The strong chromatic index of graphs with restricted Ore-degrees

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Abstract

A strong edge-coloring is a proper edge-coloring such that two edges with the same color are not allowed to lie on a path of length three. The strong chromatic index of a graph G denoted by $s'(G)$ is the minimum number of colors in a strong edge-coloring.

We denote the degree of a vertex v by $d(v)$. Let the Ore-degree of a graph G be the maximum values of $d(u)+d(v)$ where u and v are adjacent vertices in G. Let F_3 denote the graph obtained from a 5-cycle by adding a new vertex and joining it to a pair of nonadjacent vertices of the 5-cycle. In 2008, Wu and Lin [J. Wu and W. Lin, The strong chromatic index of a class of graphs, Discrete Math., 308 (2008), 6254–6261] studied the strong chromatic index with respect to the Ore-degree. Their main result states that if a connected graph G is not F_3 and its Ore-degree is 5, then $s'(G) \leq 6$. Inspired by the result of Wu and Lin, we investigate the strong edge-coloring of graphs with Ore-degree 6. We show that each graph G with Ore-degree 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$. Our results give general forms of previous results about strong chromatic indices of graphs with maximum degree 3.

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1 Introduction

Graphs in this paper are finite, undirected, and loopless, but multiple edges are allowed. We always assume that graphs are connected unless the context implies otherwise. Note that some results that we refer to may not consider multiple edges, but these results can be extended easily to graphs with multiple edges.

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G respectively. We use $d(x)$ to denote the degree of a vertex x and $\Delta(G)$ to denote the maximum degree of a graph G. Let the *Ore-degree* of a graph G be the maximum values of $d(u) + d(v)$ where u and v are adjacent vertices in G. A k-vertex is a vertex of degree k.

The *distance* between edges e_1 and e_2 in a graph G is the distance between the corresponding two vertices in the line graph of G. A *strong edge-coloring* of a graph G is an edge-coloring in which two distinct edges with distance at most 2 have different colors. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. The *strong chromatic index* $s'(G)$ is the minimum k such that G has a strong k-edge-coloring. Throughout this paper, the term coloring means strong edge-coloring, unless the coloring is specified to be other type of coloring.

Erdős and Nešetřil [4] conjectured that $s'(G) \leq 5D^2/4 - D/2 + 1/4$, if D is odd and $s'(G) \leq 5D^2/4$, if D is even, where $D = \Delta(G)$. Andersen [1] and Horák, Qing, and Trotter [7] settled the case $D = 3$ of the conjecture by showing the following.

Theorem 1.1 ([1, 7]) *If a graph G has a maximum degree three, then $s'(G) \leq 10$.*

Horák [6] showed that there is a strong 23-edge-coloring for graphs with maximum degree four. Cranston [3] improved the bound to 22. The conjecture for $D = 4$ which has $s'(G) \leq 20$ remains unsolved.

Faudree et al. [5] formulated a bipartite version of this problem. They conjectured that $s'(G) \leq D^2$, if G is a bipartite graph. Steger and Yu [14] settled the conjecture for $\Delta(G) = 3$ which is the first non-trivial case of the second conjecture by showing the following.

Theorem 1.2 ([14]) *If a bipartite graph G has a maximum degree three, then $s'(G) \leq 9$.*

A stronger version of the second conjecture, due to Brualdi and Massey [2], states that $s'(G)$ is bounded by $D_1 D_2$, where D_1 and D_2 are the maximum degrees among vertices in the two partite sets, respectively. Quinn and Benjamin [12] proved this for a special class of bipartite graphs whose partite sets are the k-sets and l-sets in $[m]$, adjacent when the two sets share exactly j elements. Quinn and Sundberg [13] proved it for the incidence bigraph of the k-sets in $[m]$. Nakprasit [10] gave the affirmative answer to the conjecture for $D_1 = 2$.

Note that there are researches focusing to colorings related to Ore-degrees of graphs instead of maximum degrees. For example, Kierstead and Kostochka [8, 9] studied the relation of ordinary coloring, equitable coloring, and nearly-equitable coloring to Ore-degrees of graphs. In [5], Faudree et al. conjectured that if G is a bipartite graph with Ore-degree at most 5, then $s'(G) \leq 6$. Let F_D denote the graph obtained from a 5-cycle by adding $D-2$ new vertices and joining them to a pair of nonadjacent vertices of the 5-cycle. Wu and Lin [15] obtained the main result in their paper which verified the conjecture in a stronger form as follows.

Theorem 1.3 ([15]) *If a graph G is not F_3 and its Ore-degree is at most 5, then $s'(G) \leq 6$.*

Let H_1 denote the graph obtained from a 8-cycle $C = v_1v_2\dots v_8$ by adding two vertices v_1' and v_5' and joining v_1' to v_2, v_8, and v_5' to v_4, v_6. Wu and Lin [15] noted that they did not know any graphs with Ore-degree 5 to have $s'(G) \geq 6$ except F_3, H_1, and $K_{2,3}$. The result of Wu and Lin is generalized by Nakprasit and Nakprasit [11] as follows.

