Spiders are antimagic

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Abstract

For a graph G, an edge labeling of G is a bijection $f : E(G) \to \{1, 2, \cdots, |E(G)|\}$. The induced vertex sum f^+ of f is a function defined on $V(G)$ given by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. And G is called antimagic if there exists an edge labeling of G such that the induced vertex sum of the edge labeling is injective. Hartsfield and Ringel conjectured in 1990 that all connected graphs except K_2 are antimagic. A spider is a connected graph of which one and only one vertex has degree exceeding 2. This paper shows that all spiders are antimagic.

1 Introduction

Let G be a finite simple graph without any isolated vertex. An *edge labeling* of G is a bijection $f : E(G) \to \{1, 2, \cdots, |E(G)|\}$ and the induced *vertex sum* f^+ of f is a function defined on $V(G)$ given by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. And $f^+(u)$ is called the *vertex sum* of u. The graph G is called *antimagic* if there exists an edge labeling of G such that the induced vertex sum of the edge labeling is injective. Hartsfield and Ringel [3] introduced antimagic graphs in 1990. They showed that paths, complete graphs (except K_2), cycles, and wheels are antimagic, and conjectured that all connected graphs except K_2 are antimagic. This conjecture is far from completely solved. Alon et al. [1] validated this conjecture for graphs having n vertices and minimum degree $\Omega(\log n)$. They also proved that a complete partite graph besides K_2 and a graph of order $n \geq 4$ with maximum degree $\geq n - 2$ are antimagic. Wang and Hsiao [5] constructed antimagic graphs through Cartesian products and lexicographic products. Let P_n denote a path of order n. They showed that $P_m \times P_n$ (the Cartesian product of P_m and P_n, $m \geq n \geq 2$) and $H \times P_n$ ($n \geq 2$), where H is a d-regular graph for some $d \geq 1$, are antimagic. They also proved that if

F is an antimagic k-regular graph, $k \geq 2$, then the lexicographic product $U[F]$ is antimagic for any graph U. Lee, Lin and Tsai [4] proved that a power of cycle C_n^2 is antimagic and the vertex sums of all vertices form a set of successive integers when n is odd. Gallian [2] comprehensively introduced many types of graph labelings and their open problems, which include the antimagic labeling. A *caterpillar* is a connected graph of order at least three which contains a path such that each vertex not on the path is adjacent to a vertex on the path. A *spider* is a connected graph of which one and only one vertex has degree exceeding 2. Till now, it is not known if all caterpillars are antimagic. The main result of this paper is that all spiders are antimagic.

2 Main result

As mentioned in section 1, a spider is a connected graph of which one and only one vertex has degree exceeding 2. The vertex with degree exceeding 2 is called the *body* of the spider. Let S be a spider and x the body of S. Then each component of $S - x$ is called a *leg* of S. Obviously each leg is a path. Suppose a spider has k legs and these legs have l_i vertices for $i = 1, 2, \cdots, k$, where $l_1 \leq l_2 \leq \cdots \leq l_k$. Then this spider is denoted by $SP(l_1, l_2, \cdots, l_k)$. For $j = 1, 2, \cdots, k$, let L_j be the leg with l_j vertices and b the body of $SP(l_1, l_2, \cdots, l_k)$. For $i = 1, 2, \cdots, l_j$, let $v_{i,j}$ denote the vertex on L_j with $d(v_{i,j}, b) = l_j - i + 1$, where $d(v_{i,j}, b)$ is the distance between $v_{i,j}$ and b. For each j where $1 \leq j \leq k$ let $e_{i,j}$ denote the edge $v_{i,j}v_{i+1,j}$ for $i = 1, 2, \cdots, l_j - 1$ and $e_{l_j,j}$ the edge $v_{l_j,j}b$. The following figure shows the notations of vertices and edges on $SP(1, 3, 3, 5)$.

