A CHARACTERIZATION OF GRAPHIC MATROIDS WHICH YIELD BIOGRAPHIC SPLITTINGS MATROIDS

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Abstract
In this paper, we consider the problem of determining precisely which graphic matroids M have the property that the splitting operation, by every pair of elements, on M yields a cographic matroid. This problem is solved by proving that there are exactly three minor-minimal graphs that do not have this property.

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1. Introduction

The matroid notations and terminology used here will follow Oxley [6]. Fleischner [3] introduced the idea of splitting a vertex of degree at least three in a connected graph and used the operation to characterize Eulerian graphs. Figure 1 shows the graph $G_{x,y}$ that is obtained from G by splitting away the edges x and y from the vertex v.

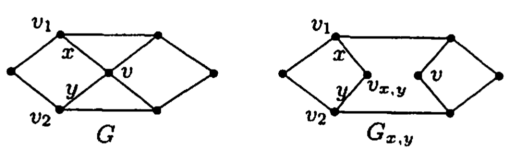


Figure 1

Fleischner [3] characterized Eulerian graphs and also developed an algorithm to find all distinct Eulerian trails in an Eulerian graph using the splitting operation. Tutte [12] characterized 3-connected graphs, and Slater [11] classified 4-connected graphs using a slight variation of this operation.

Raghunathan et al. [7] extended the notion of splitting operation from graphs to binary matroids as follows:

Definition 1.1. Let $M = M[A]$ be a binary matroid and suppose $x, y \in E(M)$. Let $A_{x,y}$ be the matrix obtained from A by adjoining the row that is zero everywhere except for entries of 1 in the columns labeled by x and y. The splitting matroid $M_{x,y}$ is defined to be the vector matroid of the matrix $A_{x,y}$. The transition from M to $M_{x,y}$ is called a splitting operation.

Alternatively, the splitting operation can be defined in terms of circuits of binary matroids as follows:

Lemma 1.2 [7]. *Let $M = (E, \mathcal{C})$ be a binary matroid on a set E together with the set $\mathcal{C}$ of circuits and let $\{x, y\} \subseteq E$. Then $M_{x,y} = (E, \mathcal{C}')$ with $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$ where*
$\mathcal{C}_0 = \{C \in \mathcal{C} \ : \ x, y \in C \text{ or } x \notin C, \ y \notin C\}$; and
$\mathcal{C}_1 = \{C_1 \cup C_2 \ : \ C_1, C_2 \in \mathcal{C}, \ x \in C_1, \ y \in C_2, \ C_1 \cap C_2 = \phi$ and $C_1 \cup C_2$ contains no member of $\mathcal{C}_0\}$.

Note that the elements x and y are in series in $M_{x,y}$. Various properties of the splitting matroids have been explored in [1, 2, 5, 7, 8, 9, 10].

The splitting operation on a graphic matroid may not yield a graphic matroid. Shikare and Waphare [8] characterized graphic matroids whose splitting matroids for every pair of element are also graphic. They proved the following theorem.

Theorem 1.3 [8]. *The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to the cycle matroid of any of the following four graphs.*

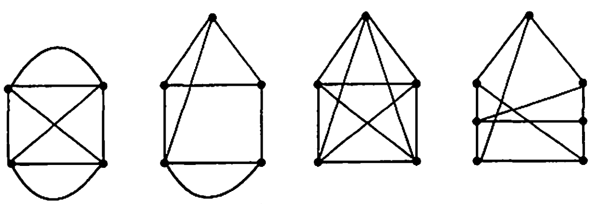


Figure 2

Cographicness of a matroid is not preserved under the splitting operation. Borse, Shikare and Dalvi [2] obtained the following result in this regard.

Theorem 1.4 [2]. *The splitting operation, by any pair of elements, on a* *cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to the cycle matroid of any of the following two graphs.*

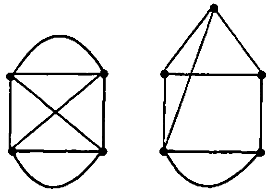


Figure 3

Let $M(G)$ denotes the cycle matroid of a graph G. If M is a graphic matroid containing a minor isomorphic to $M(K_5)$ or $M(K_{3,3})$ and there is a pair x, y of elements of M such that $M = M_{x,y}$, then $M_{x,y}$ is not cographic. In the light of Lemma 2.1(ii), we say that a pair x, y of elements of M is *non-trivial* if none of x and y is a loop or a coloop, and $M \neq M_{x,y}$. In this paper, we characterize those graphic matroids M for which $M_{x,y}$ is cographic for every non-trivial pair $x, y \in E(M)$. The following is the main theorem of the paper.

