On three types of $(2, k)$-distance Fibonacci numbers and number decompositions

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Abstract

In this paper we define new generalizations of Fibonacci numbers and Lucas numbers in the distance sense. These generalizations are closely related to the concept of $(2, k)$-distance Fibonacci numbers presented in [10]. We show some applications of these numbers in number decompositions and we also define a new type of Lucas numbers.

Keywords: generalized Fibonacci numbers, decomposition of a number, generalized Lucas numbers, distance Fibonacci numbers

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1 Introduction

The well-known Fibonacci numbers F_n are defined for $n \geq 2$ by recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial terms $F_0 = F_1 = 1$. The Lucas numbers L_n are defined by the same recurrence $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. There are many various generalizations of the Fibonacci numbers in the literature. Some types of generalizations of the Fibonacci numbers are called generalizations in the distance sense. For an arbitrary integer k, $k \geq 1$, the nth generalized Fibonacci number is defined recursively by adding two previous terms such that exactly one of these numbers is the $(n-k)$th generalized Fibonacci number. Then the second term is taken in such way that the recurrence linear equation generalizes the Fibonacci numbers in classical sense. Some types of distance Fibonacci numbers were introduced and studied quite recently, for example

1. [6] $F(k,n) = F(k,n-1)+F(k,n-k)$ for $n \geq k+1$ and $F(k,n) = n+1$ for $n \leq k$,

2. [2] $Fd(k,n) = Fd(k,n-k+1)+Fd(k,n-k)$ for $n \geq k$ and $Fd(k,n) = 1$ for $n \leq k-1$,

3. [10] $F_2^{(1)}(k,n) = F_2^{(1)}(k,n-2) + F_2^{(1)}(k,n-k)$ for $n \geq k$ and $F_2^{(1)}(k,n) = 1$ for $n \leq k-1$.

Another interesting generalizations of Fibonacci numbers can be found in [3, 4, 5, 8, 9]. Our paper is a sequel of papers [1] and [10]. We introduce two new kinds of $(2,k)$-distance Fibonacci numbers and we show that they are closely related to special number decompositions and there are interesting relations between them.

Let $k \geq 1$, $n \geq 0$ be integers, $j = 1,2,3$. Then $(2,k)$-distance Fibonacci numbers of the jth kind $F_2^{(j)}(k,n)$ we define in the following way

$$F_2^{(j)}(k,n) = F_2^{(j)}(k,n-2) + F_2^{(j)}(k,n-k) \text{ for } n \geq k+1 \tag{1}$$

and

$$F_2^{(1)}(k,n) = \begin{cases} 1 & \text{if } n \leq k-1 \quad \text{or } n = k = 1, \\ 2 & \text{if } n = k \geq 2, \end{cases}$$

$$F_2^{(2)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is odd} \quad \text{and } n \leq k-1, \\ 1 & \text{if } n \text{ is even} \quad \text{and } n \leq k-1, \end{cases}$$

$$F_2^{(2)}(k,k) = \begin{cases} 0 & \text{if } k = 1, \\ 1 & \text{if } k \text{ is odd} \quad \text{and } k \geq 3, \\ 2 & \text{if } k \text{ is even}, \end{cases}$$

$$F_2^{(3)}(k,n) = \begin{cases} 1 & \text{if } n \text{ is even} \quad \text{and } n \leq k-1, \\ 2 & \text{if } n \text{ is odd} \quad \text{and } n \leq k-1, \end{cases}$$

$$F_2^{(3)}(k,k) = \begin{cases} 3 & \text{if } k \text{ is odd} \quad \text{and} \quad k \geq 3, \\ 2 & \text{if } k \text{ is even} \quad \text{or} \quad k = 1. \end{cases}$$

Tables 1,2,3 present the initial numbers $F_2^{(j)}(k,n)$ for some values k and n.

