The smallest realization of a given vector

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Abstract

For a vector $R = (r_1, r_2, \ldots, r_m)$ of non-negative integers, a mixed hypergraph $\mathcal{H}$ is a realization of R if its chromatic spectrum is R. In this paper, we determine the minimum number of vertices of realizations of a special kind of vectors R_2. As a result, we partially solve an open problem proposed by Král in 2004.

Key words: mixed hypergraph; strict coloring; feasible set; chromatic spectrum; realization

1 Introduction

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where X is a finite set and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of X called the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. If $X' \subset X$, $\mathcal{C}' = \{C \in \mathcal{C} | C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} | D \subseteq X'\}$, then the hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ is called the *induced sub-hypergraph* of $\mathcal{H}$ on X', denoted by $\mathcal{H}[X']$.

A coloring of the vertices of $\mathcal{H}$ is *proper* if there are two vertices with a *Common* color in each $\mathcal{C}$-edge and there are two vertices with *Distinct* colors in each $\mathcal{D}$-edge. A proper coloring using exactly k-colors is called a *strict k-coloring* and a mixed hypergraph is *k-colorable* if it has a strict k-coloring. A coloring may also be viewed as a *partition* (*feasible partition*) of the vertex set, where the *color classes* (partition classes) are the sets of vertices assigned to the same color. An edge is said to be *monochromatic* (resp. *polychromatic*) if all of its vertices have the same color (resp. different colors). The maximum (resp. minimum) number of colors which can be

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used in a strict coloring of $\mathcal{H}$ is called the *upper chromatic number* (resp. *lower chromatic number*) and denoted by $\bar{\chi}(\mathcal{H})$ (resp. $\chi(\mathcal{H})$). The study of the colorings of mixed hypergraphs has made a lot of progress since its inception [6]. For more information, we refer readers to [3, 5, 7, 8].

The *feasible set* $\mathcal{F}(\mathcal{H})$ of a mixed hypergraph $\mathcal{H}$ is the set of all the values k such that $\mathcal{H}$ has a strict k-coloring. For each k, let r_k denote the number of feasible partitions of the vertex set into k nonempty subsets. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\bar{\chi}})$ is called the *chromatic spectrum* of $\mathcal{H}$, where $\bar{\chi}$ is the upper chromatic number of $\mathcal{H}$. If S is a set of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a *realization* of S if $\mathcal{F}(\mathcal{H}) = S$; and a mixed hypergraph $\mathcal{H}$ is a *one-realization* of S if it is a realization of S and all the entries of the chromatic spectrum of $\mathcal{H}$ are either 0 or 1. The concept of one-realization was introduced by Jiang et al. in [2] with an extra condition as proper realizations and further studied by Král in [4]. Moreover, for a vector R of non-negative integers, a mixed hypergraph $\mathcal{H}$ is called a *realization* of R if $R(\mathcal{H}) = R$.

Bujtás and Tuza [1] gave a necessary and sufficient condition for S to be the feasible set of an r-uniform mixed hypergraph. Jiang et al. [2] proved that the minimum number of vertices of realizations of S is $2\max(S) - \min(S)$ if $|S| = 2$ and $\max(S) - 1 \notin S$. Moreover, they also mentioned that the question of finding the minimum number of vertices in a mixed hypergraph with feasible set S of size at least 3 remains open. Král [4] proved that there exists a one-realization of S with at most $|S| + 2\max(S) - \min(S)$ vertices and proposed the following problem: what is the number of vertices of the smallest mixed hypergraph whose spectrum is equal to a given spectrum $(r_1, r_2, \ldots, r_m)$? In [9], P. Zhao et al. obtained an upper bound on the minimum number of vertices of 3-uniform bi-hypergraphs with a given feasible set. Moreover, P. Zhao et al. [10] proved that the minimum number of vertices of one-realizations of a given set is $2\max(S) - \min(S)$ if $\max(S) - 1 \notin S$ or $2\max(S) - \min(S) - 1$ otherwise. In this paper, we generalize this result to the minimum number of vertices of realizations of a special kind of vector.