Theorem 1.5. *The splitting operation, by any non-trivial pair of elements, on a graphic matroid yields a cographic matroid if and only if it has no minor isomorphic to the cycle matroid of any of the following three graphs.*

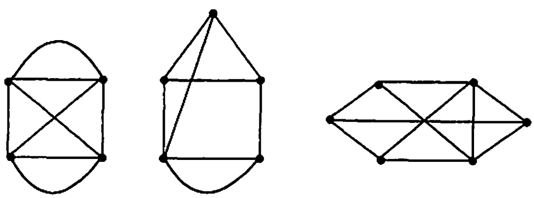


Figure 4

2. The splitting operation and minors

In this section, we provide necessary lemmas which are useful in the proof of Theorem 1.5.

Lemma 2.1 [8]. *Let x and y be elements of a binary matroid M and let $r(M)$ denote the rank of M. Then the following statements hold:*
(i) if x and y are not coloops in M, then they are in series in $M_{x,y}$;

(ii) $M_{x,y} = M$ *if and only if* x *and* y *are in series or both* x *and* y *are coloops in* M;
(iii) $r(M_{x,y}) = r(M) + 1$ *if* $\{x, y\}$ *is a non-trivial pair;*
(iv) if x_1, x_2 *are in series in* M, *then they are in series in* $M_{x,y}$.

Theorem 2.2 [6]. *A binary matroid is cographic if and only if it has no minor isomorphic to* F_7, F_7^*, $M(K_5)$ *or* $M(K_{3,3})$.

Notation. For the sake of convenience, let $\mathcal{F} = \{F_7, F_7^*, M(K_5), M(K_{3,3})\}$.

Lemma 2.3. *Let* M *be a graphic matroid and let* $x, y \in E(M)$ *be a non-trivial pair such that* $M_{x,y}$ *is not cographic. Then there is a loopless and coloopless minor* N *of* M *such that no two elements of* N *are in series and* $N_{x,y}/\{x\} \cong F$ *or* $N_{x,y}/\{x, y\} \cong F$ *for some* $F \in \mathcal{F}$.

Proof. The proof is similar to the proof of Theorem 2.3 of [8]. □

Definition 2.4. Let M be a loopless and coloopless graphic matroid in which no two elements are in series and let $F \in \mathcal{F}$. We say that M is *minimal* with respect to F if there exist two elements x and y of M such that $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x, y\} \cong F$.

Corollary 2.5. Let M be a graphic matroid. For any non-trivial pair $\{x, y\}$ of elements of M, the matroid $M_{x,y}$ is cographic if and only if M has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.

Lemma 2.6. *Let* $F \in \mathcal{F}$ *and let* M *be a graphic matroid such that either* $M_{x,y}/\{x\} \cong F$ *or* $M_{x,y}/\{x, y\} \cong F$ *for some non-trivial pair* $x, y \in E(M)$. *Then the following statements hold:*
(i) M *has neither loops nor coloops;*
(ii) x *and* y *cannot be parallel in* M;
(iii) if $x_1, x_2 \in E(M)$ *are parallel in* M, *then one of them is either* x *or* y;
(iv) if $M_{x,y}/\{x\} \cong F$, *then* M *has at most two pairs of parallel elements;*
(v) if $M_{x,y}/\{x\} \cong M(K_{3,3})$ *or* $M_{x,y}/\{x, y\} \cong M(K_{3,3})$, *then every odd circuit of* M *contains* x *or* y; *and*
(vi) if $M_{x,y}/\{x\} \cong M(K_5)$ *or* $M_{x,y}/\{x, y\} \cong M(K_5)$, *then every odd cocircuit of* M *contains* x *or* y.

Proof. The proof follows from Lemma 2.1 and from the fact that F does not contain loops, coloops and 2-circuits. □

A matroid is said to be *Eulerian* if its ground set can be expressed as a union of disjoint circuits [13].

