On the Domination Number of Generalized Petersen Graphs $P(ck,k)$ *

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Abstract

Let $G=(V(G),E(G))$ be a simple connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S\subseteq V(G)$ is a *dominating set* if for each $v\in V(G)$ either $v\in S$ or v is adjacent to some $w\in S$. That is, S is a dominating set if and only if $N[S]=V(G)$. The domination number $\gamma(G)$ is the minimum cardinalities of minimal dominating sets. In this paper, we give an

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improved upper bound on the domination number of generalized Petersen graphs $P(ck, k)$ for $c \geq 3$ and $k \geq 3$. We also prove that $\gamma(P(4k, k)) = 2k + 1$ for even k, $\gamma(P(5k, k)) = 3k$ for all $k \geq 1$, and $\gamma(P(6k, k)) = \lceil \frac{10k}{3} \rceil$ for $k \geq 1$ and $k \neq 2$.

Keywords: *Domination number, Generalized Petersen Graph*

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. For $S \subseteq V(G)$, let $\langle S \rangle$ be the subgraph induced by S.

A set $S \subseteq V(G)$ is a *dominating set* if for each $v \in V(G)$ either $v \in S$ or v is adjacent to some $w \in S$. That is, S is a dominating set if and only if $N[S] = V(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinalities of minimal dominating sets. A subset $S \subset V(G)$ is efficient dominating set or a perfect dominating set if each vertex of G is dominated by exactly one vertex in S. For a more detailed treatment of domination-related parameters and for terminology not defined here, the reader is referred to [4].

In recent years, domination and its variations on the class of generalized Petersen graph have been studied extensively [1–3, 5–9]. The generalized Petersen graph $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i, u_i : 0 \leq i \leq n - 1\}$ and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 0 \leq i \leq n - 1$, subscripts are taken modulo $n\}$. In 2009, B. Javad Ebrahimi et al [2] proved a necessary and sufficient condition for the generalized Petersen graphs to have an efficient dominating set.

Lemma 1.1. [2] If $P(n, k)$ has an efficient dominating set, then $\gamma(P(n, k))$

$= \frac{n}{2}$ and $n \equiv 0 \pmod 4$.

Theorem 1.2. [2] A generalized Petersen graph $P(n,k)$ has an efficient dominating set if and only if $n \equiv 0 \pmod 4$ and k is odd.

Recently, Weiliang Zhao et al [9] have started to study the domination number of the generalized Petersen graphs $P(ck,k)$, where $c \geq 3$ is a constant. They obtained upper bound on $\gamma(P(ck,k))$ for $c \geq 3$ as follows:

$$\gamma(P(ck,k)) \leq \begin{cases} \frac{c}{3}\lceil\frac{5k}{3}\rceil, & \text{if } c \equiv 0 \pmod 3; \\ \lceil\frac{c}{3}\rceil\lceil\frac{5k}{3}\rceil - \lceil\frac{2k}{3}\rceil, & \text{if } c \equiv 1 \pmod 3; \\ \lceil\frac{c}{3}\rceil\lceil\frac{5k}{3}\rceil - \lceil\frac{2k}{3}\rceil + \lceil\frac{k}{3}\rceil, & \text{if } c \equiv 2 \pmod 3). \end{cases}$$

They also determined the domination number of $P(3k,k)$ for $k \geq 1$ and the domination number of $P(4k,k)$ for odd k.

In this paper, we study the domination number of generalized Petersen graphs $P(ck,k)$. We give an improved upper bound on the domination number of $P(ck,k)$ for $c \geq 3$ and $k \geq 3$. We also prove that $\gamma(P(4k,k)) = 2k+1$ for even k, $\gamma(P(5k,k)) = 3k$ for all $k \geq 1$, and $\gamma(P(6k,k)) = \lceil\frac{10k}{3}\rceil$ for $k \geq 1$ and $k \neq 2$.

Throughout the paper, the subscripts are taken modulo n when it is unambiguous.

2 General upper bound of $P(ck,k)$

In this section, we shall give an improved upper bound on the domination number of $P(ck,k)$ for general c.

Theorem 2.1. For any constant $c \geq 3$ and $k \geq 3$,

$$\gamma(P(ck,k)) \leq \begin{cases} \frac{ck}{2} + \alpha, & \text{if } c \equiv 0 \pmod 4; \\ \frac{ck}{2} + \frac{k}{2} - 1 + \alpha, & \text{if } c \equiv 1,2 \pmod 4 \text{ and } k \equiv 0 \pmod 2; \\ \frac{ck-1}{2} + \frac{k+1}{2} + \alpha, & \text{if } c \equiv 1 \pmod 4 \text{ and } k \equiv 1 \pmod 2; \\ \frac{ck}{2} + \frac{k+1}{2} + \alpha, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ \frac{ck}{2} + \frac{k-1}{2} + \alpha, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 3 \pmod 4; \\ \lfloor\frac{ck}{2}\rfloor + \lfloor\frac{k}{4}\rfloor + 1 + \alpha, & \text{if } c \equiv 3 \pmod 4 \text{ and } k \neq 4,8; \\ \frac{ck}{2} + \frac{k}{4} + \alpha, & \text{if } c \equiv 3 \pmod 4 \text{ and } k = 4,8; \end{cases}$$

where

$$\alpha = \begin{cases} 0, & \text{if } k \equiv 1 \pmod 2; \\ \lfloor\frac{c}{4}\rfloor, & \text{if } k \equiv 0 \pmod 2). \end{cases}$$

Proof. To show this upper bound, it suffices to give a dominating set S with the cardinality equaling to the values mentioned in this theorem. Let $n = ck$, $m = \lfloor \frac{n}{4} \rfloor$ and $t = n \bmod 4$. Then $n = 4m + t$.