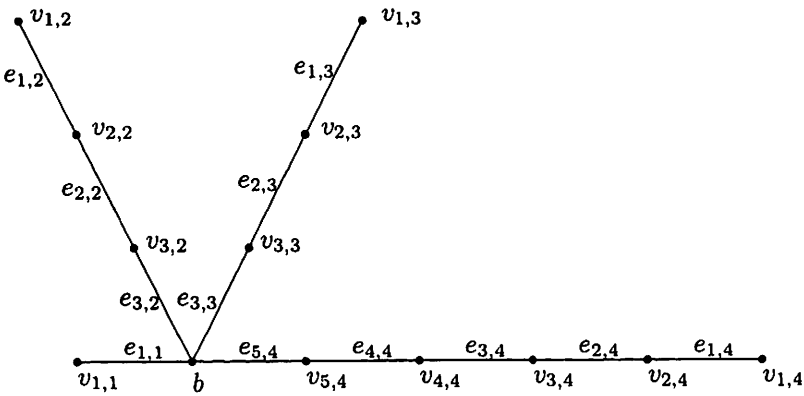

Fig. 1

Let E denote the edge set of $SP(l_1, l_2, \cdots, l_k)$. We now define an order on E as follows. For $e_{i,j}, e_{i',j'} \in E$, we have $e_{i,j} \prec e_{i',j'}$ if and only if $i < i'$, or $i = i'$ and $j < j'$. We can see that $\prec$ is a linear order on E. For example, in $SP(1,3,3,5)$, we have $e_{1,1} \prec e_{1,2} \prec e_{1,3} \prec e_{1,4} \prec e_{2,2} \prec e_{2,3} \prec e_{2,4} \prec e_{3,2} \prec e_{3,3} \prec e_{3,4} \prec e_{4,4} \prec e_{5,4}$.

Now define a function f_o on E by $f_o(e_{i,j}) = n$ if $e_{i,j}$ is the n-th edge under the linear order $\prec$. Obviously, f_o is an edge labeling of $SP(l_1, l_2, \cdots, l_k)$. As exhibited in Fig. 2, for $SP(1,3,3,5)$, we have $f_o(e_{1,1}) = 1$, $f_o(e_{1,2}) = 2$, $f_o(e_{1,3}) = 3$, $f_o(e_{1,4}) = 4$, $f_o(e_{2,2}) = 5$, $f_o(e_{2,3}) = 6$, $f_o(e_{2,4}) = 7, \cdots$, $f_o(e_{5,4}) = 12$.

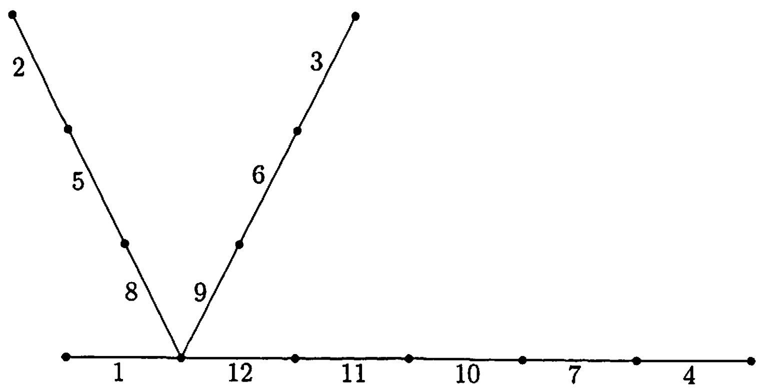

Fig. 2

The following lemma is trivial. We omit the proof.

Lemma 2.1 *Suppose $SP(l_1, l_2, \cdots, l_k)$ is a spider with the edge set E. Let $\prec$ be the linear order on E and f_o the edge labeling of $SP(l_1, l_2, \cdots, l_k)$ defined above. Then $f_o^+(v_{i,j}) < f_o^+(v_{i',j'})$ whenever $e_{i,j} \prec e_{i',j'}$; and hence all the vertex sums are distinct except that of the body.* □

Theorem 2.2 *Spiders are antimagic.*

Proof. Let $S = SP(l_1, l_2, \cdots, l_k)$ be a spider with the edge set E and the body b. Suppose $|E| = e$. If each $l_i = 1$ for $i = 1, 2, \cdots, k$ $(k \geq 3)$, then S is a star. Since a star (except K_2) is trivially antimagic, we may assume $l_k \geq 2$. Now distinguish two cases $l_k = l_{k-1}$ and $l_k \geq l_{k-1} + 1$.

Case 1: $l_k = l_{k-1}$. By the definition of f_o we have $f_o(e_{l_{k-1},k}) <$

$f_o(e_{l_{k-1},k-1})$ for $e_{l_k-1,k} \prec e_{l_{k-1},k-1}$. Then

$$\begin{aligned} f_o^+(b) &= \sum_{j=1}^{k} f_o(e_{l_j,j}) \\ &> f_o(e_{l_{k-1},k-1}) + f_o(e_{l_k,k}) \\ &> f_o(e_{l_k-1,k}) + f_o(e_{l_k,k}) \\ &= f_o^+(v_{l_k,k}), \end{aligned}$$

where $k \geq 3$ and $l_k \geq 2$. That is, $f_o^+(b) > f_o^+(v_{l_k,k})$. By Lemma 2.1, all $f_o^+(v_{i,j})$ are distinct, and $f_o^+(v_{l_k,k})$ is the maximum of all $f_o^+(v_{i,j})$, since $e_{i,j} \prec e_{l_k,k}$ for all $e_{i,j} \neq e_{l_k,k}$. We have f_o^+ is injective.