Theorem 1.4 ([11]) *If each edge xy of a graph G has $d(x)+d(y) \leq D+2$ and $\min\{d(x), d(y)\} \leq 2$, then $s'(G) \leq 2D+1$. With the further conditon that G is not F_D, we have $s'(G) \leq 2D$.*

However, the stronger form of Theorem 1.3 in terms of Ore-degrees is not known. For graphs with small Ore-degrees, we have the followings.

Observation 1.5 *(Characterization of graphs with small Ore-degrees)*

(i) The only graph with Ore-degree 0 is K_1.

(ii) No graph has Ore-degree 1.

(iii) The only graph with Ore-degree 2 is a path with one edge.

(iv) *The only graph with Ore-degree 3 is a path with two edges.*

(v) *A graph G has Ore-degree 4 if and only if G is a path of length at least 3, $K_{1,3}$, a cycle, or a graph with two vertices and two multiple edges.*

Since graphs with the above Ore-degrees can be classified explicitly, we can find their strong chromatic indices easily. Thus Theorem 1.3 by Wu and Lin is the first non-trivial result about the strong chromatic index in terms of Ore-degrees.

Inspired by the result of Wu and Lin, we show that each graph G with Ore-degree at most 6 has $s'(G) \leq 10$. With the further condition that G is bipartite, we have $s'(G) \leq 9$. Our results give general forms of Theorems 1.1 and 1.2.

2 The strong edge-colorings of graphs with restricted Ore-degrees

Note again that we assume that each graph is connected unless the context implies otherwise.

Next, we proceed to investigate strong chromatic indices in terms of Ore-degree of graphs in general.

Lemma 2.1 *Let G be a graph with Ore-degree at most R. If M is the set of vertices of G with degree $R-2$, then $s'(G) \leq \max\{s'(G-M), 3R-8\}$.*

Proof. If $\Delta(G) = R-1$, then $G = K_{1,R-1}$ which has $s'(G) = R-1$. If $\Delta(G) \leq R-3$, then $G = G-M$. Thus $\Delta(G) = R-2$ which implies $M \neq \emptyset$. Let G_1 be the graph induced by the edges incident to M.

First, note that each edge of G_1 has at most $2(R-3)+(R-2) = 3R-8$ edges within distance two. If each edge of one component of G_1 has $3R-8$ edges within distance two, then G satisfies the condition of Theorem 1.4 which implies $s'(G) \leq 2(R-2)+1 = 2R-3$. Now we may assume that every component of G_1 has an edge with at most $3R-9$ other edges within distance two.

Apply strong edge-coloring with $s'(G-M)$ colors to $E(G-M)$. It can be seen that we can use this as a partial strong edge-coloring in G. Now, we want to extend the coloring to edges in the component of $E(G_1)$ with an edge e having at most $3R-9$ edges within distance two. To greedily color them one by one, we give an ordering of the edges of this component in the following way.

If the distance from e_1 to e is greater than the distance from e_2 to e, then we color e_1 before e_2. Since every edge has at most $3R-9$ colored edges within distance two at each step, we can color every edge of such component of G_1. Using similar method to all components to complete the coloring. □

A path ww_1w_2 is a *special 2-path* if $d(w_1) = d(w_2) = 2$ and w is an $(R-2)$-vertex.

Lemma 2.2 *Let G be a graph with Ore-degree at most R with a special 2-path ww_1w_2. Then $s'(G) \leq \max\{s'(G-w_1), 2R-3\}$.*

Proof. Apply strong edge-coloring with $s'(G-w_1)$ colors to $G-w_1$. Now ww_1 has at most $2(R-3)+1 = 2R-5$ colored edges within distance two and w_1w_2 has at most $(R-3)+(R-2) = 2R-5$ colored edges within distance two. Since we have at least $2R-3$ available colors, we can extend the coloring to ww_1 and w_1w_2. As a result, we have a required coloring. □

Theorem 2.3 *If a graph G has Ore-degree at most 6, then $s'(G) \leq 10$. With the futher condition that G is bipartite, we have $s'(G) \leq 9$.*

Proof. Let G be a graph with Ore-degree 6. If $\Delta(G) = 5$, then $G = K_{1,5}$ which has $s'(G) = 5$. So we can assume that $\Delta(G) \leq 4$. Let M be the set of vertices with degree 4. Lemma 2.1 yields that $s'(G) \leq \max\{s'(G-M), 10\}$. Since $\Delta(G-M) \leq 3$, we have $s'(G-M) \leq 10$ by Theorem 1.1. Thus we have $s'(G) \leq 10$.