For $k \equiv 1 \pmod 2$, let $S_0 = A \cup B$, where

$$A = \{v_{4i} : 0 \le i \le m-1\} \quad \text{and} \quad B = \{u_{4i+2} : 0 \le i \le m-1\},$$

and let

$$S = \begin{cases} S_0, & \text{if } c \equiv 0 \pmod 4; \\ S_0 \cup \{u_{n-2-4i}, u_{n-4-4i} : 0 \le i \le \lfloor \frac{k}{4} \rfloor - 1\} \cup \{u_{n-1}\}, & \text{if } c \equiv 1 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ S_0 \cup \{u_{n-2-4i}, u_{n-4-4i} : 0 \le i \le \lceil \frac{k}{4} \rceil - 1\} \cup \{v_{n-3}\}, & \text{if } c \equiv 1 \pmod 4 \text{ and } k \equiv 3 \pmod 4; \\ S_0 \cup \{u_{n-2-4i}, u_{n-5-4i} : 0 \le i \le \lfloor \frac{k}{4} \rfloor - 1\} \cup \{u_{n-1}, u_{n-3}\}, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ S_0 \cup \{u_{n-2-4i}, u_{n-3-4i} : 0 \le i \le \lceil \frac{k}{4} \rceil - 1\}, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 3 \pmod 4; \\ S_0 \cup \{u_{n-2-4i} : 0 \le i \le \lfloor \frac{k}{4} \rfloor\} \cup \{v_{n-3}\}, & \text{if } c \equiv 3 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ S_0 \cup \{u_{n-2-4i} : 0 \le i \le \lfloor \frac{k}{4} \rfloor\}, & \text{if } c \equiv 3 \pmod 4 \text{ and } k \equiv 3 \pmod 4. \end{cases}$$

It is not hard to check that

$$|S| = \begin{cases} \frac{ck}{2}, \text{if } c \equiv 0 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + 2 \times \lfloor \frac{k}{4} \rfloor + 1 = \frac{ck-1}{2} + \frac{k+1}{2}, & \text{if } c \equiv 1 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + 2 \times \lceil \frac{k}{4} \rceil + 1 = \frac{ck-1}{2} + \frac{k+1}{2}, & \text{if } c \equiv 1 \pmod 4 \text{ and } k \equiv 3 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + 2 \times \lfloor \frac{k}{4} \rfloor + 2 = \frac{ck}{2} + \frac{k+1}{2}, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + 2 \times \lceil \frac{k}{4} \rceil = \frac{ck}{2} + \frac{k-1}{2}, & \text{if } c \equiv 2 \pmod 4 \text{ and } k \equiv 3 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + \lfloor \frac{k}{4} \rfloor + 2 = \lfloor \frac{ck}{2} \rfloor + \lfloor \frac{k}{4} \rfloor + 1, & \text{if } c \equiv 3 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ 2 \times \lfloor \frac{ck}{4} \rfloor + \lfloor \frac{k}{4} \rfloor + 1 = \lfloor \frac{ck}{2} \rfloor + \lfloor \frac{k}{4} \rfloor + 1, & \text{if } c \equiv 3 \pmod 4 \text{ and } k \equiv 3 \pmod 4. \end{cases}$$

For $k \equiv 0 \pmod 2$, let $m_2 = \lfloor \frac{c}{4} \rfloor$ and $r = c \bmod 4$. Denote

$S_{40} = A_{40} \cup B_{40} \cup C_{40} \cup D_{40} \cup E_{40}$, where

$$A_{40} = \{v_{4kj+2+4i}, u_{4kj+4i} : 0 \le i \le \tfrac{k}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$B_{40} = \{v_{4kj+k+1+4i}, u_{4kj+k+3+4i} : 0 \le i \le \tfrac{k}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$C_{40} = \{v_{4kj+2k+4i}, u_{4kj+2k+2+4i} : 0 \le i \le \tfrac{k}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$D_{40} = \{v_{4kj+3k+3+4i}, u_{4kj+3k+1+4i} : 0 \le i \le \tfrac{k}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$E_{40} = \{v_{4kj+3k} : 0 \le j \le m_2 - 1\},$$

$S_{42} = A_{42} \cup B_{42} \cup C_{42} \cup D_{42} \cup E_{42}$, where

$$A_{42} = \{v_{4kj+4i}, u_{4kj+2+4i} : 0 \le i \le \tfrac{k-2}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$B_{42} = \{v_{4kj+k+1+4i}, u_{4kj+k-1+4i} : 0 \le i \le \tfrac{k-2}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$C_{42} = \{v_{4kj+2k+2+4i}, u_{4kj+2k+4i} : 0 \le i \le \tfrac{k-2}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$D_{42} = \{v_{4kj+3k-1+4i}, u_{4kj+3k+1+4i} : 0 \le i \le \tfrac{k-2}{4} - 1,\ 0 \le j \le m_2 - 1\},$$
$$E_{42} = \{v_{4kj+k-2}, v_{4kj+2k-2}, v_{4kj+4k-3}, u_{4kj+2k-3}, u_{4kj+4k-2}, : 0 \le j \le m_2 - 1\},$$

and

$$S_4 = \begin{cases} S_{40}, & \text{if } k \equiv 0 \pmod 4; \\ S_{42}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Then

$$|S_4| = \begin{cases} 2 \times \frac{k}{4} \times \frac{c-r}{4} \times 4 + \frac{c-r}{4} = \frac{(c-r)k}{2} + \frac{c-r}{4}, & \text{if } k \equiv 0 \pmod 4; \\ 2 \times \frac{k-2}{4} \times \frac{c-r}{4} \times 4 + 5 \times \frac{c-r}{4} = \frac{(c-r)k}{2} + \frac{c-r}{4}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

If $c \equiv 0 \pmod 4$, let $S = S_4$. Then $|S| = \frac{ck}{2} + \frac{c}{4}$.

If $c \equiv 1 \pmod 4$, let

$$S = \begin{cases} S_4 \cup \{u_i : n-k+1 \leq i \leq n-1\}, & \text{if } k \equiv 0 \pmod 4; \\ S_4 \cup \{u_i : n-k+1 \leq i \leq n-4\} \cup \{v_{n-k}, v_{n-3}, u_{n-1}\}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Then

$$|S| = \begin{cases} \frac{(c-1)\times k}{2} + \frac{c-1}{4} + k - 1 = \frac{ck}{2} + \frac{k}{2} + \lfloor \frac{c}{4} \rfloor - 1, & \text{if } k \equiv 0 \pmod 4; \\ \frac{(c-1)\times k}{2} + \frac{c-1}{4} + k - 4 + 3 = \frac{ck}{2} + \frac{k}{2} + \lfloor \frac{c}{4} \rfloor - 1, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

If $c \equiv 2 \pmod 4$, let

$$S = \begin{cases} S_4 \cup \{v_{n-2k+2+4i}, u_{n-2k+4i} : 0 \leq i \leq \frac{k}{4} - 1\} & \\ \quad \cup \{u_i : n-k \leq i \leq n-1\} \setminus \{u_{n-2k}\}, & \text{if } k \equiv 0 \pmod 4; \\ S_4 \cup \{v_{n-2k+4i}, u_{n-2k+2+4i} : 0 \leq i \leq \frac{k-2}{4} - 1\} & \\ \quad \cup \{u_i : n-k-3 \leq i \leq n-5\} \cup \{v_{n-3}\}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Then

$$|S| = \begin{cases} \frac{(c-2)\times k}{2} + \frac{c-2}{4} + 2 \times \frac{k}{4} + k - 1 = \frac{ck}{2} + \frac{k}{2} + \lfloor \frac{c}{4} \rfloor - 1, & \text{if } k \equiv 0 \pmod 4; \\ \frac{(c-2)\times k}{2} + \frac{c-2}{4} + 2 \times \frac{k-2}{4} + k - 1 + 1 = \frac{ck}{2} + \frac{k}{2} + \lfloor \frac{c}{4} \rfloor - 1, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