In the rest of this paper, we always assume that n_i and l_i are two sets of integers with $2 \leq n_s < \cdots < n_1$, $l_1 = 0$ and $0 \leq l_i < n_{i-1} - n_i$ for all $i \in \{2, \ldots, s\}$; moreover, $R_2 = (r_1, r_2, \ldots, r_{n_1})$ is the vector with $r_1 = 0$ and $r_{n_i} = 2^{l_i}$, $i \in \{1, 2, \ldots, s\}$. For any positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. The main result of this paper is as follows:

Theorem 1.1 *If $\delta(R_2)$ is the minimum number of vertices of realizations of R_2, then*

$$\delta(R_2) = \begin{cases} 2n_1 - n_s, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - n_s - 1, & \text{if } n_1 = n_2 + 1, \end{cases}$$

As a result, we partially solve the open problem proposed by Král.

2 The proof of Theorem 1.1

In this section, we first show that the number $\delta(R_2)$ given in Theorem 1.1 is a lower bound on the number of vertices of the smallest realization of R_2, then construct two families of mixed hypergraphs which meet the bound in each case. For a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ and a strict coloring c of $\mathcal{H}$, let $c(v)$ denote the color of $v \in X$ under c.

Lemma 2.1

$$\delta(R_2) \geq \begin{cases} 2n_1 - n_s, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - n_s - 1, & \text{if } n_1 = n_2 + 1. \end{cases}$$

Proof. Since the first part was given by Theorem 3 in [2] and Lemma 2.2 in [10] respectively, it suffices to prove the second part. Let $n_1 = n_2 + 1$ and $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a realization of R_2. Then $l_2 = 0$, i.e., $r_{n_2} = 1$. Suppose$|X| \leq 2n_1 - (n_s + 2)$. Then for any strict n_1-coloring $c_1 = \{C_1, C_2, \dots, C_{n_1}\}$ of $\mathcal{H}$, there exist at least $n_s + 2$ color classes of size one. Assume that $C_1 = \{\alpha_1\}, C_2 = \{\alpha_2\}, \dots, C_{n_s+2} = \{\alpha_{n_s+2}\}$. For any strict n_s-coloring c_s of $\mathcal{H}$, there are the following two possible cases.

Case 1. There exist three vertices in $\{\alpha_1, \alpha_2, \dots, \alpha_{n_s+2}\}$ with the same color under c_s. Suppose $c_s(\alpha_1) = c_s(\alpha_2) = c_s(\alpha_3)$. Then $\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_2, \alpha_3\} \notin \mathcal{D}$, which follows that $\{C_1 \cup C_2, C_3, \dots, C_{n_1}\}$, $\{C_1, C_2 \cup C_3, C_4, \dots, C_{n_1}\}$ are strict n_2-colorings of $\mathcal{H}$, a contradiction to that $r_{n_2} = 1$.

Case 2. There exist two pairs of vertices in $\{\alpha_1, \alpha_2, \dots, \alpha_{n_s+2}\}$ such that each pair have the same color under c_s. Suppose $c_s(\alpha_1) = c_s(\alpha_2)$ and $c_s(\alpha_3) = c_s(\alpha_4)$. Then $\{\alpha_1, \alpha_2\}, \{\alpha_3, \alpha_4\} \notin \mathcal{D}$, which implies that $\{C_1 \cup C_2, C_3, \dots, C_{n_1}\}$ and $\{C_1, C_2, C_3 \cup C_4, C_5, \dots, C_{n_1}\}$ are strict n_2-colorings of $\mathcal{H}$, also a contradiction to that $r_{n_2} = 1$. □

In the rest of this section, we shall construct the desired mixed hypergraphs. Our construction here is based on the Construction I in [10]. In order to get the desired r_{n_k} strict n_k-colorings, we define the vertices of γ^1_{pk} and γ^2_{pk} in the construction.

Construction I. For $i \in [n_s]$, $p \in [s] \setminus \{1\}, j \in \{0, l_p + 1, l_p + 2, \dots, n_{p-1} - n_p - 1\}$ and $k \in [l_p]$, write

$$\theta_i = (\underbrace{i, i, \dots, i}_{\sum_{i=1}^{s} 2^{l_i}}),\ \beta_1 = (n_1, \underbrace{n_2, \dots, n_2}_{2^{l_2}}, \dots, \underbrace{n_s, \dots, n_s}_{2^{l_s}})$$