Lemma 2.7 [8]. *If x, y are non-adjacent edges of a graph G such that $M(G)_{x,y}/\{x,y\}$ is Eulerian, then either G is Eulerian or the end vertices of x and y are precisely of odd degree.*

3. Cographic splitting matroids

In this section, we obtain the minimal matroids corresponding to the four matroids F_7, F_7^*, $M(K_{3,3})$ and $M(K_5)$, and use them to give a proof of Theorem 1.5.

Lemma 3.1 [2]. *Let M be a graphic matroid. Then M is minimal with respect to the matroid F_7 or F_7^* if and only if M is isomorphic to one of the matroids $M(G_1)$, $M(G_2)$ and $M(G_3)$ where G_1, G_2 and G_3 are the graphs of Figure 5.*

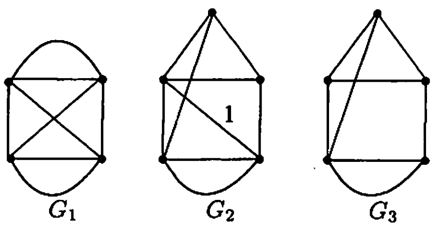


Figure 5

In the following lemma, we characterize the minimal matroids corresponding to the matroid $M(K_{3,3})$.

Lemma 3.2. *Let M be a graphic matroid. Then M is minimal with respect to the matroid $M(K_{3,3})$ if and only if M is isomorphic to one of the cycle matroids $M(G_4), M(G_5), M(G_6)$ and $M(G_7)$ where G_4, G_5, G_6 and G_7 are the graphs of Figure 6.*

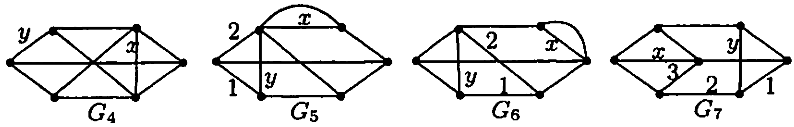


Figure 6

Proof. It follows from the matrix representations that $M(G_4)_{x,y}/\{x\} \cong M(K_{3,3}), M(G_5)_{x,y}/\{x\} \cong M(K_{3,3}), M(G_6)_{x,y}/\{x\} \cong M(K_{3,3}), M(G_7)_{x,y}/\{x, y\} \cong M(K_{3,3})$. Therefore $M(G_4), M(G_5), M(G_6)$ and $M(G_7)$ are minimal with respect to the matroid $M(K_{3,3})$.

Conversely, suppose M is a minimal matroid with respect to $M(K_{3,3})$. Then there exist elements x and y of M such that $M_{x,y}/\{x\} \cong M(K_{3,3})$ or $M_{x,y}/\{x, y\} \cong M(K_{3,3})$.

Case (i). $M_{x,y}/\{x\} \cong M(K_{3,3})$.

Since $r(M_{x,y}/\{x\}) = r(M(K_{3,3})) = 5$, $M_{x,y}$ is a matroid of rank 6 and $|E(M)| = 10$. In the light of Lemma 2.1(iii), the matroid M has rank 5 and its ground set has 10 elements. Let G be a connected graph corresponding to M. Then G has 6 vertices, 10 edges and has no vertex of degree 2. Hence, by Lemma 2.6, G has minimum degree at least 3. Thus the degree sequence of G is $(5, 3, 3, 3, 3, 3)$ or $(4, 4, 3, 3, 3, 3)$. Suppose G is a simple graph. By [4, pp. 223], each simple connected graph with these degree sequences is isomorphic to one of the graphs of Figure 7.

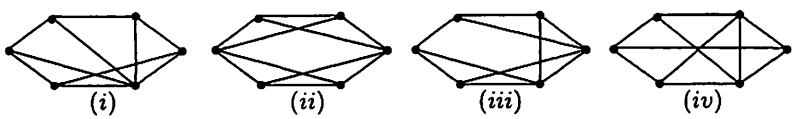


Figure 7

By the nature of circuits of $M(K_{3,3})$ or $M_{x,y}$ and by Lemma 2.6, it follows that G can not have

(i) two or more edge disjoint triangles; and

(ii) a circuit of size 3, 4 or 6 containing both x and y.

Since each of the graphs *(i)*, *(ii)* and *(iii)* of Figure 7 contains two or more edge disjoint triangles, we discard them. Therefore the graph G is isomorphic to the graph *(iv)* of Figure 7 which is nothing but the graph G_4

as stated in the lemma.