If $c \equiv 3 \pmod 4$, let

$$S = \begin{cases} S_4 \cup \{v_{n-ik}, v_{n-ik+3} : 1 \leq i \leq 3\} & \\ \quad \cup \{u_{n-2k+2}, u_{n-k+1}\} \setminus \{v_{n-k}\}, & \text{if } k = 4; \\ S_4 \cup \{v_{n-ik}, v_{n-ik+3}, v_{n-ik+6} : 1 \leq i \leq 3\} & \\ \quad \cup \{u_{n-3k+4}, u_{n-2k+2}, u_{n-2k+7}, u_{n-k+1}, u_{n-k+5}\}, & \text{if } k = 8; \\ S_4 \cup \{v_{n-3k+6+4i}, u_{n-3k+4+4i} : 0 \leq i \leq \frac{k}{4} - 2\} & \\ \quad \cup \{v_{n-2k+9+4i}, u_{n-2k+11+4i} : 0 \leq i \leq \frac{k}{4} - 3\} & \\ \quad \cup \{v_{n-k+8+4i}, u_{n-k+9+4i}, u_{n-k+10+4i} : 0 \leq i \leq \frac{k}{4} - 3\} & \\ \quad \cup \{v_{n-ik}, v_{n-ik+3} : 1 \leq i \leq 3\} & \\ \quad \cup \{v_{n-2k+6}, v_{n-k+6}, v_{n-1}\} & \\ \quad \cup \{u_{n-2k+2}, u_{n-2k+7}, u_{n-k+1}, u_{n-k+5}\}, & \text{if } k \equiv 0 \pmod 4 \\ & \text{and } k \neq 4, 8; \\ S_4 \cup \{v_{n-3k+4i}, u_{n-3k+2+4i} : 0 \leq i \leq \frac{k-2}{4} - 1\} & \\ \quad \cup \{v_{n-2k+1+4i}, u_{n-2k-1+4i} : 0 \leq i \leq \frac{k-2}{4}\} & \\ \quad \cup \{v_{n-k+3+4i}, u_{n-k+4i}, u_{n-k+1+4i} : 0 \leq i \leq \frac{k-2}{4} - 1\} & \\ \quad \cup \{v_{n-2k-2}, u_{n-2}\}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Then

$$|S| = \begin{cases} \frac{(c-3)\times k}{2} + \frac{c-3}{4} + 8 - 1 = \frac{ck}{2} + \frac{k}{4} + \lfloor \frac{c}{4} \rfloor, & \text{if } k = 4; \\ \frac{(c-3)\times k}{2} + \frac{c-3}{4} + 14 = \frac{ck}{2} + \frac{k}{4} + \lfloor \frac{c}{4} \rfloor, & \text{if } k = 8; \\ \frac{(c-3)\times k}{2} + \frac{c-3}{4} + 2 \times (\frac{k}{4} - 1) + 5 \times (\frac{k}{4} - 2) + 13 = \frac{ck}{2} + \frac{k}{4} + \lceil \frac{c}{4} \rceil, & \\ & \text{if } k \equiv 0 \pmod 4 \text{ and } k \neq 4, 8; \\ \frac{(c-3)\times k}{2} + \frac{c-3}{4} + 2 \times (\frac{k-2}{4} + 1) + 5 \times \frac{k-2}{4} + 2 = \frac{ck}{2} + \frac{k-2}{4} + \lceil \frac{c}{4} \rceil, & \\ & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

It is not hard to verify that S is a dominating set of $P(ck, k)$ with cardinality equaling to the values mentioned in this theorem. □

In Figure 2.1 and Figure 2.2, we show the dominating sets of $P(ck, k)$ for $3 \leq k \leq 10$ and $4 \leq c \leq 7$, where the vertices of dominating sets are in dark.

As an immediate consequence of Lemma 1.1, Theorem 1.2 and Theorem 2.1, we have the following

Theorem 2.2. For $k \geq 1$,

$$\gamma(P(4k, k)) = \begin{cases} 2k, & \text{if } k \equiv 1 \pmod 2; \\ 2k+1, & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

3 The domination number of $P(5k, k)$

In this section, we shall determine the exact domination number of $P(5k, k)$ for $k \geq 1$.

From Theorem 2.1, we have the following upper bound for $P(5k, k)$.

Lemma 3.1. For $k \geq 4$, $\gamma(P(5k, k)) \leq 3k$.

To prove the lower bound, we need some further notations. In the rest of the paper, let S be an arbitrary dominating set of $P(ck, k)$. For convenience, let

$$\begin{aligned} A_i &= \{v_{i+jk} : 0 \leq j \leq c-1\}, \\ B_i &= \{u_{i+jk} : 0 \leq j \leq c-1\}, \\ D_{i(j)} &= \{v_{i+jk}, u_{i+jk}\}, \quad 0 \leq j \leq c-1, \end{aligned}$$