Now, it remains to show that $s'(G) \leq 9$ when G is bipartite. Suppose that G is a minimal counterexample to the theorem. Consider the case that G contains two distinct edges e_1, e_2 with a pair of common endpoints. Since $s'(G-e_1) \leq 9$ by minimality and e_1 has at most seven edges within distance two, we have $s'(G) \leq 9$. Thus we may assume G has no multiple edges. If $\Delta(G) = 5$, then $G = K_{1,5}$ which has $s'(G) = 5$. If $\Delta(G) \leq 3$, then Theorem 1.2 yields $s'(G) \leq 9$. Consequently, we assume that $\Delta(G) = 4$. Since G is not $K_{1,4}$ which has $s'(G) = 4$, the graph G contains a 4-vertex adjacent to a 2-vertex. If G has no 3-vertices, then Theorem 1.4

yields $s'(G) \le 9$. Thus G contains a 3-vertex and a 4-vertex. Since G is connected and has Ore-degree 6, the graph G has a path of length at least two with every internal vertex is 2-vertex whereas one endpoint is 3-vertex and the other is 4-vertex. If G contains a special 2-path, then $s'(G) \le 9$ by minimality of G and Lemma 2.2. Thus G contains a path uvw with u is a 4-vertex, v is a 2-vertex, and w is a 3-vertex.

Consider such u with its four neighbors v_1, v_2, v_3, and v_4. Since G is bipartite, the set $\{v_1, v_2, v_3, v_4\}$ is independent. Suppose some v_i is a 1-vertex. Since $s'(G - uv_i) \le 9$ by minimality and uv_i has at most six edges within distance two, we have $s'(G) \le 9$. Let w_i different from u be the other neighbor of the 2-vertex v_i $(1 \le i \le 4)$. Note that w_1, w_2, w_3, w_4 are not necessarily pairwise distinct. We have $d(w_i) \ne 1$ as before. Moreover, $d(w_i) \ne 2$ because G has no special 2-edges. Combinining with the fact that G has Ore-degree at most 6, we have each $d(w_i) = 3$ or 4. From the choice of u, some w_i is a 3-vertex. Let $W = \{w_1, w_2, w_3, w_4\}$.

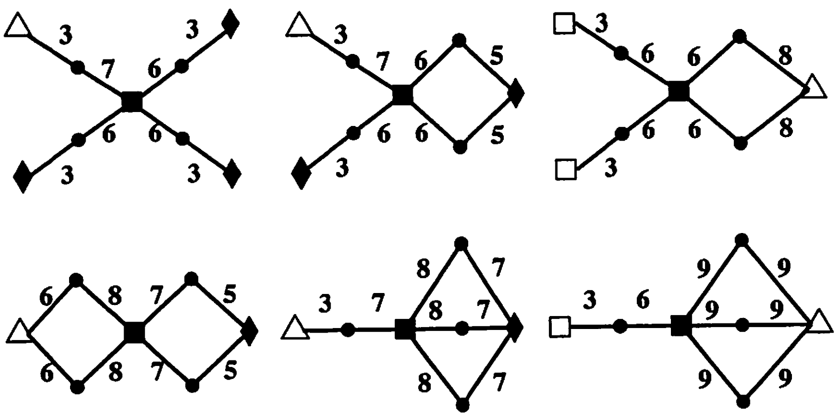

Figure 1: Configurations in a minimum counterexample.

We claim that the counterexample G must contain one of the configurations in Fig. 1, where a black square is the vertex u, a dot indicates a vertex of degree 2 (that is some v_i), a hollow triangle indicates a vertex of degree 3, a hollow square indicates a vertex of degree 4, the degree of a black diamond is at least the number of edges incident to the black diamond in the figure, and all vertices are distinct. If $|W| = 4$, that is all w_1, w_2, w_3, w_4 are distinct, then G contains the first configuration. Consider the case $|W| = 3$ where $w_3 = w_4$. If $d(w_1)$ or $d(w_2)$ is 3, then G contains the second configuration, otherwise G contains the third configuration. The case $|W| = 2$ where $w_1 = w_2$ and $w_3 = w_4$ implies G contains the fourth configuration. Consider the case $|W| = 2$ where $w_2 = w_3 = w_4$. If $d(w_1) = 3$, then G contains the fifth configuration, otherwise G contains the sixth configuration. The case that $|W| = 1$ contradicts the fact that $d(w_i) = 3$.

After some partial strong k-edge coloring on G, we use $A(e)$ denote the number of legal colors from k colors that can be assigned to e. Consider a coloring of all edges in G except edges in a configurtion. Each edge e is the figure is shown with a lower bound for $A(e)$ that is calculated from 9 minus the number of edges with distance within two from the edge e,

Since the number of legal colors for each edge not incident to u is large enough, the sets of legal colors of those edges cannot be all pairwise disjoint. Thus we can assign some color to two of those edges simultaneously. Next, we color other two edges that are not incident to the vertex u. Note that the lower bound for $A(e)$ in each uncolored edge e is now decreased by at most three. Finally, we color four edges incident to the vertex u sequentially from an edge with the least number of legal colors to the most one. Since the number of legal colors for each edge is large enough, the strong edge-coloring using at most nine colors can be completed. □

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