$$\alpha_{pj} = (\underbrace{n_p + j, \ldots, n_p + j}_{\sum_{i=1}^{p-1} 2^{l_i}}, \underbrace{1, \ldots, 1}_{\sum_{i=p}^{s} 2^{l_i}}),$$

$$\beta_{pj} = (\underbrace{n_p + j, \ldots, n_p + j}_{\sum_{i=1}^{p-1} 2^{l_i}}, \underbrace{n_p, \ldots, n_p}_{2^{l_p}}, \ldots, \underbrace{n_s, \ldots, n_s}_{2^{l_s}}),$$

$$\gamma^1_{pk} = (\underbrace{n_p + k, \ldots, n_p + k}_{\sum_{i=1}^{p-1} 2^{l_i}},$$

$$\underbrace{\underbrace{n_p, \ldots, n_p}_{2^{k-1}}, \underbrace{1, \ldots, 1}_{2^{k-1}}, \ldots, \underbrace{n_p, \ldots, n_p}_{2^{k-1}}, \underbrace{1, \ldots, 1}_{2^{k-1}}}_{2^{l_p}}, \underbrace{1, \ldots, 1}_{\sum_{i=p+1}^{s} 2^{l_i}}),$$

$$\gamma^2_{pk} = (\underbrace{n_p + k, \ldots, n_p + k}_{\sum_{i=1}^{p-1} 2^{l_i}}, \underbrace{\underbrace{1, \ldots, 1}_{2^{k-1}}, \underbrace{n_p, \ldots, n_p}_{2^{k-1}}, \ldots, \underbrace{1, \ldots, 1}_{2^{k-1}}, \underbrace{n_p, \ldots, n_p}_{2^{k-1}}}_{2^{l_p}},$$

$$\underbrace{n_{p+1}, \ldots, n_{p+1}}_{2^{l_{p+1}}}, \ldots, \underbrace{n_s, \ldots, n_s}_{2^{l_s}}),$$

$$X = \bigcup_{i=1}^{n_s} \bigcup_{p=2}^{s} \bigcup_{j=0,l_p+1}^{n_{p-1}-n_p-1} \bigcup_{k=1}^{l_p} \{\theta_i, \alpha_{pj}, \beta_{pj}, \gamma^1_{pk}, \gamma^2_{pk}, \beta_1\} \text{ and}$$

$$\mathcal{C} = \{\{\alpha_1, \alpha_2, \alpha_3\} \mid \alpha_l \in X, l \in [3], |\{\alpha_{1(j)}, \alpha_{2(j)}, \alpha_{3(j)}\}| = 2, j \in [\sum_{i=1}^{s} 2^{l_i}]\},$$

$$\mathcal{D} = \{\{\alpha_1, \alpha_2\} \mid \alpha_l \in X, l \in [2], \alpha_{1(j)} \neq \alpha_{2(j)}, j \in [\sum_{i=1}^{s} 2^{l_i}]\} \cup$$

$$(\bigcup_{p=2}^{s} \bigcup_{k=1}^{l_p} \{\{\beta_1, \gamma^1_{pk}, \gamma^2_{pk}\}\}) \cup (\bigcup_{p_1=2}^{s} \bigcup_{p_2=2}^{s} \bigcup_{k=1}^{l_{p_2}} \bigcup_{j=0,l_{p_1}+1}^{n_{p_1-1}-n_{p_1}-1} \{\{\alpha_{p_1 j}, \beta_{p_1 j}, \gamma^1_{p_2 k}\},$$

$$\{\alpha_{p_1 j}, \beta_{p_1 j}, \gamma^2_{p_2 k}\}, \{\alpha_{p_1 j}, \gamma^1_{p_2 k}, \gamma^2_{p_2 k}\}, \{\beta_{p_1 j}, \gamma^1_{p_2 k}, \gamma^2_{p_2 k}\}, \{\gamma^1_{p_1 j}, \gamma^1_{p_2 k}, \gamma^2_{p_2 k}\}\}),$$

where $\alpha_{l(j)}$ is the j^{th} entry of the vertex α_l. Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a mixed hypergraph with $2n_1 - n_s$ vertices.