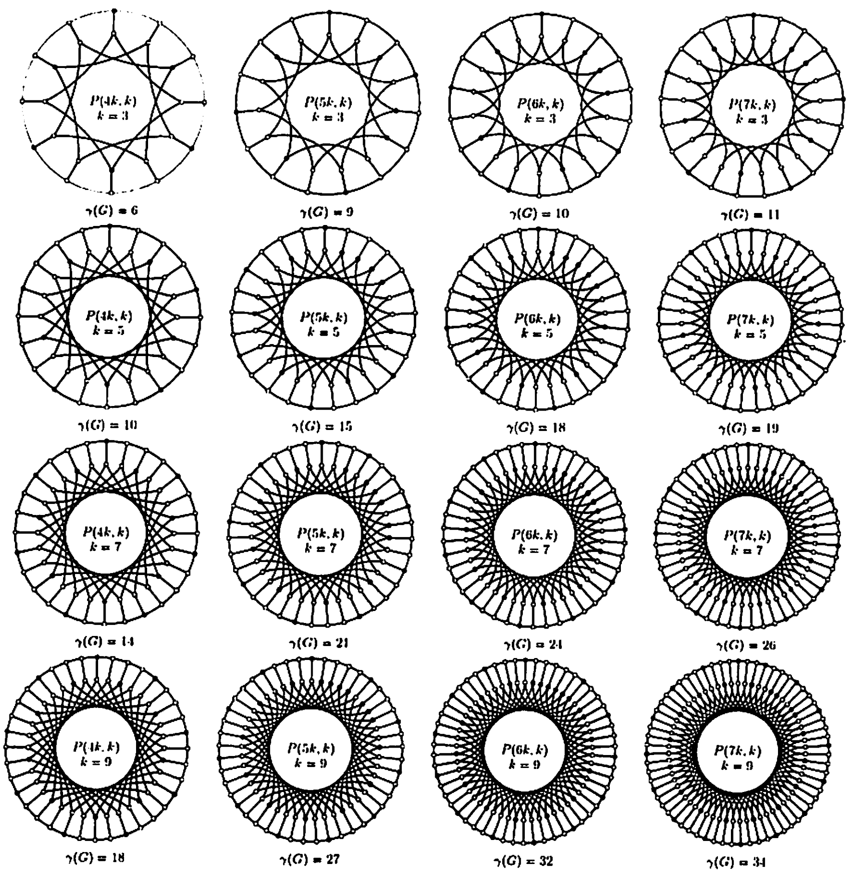

Figure 2.1: The dominating sets of $P(ck, k)$ for $k = 3, 5, 7, 9$ and $c = 4, 5, 6, 7$

for $0 \leq i \leq k-1$, where the vertices of A_i are on the outer cycle and those of B_i are on the inner cycle(s). For $0 \leq i \leq k-1$, let $G_i = \langle A_i \cup B_i \rangle$ be the ith subgraph induced by $A_i \cup B_i$ and $S_i = V(G_i) \cap S$.

Lemma 3.2. Let $\ell \in \{0, 1, \ldots, k-1\}$. If there exists two vertices $v_x, v_y \in S_\ell$ such that $|x - y| \in \{2k, 3k\}$, then $|S_\ell| \geq 4$.

Proof. Suppose to the contrary that $|S_\ell| \leq 3$. Without loss of generality, we may assume $x = \ell$ and $y = \ell + 2k$, i.e., $v_\ell, v_{\ell+2k} \in S_\ell$ (see Figure 3.1). Then at least one vertex of $\{u_{\ell+k}, u_{\ell+3k}, u_{\ell+4k}\}$ would not be dominated by S, a contradiction. $\square$

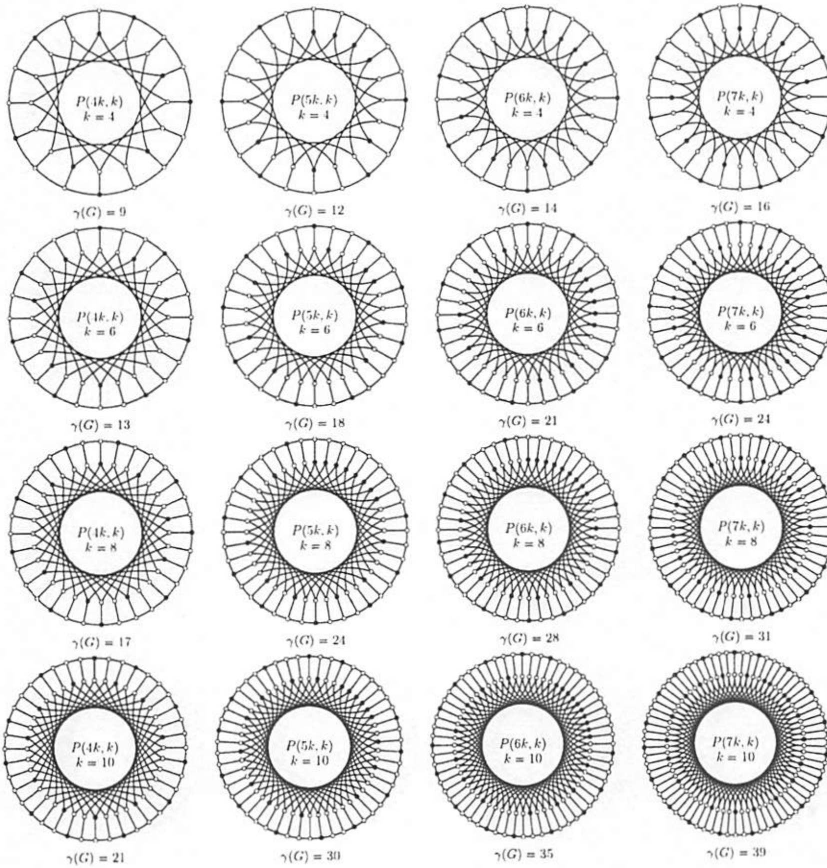

Figure 2.2: The dominating sets of $P(ck, k)$ for $k = 4, 6, 8, 10$ and $c = 4, 5, 6, 7$

Lemma 3.3. For any $i \in \{0, 1, \ldots, k-1\}$, $|S_i| \geq 2$. Moreover, if there exists an integer $\ell \in \{0, 1, \ldots, k-1\}$ such that $|S_\ell| = 2$, then $S_\ell \subseteq B_\ell$, S_ℓ is an independent set, and the following statements hold.

(i) If $|S_{\ell+1}| = 2$, then $|S_{\ell+2}| \geq 4$. Moreover, the equality holds only if $|S_{\ell+3}| \geq 4$;

(ii) If $|S_{\ell+1}| = 3$, then $|S_{\ell+2}| \geq 3$. Moreover, the equality holds only if $|S_{\ell+3}| \geq 4$;

where the subscripts are taken modulo k.

Proof. Since $\langle B_i \rangle$ is isomorphic to C_5 and every vertex of B_i must be dom-

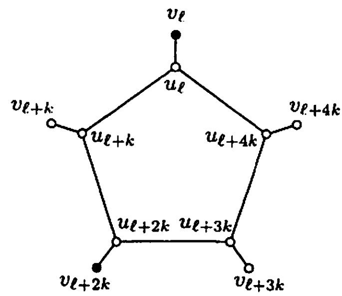

Figure 3.1: The graph for the proof of Lemma 3.1

inated by S_i, we have that $|S_i| \geq 2$ for any $i \in \{0, 1, \dots, k-1\}$.

Suppose that there exists an integer $\ell \in \{0, 1, \dots, k-1\}$ such that $|S_\ell| = 2$.

Assume to the contrary that $|S_\ell \cap B_\ell| \leq 1$, or $|S_\ell \cap B_\ell| = 2$ and S_ℓ is not an independent set. Then at least one vertex of B_ℓ would not be dominated by S, a contradiction. Hence, $S_\ell \subseteq B_\ell$ and S_ℓ is an independent set.