Case 2: $l_k \geq l_{k-1}+1$. We see that $e_{l_{k-1},k-1} \prec e_{l_{k-1},k} \prec e_{l_{k-1}+1,k} \prec e_{l_{k-1}+2,k} \prec \cdots \prec e_{l_k,k}$ and $e_{i,j} \prec e_{l_{k-1},k-1}$ for all other i,j. Now suppose $f_o(e_{l_{k-1},k-1}) = p$. Then $f_o(e_{l_{k-1},k}) = p+1$, $f_o(e_{l_{k-1}+1,k}) = p+2$, $f_o(e_{l_{k-1}+2,k}) = p+3, \cdots, f_o(e_{l_k,k}) = e$. See Fig. 3.

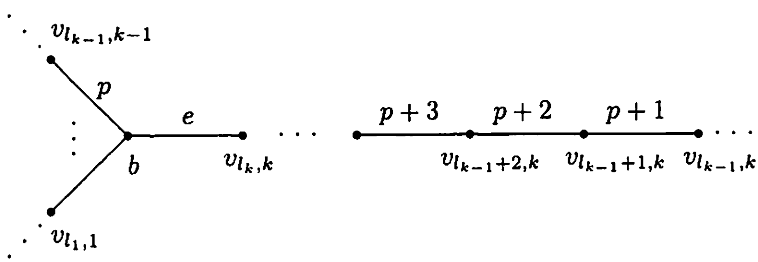

Fig. 3

Note that $e \geq p+2$ for $l_k \geq l_{k-1}+1$. By Lemma 2.1 $f_o^+(v_{i,j}) < f_o^+(v_{l_{k-1}+1,k})$ for all $v_{i,j} \in V(S) - \{b, v_{l_{k-1}+1,k}, v_{l_{k-1}+2,k}, \cdots, v_{l_k,k}\}$. Now $f_o^+(v_{l_{k-1}+1,k}) = 2p+3$ and $f_o^+(b) \geq f_o(e_{l_1,1}) + f_o(e_{l_{k-1},k-1}) + f_o(e_{l_k,k}) \geq 2p+3$. Hence if $f_o^+(b) = f_o^+(v_{i,j})$ for some $v_{i,j}$, then $v_{i,j} \in \{v_{l_{k-1}+1,k}, v_{l_{k-1}+2,k}, \cdots, v_{l_k,k}\}$. Distinguish two cases for discussion.

Subcase 2.1: $f_o^+(b) = f_o^+(v_{l_{k-1}+1,k})$. That is, $f_o^+(b) = 2p+3$. It is easily seen that $S = SP(1, l_2, l_2+1)$. Now $f_o(e_{l_2,2}) = p$, $f_o(e_{l_2,3}) = p+1$, $f_o(e_{l_2+1,3}) = p+2$. By Lemma 2.1, all $f_o^+(v_{i,j})$ are distinct, and $f_o^+(v_{i,j}) < f_o^+(v_{l_2,3})$ for $v_{i,j} \in V(S) - \{b, v_{l_2,3}, v_{l_2+1,3}\}$, since $e_{i,j} \prec e_{l_2,3}$ for all $e_{i,j} \in E - \{e_{l_2,3}, e_{l_2+1,3}\}$. We now define an edge labeling f of S by $f(e_{l_2,3}) = p+2$, $f(e_{l_2+1,3}) = p+1$ and $f(e_{i,j}) = f_o(e_{i,j})$ for all other i,j. Then $f^+(v_{l_2,3}) = f_o^+(v_{l_2,3}) + 1$ and $f^+(v_{i,j}) = f_o^+(v_{i,j})$ for all other i,j. Fig. 4 shows several edges' labels of f. Clearly, for all $v_{i,j} \in V(S) - \{b, v_{l_2,3}, v_{l_2+1,3}\}$, the vertex sums $f^+(v_{i,j})$ are distinct and $f^+(v_{i,j}) = f_o^+(v_{i,j}) < f_o^+(v_{l_2,3}) < f^+(v_{l_2,3})$. Next evaluate $f^+(v_{l_2,3})$. If

$l_2 = 1$, then $f^+(v_{l_2,3}) = f(e_{l_2,3}) = p+2$; if $l_2 \geq 2$, we see that $f(e_{l_2-1,3}) = p-1$ and $f^+(v_{l_2,3}) = f(e_{l_2-1,3}) + f(e_{l_2,3}) = 2p+1$. And $f^+(b) = 2p+2, f^+(v_{l_2+1,3}) = 2p+3$. We see that $f^+(v_{i,j})$ $(1 \leq i \leq l_j, 1 \leq j \leq k)$ and $f^+(b)$ are all distinct. Hence f^+ is injective.