For example, when $S = \{2, 4\}$ and $l_2 = 1$, then $X = \{\theta_1, \alpha_{20}, \beta_{20}, \gamma^1_{21}, \gamma^2_{21}, \beta_1\}$, where $\theta_1 = (1,1,1), \theta_2 = (2,2,2)$, $\alpha_{20} = (2,1,1), \beta_{20} = (2,2,2)$, $\gamma^1_{21} = (3,2,1), \gamma^2_{21} = (3,1,2)$, $\beta_1 = (4,2,2)$ and

$$\mathcal{C} = \{\{\theta_1, \alpha_{20}, \beta_{20}\}, \{\theta_1, \gamma^1_{21}, \gamma^2_{21}\}\}, \{\alpha_{20}, \gamma^1_{21}, \gamma^2_{21}\}\}, \{\beta_{20}, \gamma^1_{21}, \gamma^2_{21}\},$$
$$\{\gamma^1_{21}, \alpha_{20}, \beta_{20}\}, \{\gamma^2_{21}, \alpha_{20}, \beta_{20}\}, \{\beta_1, \alpha_{20}, \beta_{20}\}, \{\beta_1, \gamma^1_{21}, \gamma^2_{21}\}\},$$
$$\mathcal{D} = \{\{\theta_1, \beta_{20}\}, \{\theta_1, \beta_1\}, \{\alpha_{20}, \beta_1\}, \{\beta_1, \gamma^1_{21}, \gamma^2_{21}\},$$

$$\{\alpha_{20},\beta_{20},\gamma_{21}^1\},\{\alpha_{20},\beta_{20},\gamma_{21}^2\},\{\alpha_{20},\gamma_{21}^1,\gamma_{21}^2\},\{\beta_{20},\gamma_{21}^1,\gamma_{21}^2\}\}.$$

Moreover, when $S=\{2,t\}$ and $l_2=0$, our construction is similar to the first construction in [2], but our edge-set gives a one-realizer whereas the construction in [2] does not. The later construction in [2] of their proper realizers uses many more than the minimum number of vertices.

Note the inspiration for our construction is so that for each $i\in[s]$ and $h\in[2^{l_i}]$, $c_{ih}=\{X_{ih1},X_{ih2},\dots,X_{ihn_i}\}$ is a strict n_i-coloring of $\mathcal{H}$, where X_{iht} consists of vertices

$$(x_1^1,x_2^1,\dots,x_2^{2^{l_2}},\dots,x_i^1,\dots,x_i^{h-1},t,x_i^{h+1},\dots,x_i^{2^{l_i}},\dots,x_s^1,\dots,x_s^{2^{l_s}})\in X.$$

Lemma 2.2 *Let $c=\{C_1,C_2,\dots,C_m\}$ be a strict coloring of $\mathcal{H}$. Then we may reorder the color classes such that the following conditions hold:*

(i) $\theta_i\in C_i$, $i\in[n_s]$;

(ii) $\alpha_{pj}\notin C_i$, $i\in[n_s-1]\setminus\{1\}$;

(iii) $\beta_1,\beta_{pj}\notin C_i, i\in[n_s-1]$;

(iv) $\gamma_{pk}^1,\gamma_{pk}^2\notin C_i, i\in[n_s-1]\setminus\{1\}$;

(v) $\alpha_{s0}\in C_1\cup C_{n_s}$.

Proof. The $\mathcal{D}$-edge $\{\theta_i,\theta_j\}$ implies that $c(\theta_i)\neq c(\theta_j)$ if $i\neq j$. Hence, we may reorder the color classes such that $\theta_i\in C_i$, $i\in[n_s]$. It follows that (i) holds.

For $i\in[n_s-1]\setminus\{1\}$, from the $\mathcal{D}$-edges $\{\alpha_{pj},\theta_i\}$, $\{\gamma_{pk}^1,\theta_i\}$, $\{\gamma_{pk}^2,\theta_i\}$, one gets $\alpha_{pj},\gamma_{pk}^1,\gamma_{pk}^2\notin C_i$. Hence, (ii) and (iv) hold.

