}{n!}.$$

From (19) and (20), we have

$$(21) \quad 2 \sum_{j=0}^{w_2-1} (-1)^j Bl_n (w_1 w_2 x + w_1 j | \, w_2)$$
$$= \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} (w_1 w_2 x + w_1 j + w_2 y)_n \, d\mu_{-1}(y).$$

By the same method as (21), we get

$$2\sum_{j=0}^{w_1-1}(-1)^j Bl_n(w_1w_2x+w_2j|\,w_1) = \sum_{j=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_n\,d\mu_{-1}(y). \tag{22}$$

Therefore, by Corollary 2.2, (21) and (22), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1$ (mod 2), $w_2 \equiv 1$(mod 2), $w_3 \equiv 1$ (mod 2), we have*

$$\sum_{j=0}^{w_2-1}(-1)^j Bl_n(w_1w_2x+w_1j|\,w_2) = \sum_{j=0}^{w_1-1}(-1)^j Bl_n(w_1w_2x+w_2j|\,w_1).$$

Now, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_n\,d\mu_{-1}(y) &= \sum_{i=0}^{n} S_1(n,i)\int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)^i\,d\mu_{-1}(y)\\ &= \sum_{i=0}^{n} S_1(n,i)\,w_1^i\int_{\mathbb{Z}_p}\left(w_2x+\frac{w_2}{w_1}j+y\right)^i d\mu_{-1}(y)\\ &= \sum_{i=0}^{n} S_1(n,i)\,w_1^i E_i\left(w_2x+\frac{w_2}{w_1}j\right).\end{aligned} \tag{23}$$

Thus, by (23), we get

$$\sum_{j=0}^{w_1-1}(-1)^j\int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_n\,d\mu_{-1}(y) = \sum_{i=0}^{n} S_1(n,i)\,w_1^i\sum_{j=0}^{w_1-1}(-1)^j E_i\left(w_2x+\frac{w_2}{w_1}j\right). \tag{24}$$

By the same method as (24), we get

$$\sum_{j=0}^{w_2-1}(-1)^j\int_{\mathbb{Z}_p}(w_1w_2x+w_1j+w_2y)_n\,d\mu_{-1}(y)$$
$$=\sum_{i=0}^{n}S_1(n,i)\,w_2^i\sum_{j=0}^{w_2-1}(-1)^j E_i\left(w_1x+\frac{w_1}{w_2}j\right).$$

Therefore, by Corollary 2.2, (23) and (24), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, we have*

$$\sum_{i=0}^{n}S_1(n,i)\,w_1^i\sum_{j=0}^{w_1-1}(-1)^j E_i\left(w_2x+\frac{w_2}{w_1}j\right)$$
$$=\sum_{i=0}^{n}S_1(n,i)\,w_2^i\sum_{j=0}^{w_2-1}(-1)^j E_i\left(w_1x+\frac{w_1}{w_2}j\right).$$

Remark. When $\lambda = 1$, we note that $2Bl_n(x|\,1) = Ch_n(x)$. By Theorem 2.3, we get

$$Ch_n(w_2x)=\sum_{j=0}^{w_2-1}(-1)^j Bl_n(w_2x+j|\,w_2),$$

where $Ch_n(x)$ are the Changhee polynomials which are defined by

$$\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}=\frac{2}{t+2}(1+t)^x, \quad \text{(see [7])}.$$

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