(i) Suppose $|S_{\ell+1}| = 2$. Then $S_\ell \cap A_\ell = \emptyset$, $S_{\ell+1} \cap A_{\ell+1} = \emptyset$ and $S_{\ell+1}$ is an independent set. Without loss of generality, we may assume $S_{\ell+1} = \{u_{\ell+1}, u_{\ell+1+2k}\}$. Since $S_\ell \cap A_\ell = \emptyset$, to dominate $\{v_{\ell+1+k}, v_{\ell+1+3k}, v_{\ell+1+4k}\}$, we have $v_{\ell+2+k}, v_{\ell+2+3k}, v_{\ell+2+4k} \in S_{\ell+2}$. It follows from Lemma 3.2 that $S_{\ell+2} \geq 4$.

If $S_{\ell+2} = 4$, to dominate $\{u_{\ell+2}, u_{\ell+2+2k}\}$, then $u_{\ell+2+k} \in S_{\ell+2}$, which implies that $S_{\ell+2} = \{v_{\ell+2+k}, v_{\ell+2+3k}, v_{\ell+2+4k}, u_{\ell+2+k}\}$ and $|D_{\ell+2(0)} \cap S_{\ell+2}|$ $= |D_{\ell+2(2)} \cap S_{\ell+2}| = 0$. Since $S_{\ell+1} \cap A_{\ell+1} = \emptyset$, to dominate $\{v_{\ell+2}, v_{\ell+2+2k}\}$, we have $v_{\ell+3}, v_{\ell+3+2k} \in S_{\ell+3}$ (see Figure 3.2 (1)). It follows from Lemma 3.2 that $S_{\ell+3} \geq 4$.

(ii) Suppose $|S_{\ell+1}| = 3$. If $|S_{\ell+2}| = 2$, then $S_\ell \cap A_\ell = \emptyset$ and $S_{\ell+2} \cap A_{\ell+2} = \emptyset$. To dominate all the vertices in $A_{\ell+1}$, we have that $|D_{\ell+1(j)} \cap S_{\ell+1}| \geq 1$ for every $j \in \{0, 1, 2, 3, 4\}$. It follows that $|S_{\ell+1}| \geq 5$, a contradiction with $|S_{\ell+1}| = 3$. Hence, $|S_{\ell+2}| \geq 3$.

Now suppose $|S_{\ell+2}| = 3$. It is easy to see that there exist at least

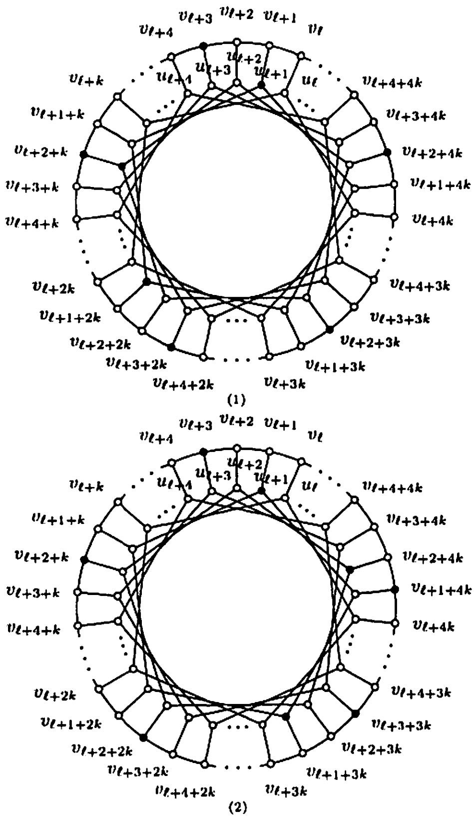

Figure 3.2: The graph for the proof of Lemma 3.2

two different index $j_1, j_2 \in \{0,1,2,3,4\}$ such that $D_{\ell+1(j_1)} \cap S_{\ell+1} = \emptyset$ and $D_{\ell+1(j_2)} \cap S_{\ell+1} = \emptyset$.

If $|j_1 - j_2| \notin \{1,4\}$, that is, $|j_1 - j_2| \in \{2,3\}$, say $j_1 = 1$ and $j_2 = 3$, since $S_\ell \cap A_\ell = \emptyset$, to dominate $\{v_{\ell+1+k}, v_{\ell+1+3k}\}$, we have that $v_{\ell+2+k}, v_{\ell+2+3k} \in S_{\ell+2}$. It follows from Lemma 3.2 that $|S_{\ell+2}| \geq 4$, a contradiction with $|S_{\ell+2}| = 3$. Hence, we conclude that $|j_1 - j_2| \in \{1,4\}$ and $|D_{\ell+1(t)} \cap S_{\ell+1}| = 1$ for $t \in \{0,1,2,3,4\} \setminus \{j_1, j_2\}$.

Without loss of generality, we may assume $j_1 = 1$ and $j_2 = 2$. To dominate $\{u_{\ell+1+k}, u_{\ell+1+2k}\}$, we have that $u_{\ell+1}, u_{\ell+1+3k} \in S_{\ell+1}$ and $v_{\ell+1}$, $v_{\ell+1+3k} \notin S_{\ell+1}$. Since $S_\ell \cap A_\ell = \emptyset$, to dominate $\{v_{\ell+1+k}, v_{\ell+1+2k}\}$, we have $v_{\ell+2+k}, v_{\ell+2+2k} \in S_{\ell+2}$. Since $S_{\ell+2} = 3$, to dominate $\{u_{\ell+2}, u_{\ell+2+3k}\}$, we have that $u_{\ell+2+4k} \in S_{\ell+2}$. It follows that $D_{\ell+2(0)} \cap S_{\ell+2} = \emptyset$ and $D_{\ell+2(3)} \cap S_{\ell+2} = \emptyset$. Since $v_{\ell+1}, v_{\ell+1+3k} \notin S_{\ell+1}$, we have $v_{\ell+3}, v_{\ell+3+3k} \in S_{\ell+3}$ (see Figure 3.2 (2) for $v_{\ell+1+4k} \in S_{\ell+1}$). It follows from Lemma 3.2 that $|S_{\ell+3}| \geq 4$.

Lemma 3.4. For $k \geq 4$, $\gamma(P(5k,k)) \geq 3k$.

Proof. Let S be a dominating set of $P(5k,k)$ with the minimum cardinality. If $|S_i| \geq 3$ for every $i \in \{0,1,\ldots,k-1\}$, then $\gamma(P(5k,k)) = |S| = \sum_{i=0}^{k-1} |S_i| \geq 3k$, and we are done. Hence, we may assume that there exists at least one index $\ell \in \{0,1,\ldots,k-1\}$ such that $|S_\ell| = 2$.