Since $\{\beta_1,\theta_i\}$, $\{\beta_{pj},\theta_i\}$ are $\mathcal{D}$-edges, we have $\beta_1,\beta_{pj}\notin C_i$ for $i\in[n_s-1]$, which implies that (iii) holds. The $\mathcal{C}$-edge $\{\alpha_{s0},\theta_{n_s},\theta_1\}$ implies that $\alpha_{s0}\in C_1\cup C_{n_s}$, as desired. □

Lemma 2.3 *Let $c=\{C_1,C_2,\dots,C_m\}$ be a strict coloring of $\mathcal{H}$ satisfying the conditions (i)-(v) in Lemma 2.2.*

(i) *Suppose $c(\alpha_{tj_t})\neq c(\beta_{tj_t})$ for some $t\in[s]\setminus\{1\}$ and $j_t\in\{0,l_t+1,\dots,n_{t-1}-n_t-1\}$. Then there exists an $a\in[m]\setminus[n_s-1]$ such that $\alpha_{pj}\in C_1$ and $\beta_1,\beta_{pj}\in C_a$ for all $p\in[t]\setminus\{1\}$;*

(ii) *Suppose $c(\alpha_{qj_q})=c(\beta_{qj_q})$ for some $q\in[s]\setminus\{1\}$ and $j_q\in\{0,l_q+1,\dots,n_{q-1}-n_q-1\}$. Then $c(\alpha_{pj})=c(\beta_{pj})$ and $c(\gamma_{pk}^1)=c(\gamma_{pk}^2)$ for all $p\in\{q,\dots,s\}$.*

Proof. (i) From the $\mathcal{C}$-edge $\{\alpha_{tj_t}, \beta_{tj_t}, \theta_1\}$, we have $\alpha_{tj_t} \in C_1$. From Lemma 2.2, there is an $a \in [m] \setminus [n_s - 1]$ with $\beta_{tj_t} \in C_a$. For $p \in [t] \setminus \{1\}$, if $n_p + j \neq n_t + j_t$, then the $\mathcal{C}$-edges $\{\beta_1, \alpha_{tj_t}, \beta_{tj_t}\}$ and $\{\beta_{pj}, \alpha_{tj_t}, \beta_{tj_t}\}$ imply that $\beta_1, \beta_{pj} \in C_a$; moreover, from the $\mathcal{C}$-edge $\{\alpha_{pj}, \beta_{pj}, \theta_1\}$ and the $\mathcal{D}$-edge $\{\alpha_{pj}, \beta_{tj_t}\}$, we have $\alpha_{pj} \in C_1$, as desired.

(ii) If $c(\alpha_{tj_t}) \neq c(\beta_{tj_t})$ holds for some $t \in \{q, \ldots, s\}$ and $j_t \in \{0, l_t + 1, \ldots, n_{t-1} - n_t - 1\}$, then by (i) we have $c(\alpha_{pj}) \neq c(\beta_{pj})$ for all $p \in [t] \setminus \{1\}$. It follows that $c(\alpha_{qj_q}) \neq c(\beta_{qj_q})$, a contradiction. Hence, $c(\alpha_{pj}) = c(\beta_{pj})$ for $p \in \{q, \ldots, s\}$. Moreover, for $p \in \{q, \ldots, s\}$, from the $\mathcal{D}$-edges $\{\gamma^1_{pk}, \alpha_{p0}, \beta_{p0}\}$ and $\{\gamma^2_{pk}, \alpha_{p0}, \beta_{p0}\}$, we have $c(\gamma^1_{pk}), c(\gamma^2_{pk}) \neq c(\alpha_{p0})$; and the $\mathcal{C}$-edge $\{\gamma^1_{pk}, \gamma^2_{pk}, \alpha_{p0}\}$ implies that $c(\gamma^1_{pk}) = c(\gamma^2_{pk})$. □

Lemma 2.4 *Let $c = \{C_1, C_2, \ldots, C_m\}$ be a strict coloring of $\mathcal{H}$ satisfying that $c(\alpha_{t0}) \neq c(\beta_{t0})$ for some $t \in [s] \setminus \{1\}$. Then there exists an $a \in [m] \setminus [n_s - 1]$ such that*

(i) $\gamma^1_{tk}, \gamma^2_{tk} \in C_1 \cup C_a$ *and* $c(\gamma^1_{tk}) \neq c(\gamma^2_{tk})$;

(ii) $\gamma^1_{pk} \in C_1$ *and* $\gamma^2_{pk} \in C_a$ *for each* $p \in [t-1] \setminus \{1\}$.