Suppose G is not a simple graph. Then, by of Case (i) of [2, Lemma 3.3], G is isomorphic to the graph G_5 or G_6 of Figure 6.

Case (ii). $M_{x,y}/\{x,y\} \cong M(K_{3,3})$.

As $r(M(K_{3,3})) = 5$, $r(M_{x,y}) = 7$. Hence $r(M) = 6$ and $|E(M)| = 11$. Let G be a connected graph corresponding to M. Then G has 7 vertices, 11 edges and has minimum degree at least 3. Therefore the degree sequence of G is $(4,3,3,3,3,3,3)$. It follows from Lemma 2.6 that G cannot have

(i) more than two edge disjoint triangles;

(ii) a cycle of size other than 6 which contains both x and y;

(iii) a triangle and a 2-circuit which are edge disjoint.

Then, by Case (ii) of [2, Lemma 3.3], G is isomorphic to G_7 of Figure 6.

□

Finally, we characterize minimal matroids corresponding to the matroid $M(K_5)$ in the following lemma.

Lemma 3.3. *Let M be a graphic matroid. Then M is minimal with respect to the matroid $M(K_5)$ if and only if M is isomorphic to one of the matroids $M(G_8), M(G_9), M(G_{10}), M(G_{11})$ and $M(G_{12})$ where G_8, G_9, G_{10}, G_{11} and G_{12} are the graphs of Figure 8.*

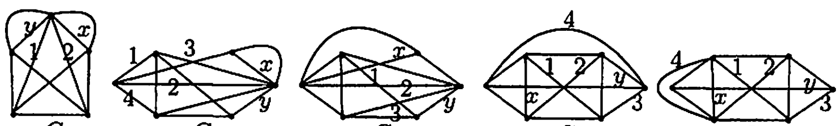


Figure 8

Proof. One can check that $M(G_8)_{x,y}/\{x\} \cong M(K_5), M(G_9)_{x,y}/\{x,y\} \cong M(K_5), M(G_{10})_{x,y}/\{x,y\} \cong M(K_5), M(G_{11})_{x,y}/\{x,y\} \cong M(K_5)$, $M(G_{12})_{x,y}/\{x,y\} \cong M(K_5)$. Therefore $M(G_8), M(G_9), M(G_{10}), M(G_{11})$ and $M(G_{12})$ are minimal matroids with respect to the matroid $M(K_5)$.

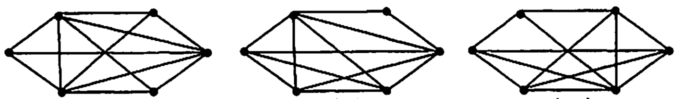
Conversely, suppose M is a minimal matroid with respect to the matroid $M(K_5)$, and let x and y be the elements of M with the property that $M_{x,y}/\{x\} \cong M(K_5)$ or $M_{x,y}/\{x,y\} \cong M(K_5)$.

Case (i). $M_{x,y}/\{x\} \cong M(K_5)$.

Then, by Case (i) of [2, Lemma 3.4], G is isomorphic to the graph G_8 of Figure 8.

Case (ii). $M_{x,y}/\{x,y\} \cong M(K_5)$.

Then $r(M(K_5)) = 4, r(M_{x,y}) = 6$ and $|E(M)| = 12$. Let G be a connected graph corresponding to M. Then G has 6 vertices, 12 edges and has minimum degree at least 3. By [4, pp. 224], there are five non-isomorphic simple graphs each with 6 vertices and 12 edges of which two graphs are planar and hence are discarded as in Case (ii) of [2, Lemma 3.4]. The remaining three non-planar graphs are as shown in Figure 9 below.

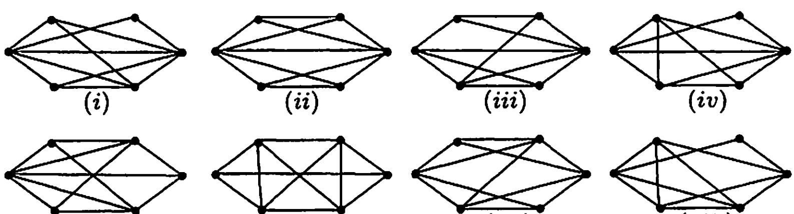


(i) *(ii)* *(iii)*

Figure 9

Suppose G is one of these three graphs. In graph *(i)* of Figure 9, each odd cocircuit of M doesn't contain x or y, a contradiction to Lemma 2.6(vi). In each of the graphs *(ii)* and *(iii)* of Figure 9, x and y together belong to a 3-circuit or a 4-circuit, a contradiction to Lemma 2.6(iv) and (vi).