Let $H = \{0 \leq i \leq n-1 : |S_i| = 2, |S_{i-1}| > 2\}$ and let $h = |H|$. Let $t_1, t_2, \ldots, t_h$ be all the integers of H, where $0 \leq t_1 < t_2 < \cdots < t_h \leq n-1$. Let $N_i = \{0 \leq x \leq n-1 : t_i \leq x \leq t_{i+1} - 1\}$ for $i = 1,2,\ldots,h$ (In particular, $t_{h+1} = t_1$). Clearly, $\{0,1,\ldots,n-1\} = \bigcup_{i=1}^{h} N_i$. By Lemma 3.3, we conclude that for any $1 \leq i \leq h$, N_i satisfies one of the following conditions:

(a) $|S_{t_i}| = 2, |S_{t_i+1}| = 2, |S_{t_i+2}| \geq 5$ and $|S_x| \geq 3$ for any $t_i + 3 \leq x \leq t_{i+1} - 1$;

(b) $|S_{t_i}| = 2, |S_{t_i+1}| = 2, |S_{t_i+2}| = 4, |S_{t_i+3}| \geq 4$, $|S_x| \geq 3$ for any $t_i + 4 \leq x \leq t_{i+1} - 1$;

(c) $|S_{t_i}| = 2, |S_{t_i+1}| = 3, |S_{t_i+2}| \geq 4$, $|S_x| \geq 3$ for any $t_i + 3 \leq x \leq t_{i+1} - 1$;

(d) $|S_{t_i}| = 2, |S_{t_i+1}| = 3, |S_{t_i+2}| = 3, |S_{t_i+3}| \geq 4$, $|S_x| \geq 3$ for any $t_i + 4 \leq x \leq t_{i+1} - 1$;

(e) $|S_{t_i}| = 2, |S_{t_i+1}| \geq 4$, $|S_x| \geq 3$ for any $t_i + 2 \leq x \leq t_{i+1} - 1$.

It is easy to check that $\sum_{x \in N_i} |S_x| \geq 3|N_i|$ for every $i \in \{1,2,\ldots,h\}$. It follows that $\gamma(P(5k,k)) = |S| = \sum_{0 \leq x \leq k-1} |S_x| = \frac{1}{5} \sum_{0 \leq x \leq n-1} |S_x| = \frac{1}{5} \sum_{i=1}^{h} \sum_{x \in N_i} |S_x| \geq \frac{1}{5} \sum_{i=1}^{h} 3|N_i| = \frac{3}{5} \sum_{i=1}^{h} |N_i| = \frac{3n}{5} = 3k$. □

As an immediate consequence of Lemma 3.1 and Lemma 3.4, we have the following

Theorem 3.5. For $k \geq 4$, $\gamma(P(5k,k)) = 3k$.

It was shown in [2] that $\gamma(P(n,1)) = \lceil \frac{n}{2} \rceil$ for $n \not\equiv 2 \pmod 4$, $\gamma(P(n,2)) = \lceil \frac{3n}{5} \rceil$, and $\gamma(P(n,3)) = \lceil \frac{n}{2} \rceil + 1$ for $n \equiv 3 \pmod 4$ and $n \neq 11$. Then, we have that $\gamma(P(5,1)) = 3$, $\gamma(P(10,2)) = 6$ and $P(15,3) = 9$, which implies that $P(5k,k) = 3k$ for $k \in \{1,2,3\}$. Hence, we have the following corollary.

Corollary 3.6. For $k \geq 1$, $\gamma(P(5k,k)) = 3k$.

4 The domination number of $P(6k,k)$

In this section, we shall determine the exact domination number of $P(6k,k)$ for $k \geq 1$.

Lemma 4.1. For $k \geq 4$, $\gamma(P(6k,k)) \leq \lceil \frac{10k}{3} \rceil$.

Proof. To show that $\gamma(P(6k,k)) \leq \lceil \frac{10k}{3} \rceil$ for $k \geq 4$, it suffices to construct a set S that uses $\lceil \frac{10k}{3} \rceil$ vertices to dominate $P(6k,k)$.

Let $m = \lfloor \frac{k}{3} \rfloor$ and $t = k \bmod 3$. Then $k = 3m + t$. Denote

$$S = \begin{cases} \{u_i : 0 \leq i \leq k-1\} \cup \{u_i : 3k \leq i \leq 4k-1\} \cup \\ \{v_{k+3i+1} : 0 \leq i \leq \frac{2k}{3} - 1\} \cup \{v_{4k+3i+1} : 0 \leq i \leq \frac{2k}{3} - 1\}, & \text{if } t = 0; \\ \\ \{u_i : 0 \leq i \leq k-1\} \cup \{u_i : 3k-2 \leq i \leq 4k-3\} \cup \\ \{v_{k+3i+1} : 0 \leq i \leq \frac{2k-2}{3} - 1\} \cup \{v_{4k+3i-1} : 0 \leq i \leq \frac{k-1}{3} - 1\} \cup \\ \{v_{5k-2}, v_{5k-1}\} \cup \{v_{5k+3i+2} : 0 \leq i \leq \frac{k-1}{3} - 1\}, & \text{if } t = 1; \\ \\ \{u_i : 0 \leq i \leq k-1\} \cup \{v_{k+3i+1} : 0 \leq i \leq \frac{k-2}{3}\} \cup \\ \{v_{2k+3i+2} : 0 \leq i \leq \frac{k-5}{3} - 1\} \cup \{u_i : 3k \leq i \leq 4k-5\} \cup \\ \{v_{4k+3i+1} : 0 \leq i \leq \frac{k-5}{3} - 1\} \cup \{v_{5k+3i} : 0 \leq i \leq \frac{k-2}{3}\} \cup \\ \{u_{3k-4}, v_{3k-2}, u_{4k-3}, u_{4k-1}, v_{4k-3}, u_{5k-2}, v_{5k-4}\}, & \text{if } t = 2. \end{cases}$$

It is easy to check that

$$|S| = \begin{cases} 2 \times 3m + 2 \times \frac{2 \times 3m}{3} = \lceil \frac{10k}{3} \rceil, & \text{if } t = 0; \\ 2 \times (3m+1) + \frac{2 \times (3m+1) - 2}{3} + 2 \times \frac{3m}{3} + 2 = \lceil \frac{10k}{3} \rceil, & \text{if } t = 1; \\ 2 \times (3m+2) - 4 + 2 \times (\frac{3m}{3} + 1) + 2 \times \frac{3m-3}{3} + 7 = \lceil \frac{10k}{3} \rceil, & \text{if } t = 2. \end{cases}$$

For $k \equiv 0, 1 \pmod 3$, it is not hard to verify that each vertex in $V(P(6k,k)) \setminus S$ can be dominated by S.