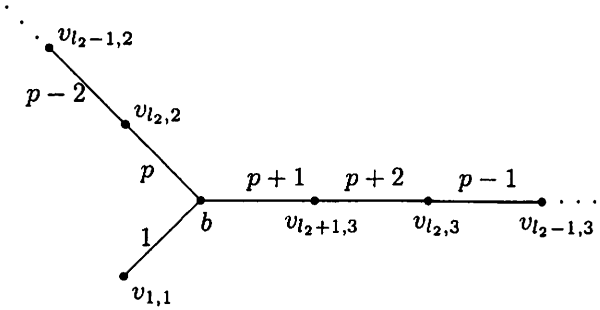

Fig. 4

Subcase 2.2: $f_o^+(b) = f_o^+(v_{t,k})$ for some $l_{k-1}+2 \leq t \leq l_k$. Note that $f_o(e_{l_{k-1},k}) = p+1$, $f_o(e_{l_{k-1}+1,k}) = p+2$, $f_o(e_{l_{k-1}+2,k}) = p+3$, $\cdots, f_o(e_{l_k,k}) = e$. Suppose $f_o(e_{t,k}) = m$. Then $f_o(e_{t-1,k}) = m-1$ and $f_o(e_{t-2,k}) = m-2$. And $f_o^+(b) = 2m-1$ for $f_o^+(v_{t,k}) = f_o(e_{t-1,k}) + f_o(e_{t,k}) = 2m-1$. By Lemma 2.1, all $f_o^+(v_{i,j})$ are distinct, and $f_o^+(v_{i,j}) < f_o^+(v_{t-2,k})$ for $v_{i,j} \in V(S) - \{b, v_{t-2,k}, v_{t-1,k}, v_{t,k}, \cdots, v_{l_k,k}\}$, since $e_{i,j} \prec e_{t-2,k}$ for all $e_{i,j} \in E - \{e_{t-2,k}, e_{t-1,k}, e_{t,k}, \cdots, e_{l_k,k}\}$. Now define an edge labeling f of S by $f(e_{t-2,k}) = m-1$, $f(e_{t-1,k}) = m-2$ and $f(e_{i,j}) = f_o(e_{i,j})$ for all other i,j. Then $f^+(v_{t-2,k}) = f_o^+(v_{t-2,k})+1$, $f^+(v_{t,k}) = f_o^+(v_{t,k})-1$ and $f^+(v_{i,j}) = f_o^+(v_{i,j})$ for all other i,j. Fig. 5 shows several edges' labels of f. Clearly, for all $v_{i,j} \in V(S) - \{b, v_{t-2,k}, v_{t-1,k}, v_{t,k}, \cdots, v_{l_k,k}\}$, the vertex sums $f^+(v_{i,j})$ are distinct and $f^+(v_{i,j}) = f_o^+(v_{i,j}) < f_o^+(v_{t-2,k}) < f^+(v_{t-2,k})$. Next evaluate $f^+(v_{t-2,k})$. If $t=3$, then $f^+(v_{t-2,k}) = f(e_{t-2,k}) = m-1$; if $t \geq 4$, we see that $f(e_{t-3,k}) \leq m-3$ and $f^+(v_{t-2,k}) = f(e_{t-3,k}) + f(e_{t-2,k}) \leq 2m-4$. And $f^+(v_{t-1,k}) = 2m-3, f^+(v_{t,k}) = 2m-2, f^+(b) = 2m-1$. And $f^+(v_{t+1,k}) = 2m+1$ if $t+1 \leq l_k$ and $f^+(v_{i,k}) = f^+(v_{i-1,k})+2$ if $i \geq t+2$. We see that $f^+(v_{i,j})$ $(1 \leq i \leq l_j, 1 \leq j \leq k)$ and $f^+(b)$ are all distinct. Hence f^+ is injective.

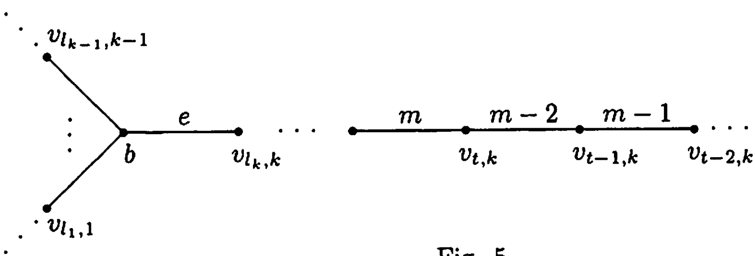

Fig. 5

From above we see that for every spider there exists an edge labeling such that the vertex sums are all distinct. Thus spiders are antimagic. □

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