Proof. For $p \in [t-1] \setminus \{1\}$, from Lemma 2.3, there is an $a \in [m] \setminus [n_s - 1]$ with $\alpha_{p0}, \alpha_{t0} \in C_1$ and $\beta_{p0}, \beta_{t0} \in C_a$. For $p \in [t] \setminus \{1\}$, since $\{\gamma^1_{pk}, \alpha_{p0}, \beta_{p0}\}$ and $\{\gamma^2_{pk}, \alpha_{p0}, \beta_{p0}\}$ are $\mathcal{C}$-edges, $\gamma^1_{pk}, \gamma^2_{pk} \in C_1 \cup C_a$; and the $\mathcal{D}$-edges $\{\gamma^1_{pk}, \gamma^2_{pk}, \beta_{p0}\}$ and $\{\gamma^1_{pk}, \gamma^2_{pk}, \alpha_{p0}\}$ imply that $c(\gamma^1_{pk}) \neq c(\gamma^2_{pk})$. Specially, $\gamma^1_{tk}, \gamma^2_{tk} \in C_1 \cup C_a$ and $c(\gamma^1_{tk}) \neq c(\gamma^2_{tk})$. Hence, (i) holds.

For $p \in [t-1] \setminus \{1\}$, from the $\mathcal{C}$-edge $\{\gamma^1_{pk}, \alpha_{t0}, \beta_{t0}\}$ and the $\mathcal{D}$-edge $\{\gamma^1_{pk}, \beta_{t0}\}$, one gets $\gamma^1_{pk} \in C_1$. Then by (i) we have $\gamma^2_{pk} \in C_a$, which implies that (ii) holds. □

Theorem 2.5 *$\mathcal{H}$ is a realization of R_2.*

Proof. It suffices to prove that $c_{11}, c_{21}, \ldots, c_{22^{l_2}}, \ldots, c_{s1}, \ldots, c_{s2^{l_s}}$ are all of the strict colorings of $\mathcal{H}$. Suppose $c = \{C_1, C_2, \ldots, C_m\}$ is a strict coloring of $\mathcal{H}$ satisfying the conditions (i)-(v) in Lemma 2.2. In particular, $\alpha_{s0} \in C_1 \cup C_{n_s}$.

Case 1 $\alpha_{s0} \in C_1$.

Note that $\beta_{s0} \in C_{n_s}$. By Lemma 2.3 we have $\alpha_{pj} \in C_1$ and $\beta_1, \beta_{pj} \in C_{n_s}$ for all $p \in [s] \setminus \{1\}$. Then by Lemma 2.4 one gets that

(i) $\gamma^1_{sk}, \gamma^2_{sk} \in C_1 \cup C_{n_s}$ and $c(\gamma^1_{sk}) \neq c(\gamma^2_{sk})$;

(ii) $\gamma^1_{pk} \in C_1$ and $\gamma^2_{pk} \in C_{n_s}$ for $p \in [s-1] \setminus \{1\}$.

It follows that $c \in \{c_{sh} \mid h \in [2^{l_s}]\}$.

Case 2 $\alpha_{s0} \in C_{n_s}$.