For $k \equiv 2 \pmod 3$, we have that

$$v_j \in \begin{cases} N[\{u_i : 0 \le i \le k-1\}], & \text{if } 0 \le j \le k-1; \\ N[\{v_{k+3i+1} : 0 \le i \le \frac{k-2}{3}\}], & \text{if } k \le j \le 2k-1; \\ N[\{v_{2k+3i+2} : 0 \le i \le \frac{k-5}{3} - 1\} \cup \{u_{3k-4}, v_{3k-2}\}], & \text{if } 2k \le j \le 3k-1; \\ N[\{u_i : 3k \le i \le 4k-5\} \cup \{u_{4k-3}, u_{4k-1}, v_{4k-3}\}], & \text{if } 3k \le j \le 4k-1; \\ N[\{v_{4k+3i+1} : 0 \le i \le \frac{k-5}{3} - 1\} \cup \{u_{5k-2}, v_{5k-4}\}], & \text{if } 4k \le j \le 5k-1; \\ N[\{v_{5k+3i} : 0 \le i \le \frac{k-2}{3}\}], & \text{if } 5k \le j \le 6k-1; \end{cases}$$

and

$$u_j \in \begin{cases} N[\{u_i : 0 \le i \le k-1\}], & \text{if } j \in \{\ell k, \ell k+1, \dots, \ell k+k-1\} \\ & \text{and } \ell \in \{0,1,5\}; \\ N[\{u_i : 3k \le i \le 4k-5\}], & \text{if } j \in \{\ell k, \ell k+1, \dots, \ell k+k-5\} \\ & \text{and } \ell \in \{2,3,4\}; \\ N[\{u_{3k-4}, u_{4k-3}, v_{3k-2}, u_{4k-1}\}], & \text{if } 3k-4 \le j \le 3k-1; \\ N[\{u_{3k-4}, u_{4k-3}, u_{5k-2}, u_{4k-1}\}], & \text{if } 4k-4 \le j \le 4k-1; \\ N[\{v_{5k-4}, u_{4k-3}, u_{5k-2}, u_{4k-1}\}], & \text{if } 5k-4 \le j \le 5k-1. \end{cases}$$

Hence, S is a dominating set of $P(6k,k)$ for $k \ge 4$ with $|S| = \lceil \frac{10k}{3} \rceil$. □

In Figure 4.1, we show the dominating sets of $P(6k,k)$ for $4 \le k \le 12$, where the vertices of dominating sets are in dark.

Lemma 4.2. For $i \in \{0,1,\dots,k-1\}$, $|S_i| \ge 2$. If there exists an integer $\ell \in \{0,1,\dots,k-1\}$ such that $|B_\ell \cap S_\ell| = 1$, then $|S_\ell| \ge 4$.

Proof. Since $\langle B_i \rangle$ is isomorphic to C_6 and every vertex of B_i must be dominated by S_i, we have that $|S_i| \ge 2$ for every $i \in \{0,1,\dots,k-1\}$. If there exists an integer $\ell \in \{0,1,\dots,k-1\}$ such that $|B_\ell \cap S_\ell| = 1$, say $u_\ell \in S_\ell$, to dominate $\{u_{\ell+2k}, u_{\ell+3k}, u_{\ell+4k}\}$, we have $v_{\ell+2k}, v_{\ell+3k}, v_{\ell+4k} \in S_\ell$. It follows that $|S_\ell| \ge 4$. The lemma follows. □

Lemma 4.3. For every $i \in \{0,1,\dots,k-1\}$, $|S_{i-1} \cup S_i \cup S_{i+1}| \ge 10$, where the subscripts are taken modulo k.

Proof. Suppose to the contrary that there exists an integer $\ell \in \{0,1,\dots,k-1\}$ such that $|S_{\ell-1} \cup S_\ell \cup S_{\ell+1}| \le 9$. Combining with Lemma 4.2, we have

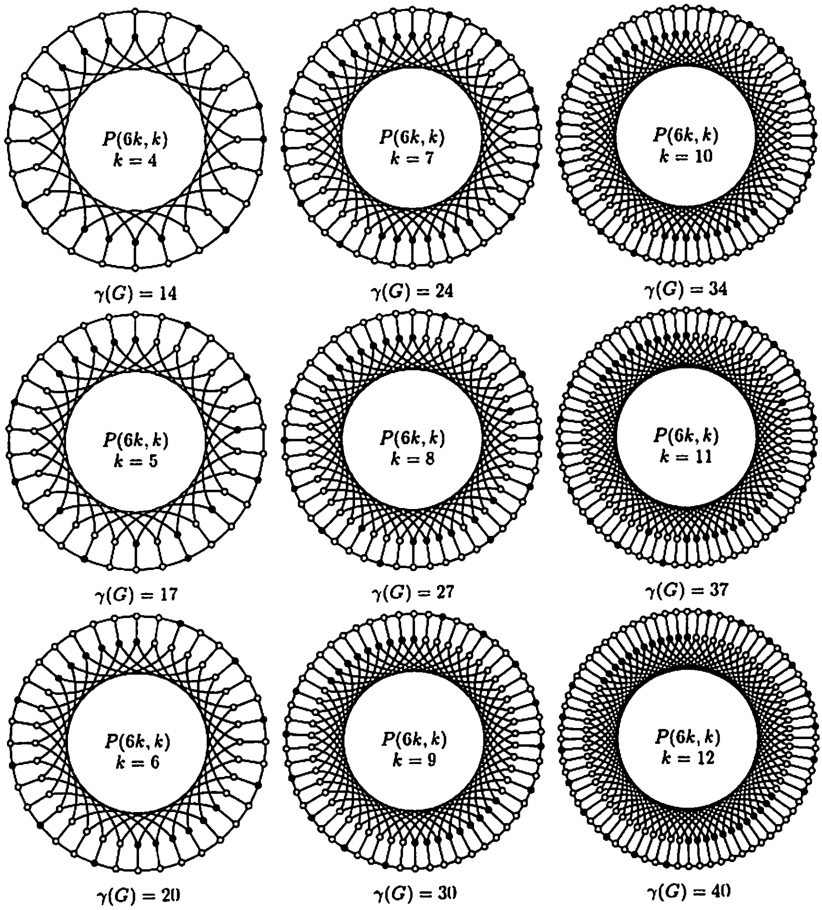

Figure 4.1: The dominating sets of $P(6k, k)$ for $4 \leq k \leq 12$

that

$$2 \leq |S_t| \leq 5 \tag{1}$$

for every $t \in \{\ell - 1, \ell, \ell + 1\}$.