Hence G is not a simple graph. By Lemma 2.6(iv), G has exactly one pair of parallel edges. Then G can be obtained from a simple graph on 6 vertices and 11 edges by adding an edge in parallel. There are eight non-isomorphic connected simple graphs each with 6 vertices and 11 edges as shown in Figure 10 (*see* [4, pp. 223]).



(i) *(ii)* *(iii)* *(iv)* *(v)* *(vi)* *(vii)* *(viii)*

Figure 10

It follows by Lemma 2.6(i), (iv) and Lemma 2.7 that G cannot be obtained from graphs *(ii)*, *(iii)* and *(vii)* of Figure 10. Suppose G is obtained from graph *(i)* or *(iv)*. Then G is isomorphic to one of the four graphs of

Figure 11. By Lemma 2.6(ii), (iii) and (vi), G is not isomorphic to each of the two graphs *(i)* and *(ii)* of Figure 11. Hence G is isomorphic to graphs *(iii)* and *(iv)* of Figure 11 which are nothing but the graphs G_9 and G_{10} of Figure 8.

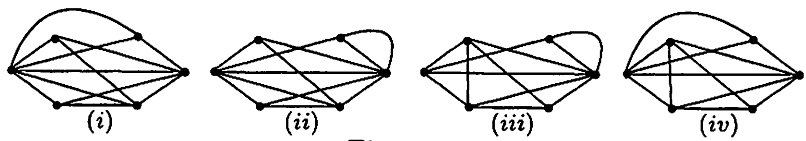


Figure 11

By Lemma 2.6(ii), (iii) and (vi), G cannot be obtained from graph *(v)* of Figure 10. Suppose G is obtained from graph *(viii)* of Figure 10. Then G is isomorphic to one of the two graphs of Figure 12. By Lemma 2.6(iv) and Lemma 2.7, G is not isomorphic to graph *(i)* of Figure 12. By Lemma 2.6(ii) and (iv) and the fact that $M(K_5)$ does not contain odd cocircuit, G cannot be isomorphic to the graph *(ii)* of Figure 12.

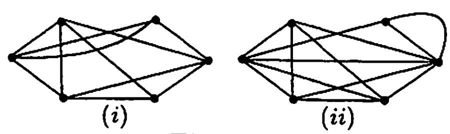


Figure 12

Suppose G is obtained from graph *(vi)* of Figure 10. Then G is isomorphic to one of the two graphs G_{11} and G_{12} of Figure 8.

Now, we use Lemmas 3.1, 3.2 and 3.3 to prove Theorem 1.5.

Proof of Theorem 1.5. Let M be a graphic matroid. On combining Corollary 2.5 and Lemmas 3.1, 3.2 and 3.3, it follows that $M_{x,y}$ is cographic for every non-trivial pair $\{x, y\}$ of elements of M if and only if M has no minor isomorphic to any of the matroids $M(G_i)$, $i = 1, 2, \ldots, 12$ where the graphs G_i's are as shown in the statement of Lemmas 3.1, 3.2 and 3.3. However, we have $M(G_3) \cong M(G_2) \setminus \{1\} \cong M(G_5)/\{1\} \setminus \{2\} \cong M(G_6)/\{1\}\setminus\{2\} \cong M(G_7)/\{1,2\}\setminus\{3\} \cong M(G_8)\setminus\{y,1,2\} \cong M(G_9)/\{1\}\setminus \{2,3,4\} \cong M(G_{10})/\{y\}\setminus\{1,2,3\} \cong M(G_{11})/\{3\}\setminus\{1,2,4\} \cong M(G_{12})/\{3\}\setminus \{1,2,4\}$. Thus $M_{x.y}$ is cographic if and only if M has no minor isomorphic to any of the matroids $M(G_i), i = 1,3,4$. But the graphs G_1, G_3 and G_4 are precisely the graphs given in the statement of the theorem. This completes the proof of the theorem. □

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