In this case, we shall prove that $c \in \{c_{ih} \mid i \in [s-1], h \in [2^{l_i}]\}$. Let $t \in [s] \setminus \{1\}$ be the minimum number such that $c(\alpha_{t0}) = c(\beta_{t0})$. By Lemma 2.3 we have $c(\alpha_{pj}) = c(\beta_{pj})$ and $c(\gamma^1_{pk}) = c(\gamma^2_{pk})$ for each $p \in \{t, \ldots, s\}$. For $p_1, p_2 \in \{t, \ldots, s\}$ and $p_1 \neq p_2$, from the $\mathcal{D}$-edges $\{\alpha_{p_1 j_1}, \beta_{p_2 j_2}\}$, $\{\alpha_{p_1 j_1}, \beta_{p_1 j_1}, \gamma^2_{p_2 k_2}\}$ and $\{\alpha_{p_2 j_2}, \beta_{p_2 j_2}, \gamma^1_{p_1 k_1}\}$, $\{\gamma^1_{p_1 k_1}, \gamma^1_{p_2 k_2}, \gamma^2_{p_2 k_2}\}$, one gets $c(\alpha_{p_1 j_1}) \neq c(\beta_{p_2 j_2})$, $c(\alpha_{p_1 j_1}) \neq c(\gamma^2_{p_2 k_2})$ and $c(\gamma^1_{p_1 k_1}) \neq c(\beta_{p_2 j_2})$, $c(\gamma^1_{p_1 k_1}) \neq c(\gamma^2_{p_2 k_2})$ for $j_i \in \{0, l_{p_i}+1, \ldots, n_{p_i-1} - n_{p_i} - 1\}$, $k_i \in [l_{p_i}]$, $i = 1, 2$. Moreover, for $p \in \{t, \ldots, s\}$, the $\mathcal{D}$-edge $\{\beta_{pj_1}, \alpha_{pj_2}\}$ implies that $c(\alpha_{pj_1}) \neq c(\alpha_{pj_2})$ if $j_1 \neq j_2$; the $\mathcal{D}$-edge $\{\alpha_{pj}, \beta_{pj}, \gamma^1_{pk}\}$ implies that $c(\alpha_{pj}) \neq c(\gamma^1_{pk})$; and the $\mathcal{D}$-edge $\{\gamma^1_{pk_1}, \gamma^1_{pk_1}, \gamma^2_{pk_2}\}$ implies that $c(\gamma^1_{pk_1}) \neq c(\gamma^1_{pk_2})$ if $k_1 \neq k_2$. Hence, we may assume that $\alpha_{pj}, \beta_{pj} \in C_{n_p+j}$ and $\gamma^1_{pk}, \gamma^2_{pk} \in C_{n_p+k}$ for $p \in \{t, \ldots, s\}$, $j \in \{0, l_p+1, \ldots, n_{p-1} - n_p - 1\}$ and $k \in [l_p]$.

Case 2.1 $t > 2$. That is to say, $c(\alpha_{t-1,0}) \neq c(\beta_{t-1,0})$. By Lemma 2.3 we have $\alpha_{t-1,0} \in C_1$ and $\beta_{t-1,0} \notin C_1$. For $p \in \{t, \ldots, s\}$, from the $\mathcal{D}$-edges $\{\beta_{t-1,0}, \alpha_{pj}\}$ and $\{\beta_{t-1,0}, \gamma^1_{pk}\}$, we have $\beta_{t-1,0} \notin C_{n_p+j} \cup C_{n_p+k}$. Hence, we may assume that $\beta_{t-1,0} \in C_{n_{t-1}}$. By Lemma 2.3 we have $\alpha_{pj} \in C_1$ and $\beta_1, \beta_{pj} \in C_{n_{t-1}}$ for $p \in [t-1] \setminus \{1\}$. Moreover, by Lemma 2.4 we have

(i) $\gamma^1_{t-1,k}, \gamma^2_{t-1,k} \in C_1 \cup C_{n_{t-1}}$ and $c(\gamma^1_{t-1,k}) \neq c(\gamma^2_{t-1,k})$;

(ii) $\gamma^1_{pk} \in C_1$ and $\gamma^2_{pk} \in C_{n_{t-1}}$ for any $p \in [t-2] \setminus \{1\}$.

Therefore, $c \in \{c_{t-1,h} \mid h \in [2^{l_{t-1}}]\}$.

Case 2.2 $t = 2$. From the $\mathcal{D}$-edges $\{\beta_1, \alpha_{pj}\}$ and $\{\beta_1, \gamma^1_{pk}, \gamma^2_{pk}\}$, we have $\beta_1 \notin C_{n_p+j} \cup C_{n_p+k}$ for all $p \in [s] \setminus \{1\}$. It follows that $c = c_{11}$.

The proof is completed. □

For the case of $n_1 = n_2 + 1$, we have the following construction.

Construction II. Let $X' = X \setminus \{\alpha_{20}\}$ and $\mathcal{H}' = \mathcal{H}[X']$. Then for each $i \in [s], h \in [2^{l_i}]$, $c'_{ih} = \{X'_{ih1}, X'_{ih2}, \ldots, X'_{ihn_i}\}$ is a strict n_i-coloring of $\mathcal{H}'$, where $X'_{iht} = X_{iht} \cap X'$.