It is easy to see that $V(G_{\ell-1}) \cup V(G_\ell) \cup V(G_{\ell+1}) = (\bigcup_{j=0}^{5} N[v_{\ell+jk}]) \cup B_{\ell-1} \cup B_{\ell+1}$. To dominate each vertex in A_ℓ, we have that

$$|N[v_{\ell+jk}] \cap (S_{\ell-1} \cup S_\ell \cup S_{\ell+1})| \geq 1 \tag{2}$$

for $0 \leq j \leq 5$. It follows that $\sum_{j=0}^{5} |N[v_{\ell+jk}] \cap (S_{\ell-1} \cup S_\ell \cup S_{\ell+1})| \geq 6$. From the assumption, we have $|(B_{\ell-1} \cap S_{\ell-1}) \cup (B_{\ell+1} \cap S_{\ell+1})| \leq 3$. It follows that

$$|B_{\ell-1} \cap S_{\ell-1}| \leq 1 \quad \text{or} \quad |B_{\ell+1} \cap S_{\ell+1}| \leq 1. \tag{3}$$

If $B_{\ell-1} \cap S_{\ell-1} = \emptyset$ or $B_{\ell+1} \cap S_{\ell+1} = \emptyset$, say $B_{\ell-1} \cap S_{\ell-1} = \emptyset$, to dominate each vertex in $B_{\ell-1}$, we have $A_{\ell-1} \subset S_{\ell-1}$, i.e., $|S_{\ell-1}| = 6$, a contradiction with (1). Hence,

$$|B_{\ell-1} \cap S_{\ell-1}| \geq 1 \quad \text{and} \quad |B_{\ell+1} \cap S_{\ell+1}| \geq 1. \tag{4}$$

It follows from (3) and (4) that $|B_{\ell-1} \cap S_{\ell-1}| = 1$ or $|B_{\ell+1} \cap S_{\ell+1}| = 1$, say $|B_{\ell-1} \cap S_{\ell-1}| = 1$. Without loss of generality, we may assume $u_{\ell-1} \in S_{\ell-1}$. To dominate $\{u_{\ell-1+2k}, u_{\ell-1+3k}, u_{\ell-1+4k}\}$, we have $v_{\ell-1+2k}, v_{\ell-1+3k}, v_{\ell-1+4k} \in S_{\ell-1}$, which implies

$$|S_{\ell-1}| \geq 4.$$

To dominate $u_{\ell+3k}$, we have that $|\{u_{\ell+2k}, u_{\ell+3k}, u_{\ell+4k}, v_{\ell+3k}\} \cap S_\ell| \geq 1$. It follows that $\sum_{j=2}^{4} |N[v_{\ell+jk}] \cap (S_{\ell-1} \cup S_\ell \cup S_{\ell+1})| \geq 3 + 1 = 4$. Combining with (2), we conclude that $|(V(G_{\ell-1}) \cup V(G_\ell) \cup V(G_{\ell+1}) \setminus B_{\ell+1}) \cap (S_{\ell-1} \cup S_\ell \cup S_{\ell+1})| = |B_{\ell-1} \cap S_{\ell-1}| + \sum_{j=0}^{5} |N[v_{\ell+jk}] \cap (S_{\ell-1} \cup S_\ell \cup S_{\ell+1})| \geq 1 + 7 = 8$. Hence, we have

$$|B_{\ell+1} \cap S_{\ell+1}| \leq 1.$$

By (4), we have $|B_{\ell+1} \cap S_{\ell+1}| = 1$. It follows from Lemma 4.2 that $|S_\ell| \geq 2$ and $|S_{\ell+1}| \geq 4$. Since $|S_{\ell-1}| \geq 4$, we have $|S_{\ell-1} \cup S_\ell \cup S_{\ell+1}| \geq 4+2+4 = 10$, a contradiction with assumption. The lemma follows. □

Lemma 4.4. For $k \geq 4$, $\gamma(P(6k,k)) \geq \lceil \frac{10k}{3} \rceil$.

Proof. Let S be a dominating set of $P(6k,k)$ with the minimum cardinality. Notice that each subset S_i is counted 18 times in $\sum_{i=0}^{6k-1} (|S_i| + |S_{i+1}| + |S_{i+2}|)$. By Lemma 4.3, we have

$$18 \times |S| = \sum_{i=0}^{6k-1} (|S_i| + |S_{i+1}| + |S_{i+2}|) \geq 6k \times 10 = 60k,$$

which implies that $\gamma(P(6k,k)) = |S| \geq \lceil \frac{10k}{3} \rceil$.

As an immediate consequence of Lemma 4.1 and Lemma 4.4, we have the following

Theorem 4.5. For $k \geq 4$, $\gamma(P(5k, k)) = 3k$.

It was shown in [2] that $\gamma(P(n,1)) = \frac{n}{2}+1$ for $n \equiv 2 \pmod 4$, $\gamma(P(n,2)) = \lceil\frac{3n}{5}\rceil$ and $\gamma(P(n,3)) = \frac{n}{2}+1$ for $n \equiv 2 \pmod 4$. Then, we have that $\gamma(P(6,1)) = 4$, $\gamma(P(12,2)) = 8$ and $P(18,3) = 10$, which implies that $P(6k,k) = \lceil\frac{10k}{3}\rceil$ for $k \in \{1,3\}$ and $P(6k,k) = \lceil\frac{10k}{3}\rceil + 1$ for $k = 2$. Hence, we have the following corollary.

Corollary 4.6. For $k \geq 1$,

$$\gamma(P(6k,k)) = \begin{cases} \lceil\frac{10k}{3}\rceil, & \text{if } k \neq 2; \\ \lceil\frac{10k}{3}\rceil + 1, & \text{if } k = 2. \end{cases}$$

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References

[1] A. Behzad, M. Behzad, C.E. Praeger, On the domination number of the generalized Petersen graphs, *Discrete Math.*, 308(2008) 603-610.

[2] B. Javad Ebrahimi, Nafiseh Jahanbakht, E.S. Mahmoodianc, Vertex domination of generalized Petersen graphs, *Discrete Math.*, 309 (2009) 4355-4361.

[3] J. Fox, R. Gera, P. Stănică, The independence number for the generalized Petersen graphs, *Ars Combin.*, in press.

[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs, *Marcel Dekker, Inc., NewYork,* 1998.

[5] Fu Xueliang, Yang Yuansheng, Jiang Baoqi, On the domination number of generalized Petersen graph $P(n,3)$, *Ars Combin.*, 84(2007), 373-383.

[6] Fu Xueliang, Yang Yuansheng, Jiang Baoqi, On the domination number of generalized Petersen graph $P(n, 2)$, *Discrete Math.*, in press.

[7] H. Yan, L.Y. Kang, G.J. Xu, The exact domination number of the generalized Petersen graphs, *Discrete Math.*, in press.

[8] B. Zelinka, Domination in generalized Petersen graphs, *Czechoslovak Math. J.*, 52(2002) 11-16.

[9] Weiliang Zhao, Meifang Zheng, Lirong Wu, Domination in the generalized Petersen graph $P(ck, k)$, *Utilitas Mathematica,* in press.