Theorem 2.6 *If $n_1 = n_2 + 1$, then $\mathcal{H}'$ is a realization of $R_2 = (0, r_2, \ldots, r_{n_2-1}, 1, 1)$.*

Proof. Let $\mathcal{H}'' = (X'', \mathcal{C}'', \mathcal{D}'')$ be the realization of $R''_2 = (0, r_2, \ldots, r_{n_2-1}, 1)$ with $n''_1 = n_2$ given by Construction I. Let $Y = \{\alpha \mid \alpha \in X, \alpha_{(1)} = \alpha_{(2)}\}$, where $\alpha_{(j)}$ is the j^{th} entry of the vertex α. Then

$$\begin{array}{lll} \phi : & Y & \to X'' \\ (x^1_2, x^1_2, x^1_3, \ldots, x^{2^{l_3}}_3, \ldots, x^1_s, \ldots, x^{2^{l_s}}_s) & & \to (x^1_2, x^1_3, \ldots, x^{2^{l_3}}_3, \ldots, x^1_s, \ldots, x^{2^{l_s}}_s) \end{array}$$

is an isomorphism from $\mathcal{G} = \mathcal{H}[Y]$ to $\mathcal{H}''$. By Theorem 2.5, all of the strict colorings of $\mathcal{G}$ are as follows:

$$e_{ih} = \{Y_{ih1}, Y_{ih2}, \ldots, Y_{ihn_i}\}, i \in [s] \setminus \{1\}, h \in [2^{l_i}],$$

where Y_{iht} consists of vertices

$$(x_2^1, x_2^1, x_3^1, \ldots, x_3^{2^{l_3}}, \ldots, x_i^1, \ldots, x_i^{h-1}, t, x_i^{h+1}, \ldots, x_i^{2^{l_i}}, \ldots, x_s^1, \ldots, x_s^{2^{l_s}}) \in X.$$

Note that the restriction on Y of any strict coloring of $\mathcal{H}'$ corresponds to a strict coloring of $\mathcal{G}$. For any strict coloring $c = \{C_1, C_2, \ldots, C_m\}$ of $\mathcal{H}'$, there are the following two possible cases.

Case 1 $c|_Y = e_{21}$.

By the proof of Theorem 2.5, we have $\beta_1 \notin C_j$ for any $j \in [n_2 - 1]$. Then, it is immediate that $c = c'_{21}$ if $\beta_1 \in C_{n_2}$, or $c = c'_{11}$ if $\beta_1 \notin C_{n_2}$.

Case 2 $c|_Y = e_{ih}$ for some $i \in [s] \setminus \{1, 2\}$ and $h \in [2^{l_i}]$.

Note that $\beta_{i0} \in C_{n_i}$ and $\alpha_{i0} \in C_1$. From the $\mathcal{C}$-edge $\{\alpha_{i0}, \beta_{i0}, \beta_1\}$ and the $\mathcal{D}$-edge $\{\beta_1, \theta_1\}$, we observe that $\beta_1 \in C_{n_i}$. Therefore, $c = c'_{ih}$.

Hence, the desired result follows. □

Combining Lemma 2.1, Theorem 2.5 and Theorem 2.6, the proof of Theorem 1.1 is completed.

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References

[1] C. Bujtás, Zs. Tuza, Uniform mixed hypergraphs: the possible numbers of colors, Graphs and Combin. 24 (2008), 1-12.

[2] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. West, The chromatic spectrum of mixed hypergraphs, Graphs and Combin. 18 (2002), 309-318.

[3] D. Kobler and A. Kündgen, Gaps in the chromatic spectrum of face-constrained plane graphs, Electronic J. Combin. 8(1) (2001), ♯N3.

[4] D. Král, On feasible sets of mixed hypergraphs, Electronic J. Combin. 11 (2004), ♯R19.

[5] Zs. Tuza and V. Voloshin, Problems and results on colorings of mixed hypergraphs, Horizons of Combinatorics, Bolyai Society Mathematical Studies 17, Springer-Verlag, 2008, pp. 235-255.

[6] V. Voloshin, On the upper chromatic number of a hypergraph, Australasian J. Comb. 11 (1995), 25-45.

[7] V. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, AMS, Providence, 2002.

[8] V. Voloshin, Introduction to Graph and Hypergraphs Theory, Nova Scinece Publishers, Inc., New York, 2009.

[9] P. Zhao, K. Diao and K. Wang, The chromatic spectrum of 3-uniform bi-hypergraphs, Discrete Math. 311 (2011) 2650-2656.

[10] P. Zhao, K. Diao and K. Wang, The smallest one-realization of a given set, Electronic J. Combin. 19 (2012), #P19.