Tab.1. $(2,k)$-distance Fibonacci numbers $F_2^{(1)}(k,n)$ of the first kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_2^{(1)}(1,n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377
$F_2^{(1)}(2,n)$	1	1	2	2	4	4	8	8	16	16	32	32	64	64
$F_2^{(1)}(3,n)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28
$F_2^{(1)}(4,n)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13
$F_2^{(1)}(5,n)$	1	1	1	1	1	2	2	3	3	4	5	6	8	9
$F_2^{(1)}(6,n)$	1	1	1	1	1	1	2	2	3	3	4	4	6	6

Tab.2. $(2,k)$-distance Fibonacci numbers $F_2^{(2)}(k,n)$ of the second kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2^{(2)}(1,n)$	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233
$F_2^{(2)}(2,n)$	1	0	2	0	4	0	8	0	16	0	32	0	64	0	128
$F_2^{(2)}(3,n)$	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21
$F_2^{(2)}(4,n)$	1	0	1	0	2	0	3	0	5	0	8	0	13	0	21
$F_2^{(2)}(5,n)$	1	0	1	0	1	1	1	2	1	3	2	4	4	5	7
$F_2^{(2)}(6,n)$	1	0	1	0	1	0	2	0	3	0	4	0	6	0	9

Tab.3. $(2,k)$-distance Fibonacci numbers $F_2^{(3)}(k,n)$ of the third kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$F_2^{(3)}(1,n)$	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_2^{(3)}(2,n)$	1	2	2	4	4	8	8	16	16	32	32	64	64	128
$F_2^{(3)}(3,n)$	1	2	1	3	3	4	6	7	10	13	17	23	30	40
$F_2^{(3)}(4,n)$	1	2	1	2	2	4	3	6	5	10	8	16	13	26
$F_2^{(3)}(5,n)$	1	2	1	2	1	3	3	4	5	5	8	8	12	13
$F_2^{(3)}(6,n)$	1	2	1	2	1	2	2	4	3	6	4	8	6	12

We recall that numbers $F_2^{(1)}(1,n)$ are the classical Fibonacci numbers F_n. For $k=2$ we obtain known sequence with powers of 2 which double up. If $k=3$ then $\{F_2^{(1)}(3,n)\}$ is the well-known Padovan sequence $\{Pv(n)\}$ defined by the recurrence relation $Pv(n)=Pv(n-2)+Pv(n-3)$ for $n\geq 3$ with $Pv(0)=Pv(1)=Pv(2)=1$.

For introduced in this paper numbers $F_2^{(2)}(k,n)$ and $F_2^{(3)}(k,n)$ it is easily seen that $F_2^{(2)}(1,n)=F_{n-2}$ for $n\geq 3$ and $F_2^{(2)}(4,2n)=F_n$ for $n\geq 1$. The numbers $F_2^{(2)}(4,2n)$ for $n\geq 1$ are Fibonacci numbers interspersed with zeros ([7]). Moreover, $F_2^{(2)}(3,n)=Pv(n)$.

By the definition of numbers $F_2^{(3)}(k,n)$, we obtain that $F_2^{(3)}(1,n)=F_{n+1}$ and for $n\geq 1$ $F_2^{(3)}(4,2n)=F_n$. Moreover, by the definition of numbers $F_2^{(j)}(k,n)$, $j=1,2,3$ we get for $k\geq 1$ and $n\geq 0$ the following relations

$$\begin{aligned} F_2^{(3)}(k,n) &= F_2^{(1)}(k,n) \text{ for even } n \text{ and even } k, \\ F_2^{(3)}(k,n) &= 2F_2^{(1)}(k,n) \text{ for odd } n \text{ and even } k, \\ F_2^{(2)}(k,2n) &= F_2^{(3)}(k,2n) \text{ for even } k, \\ F_2^{(2)}(k,2n+1) &= 0 \text{ for even } k. \end{aligned}$$

2 Interpretations of numbers $F_2^{(j)}(k,n)$ for $j = 1,2,3$

In this section we present some interpretations of $(2,k)$-distance Fibonacci numbers with respect to special number decompositions. In [10] the combinatorial interpretations of the number $F_2^{(1)}(k,n)$ with respect to special number decompositions of an integer n into parts 2 and k were studied. We shall show combinatorial interpretations for the other kinds of $(2,k)$-distance Fibonacci numbers which are closely related to number decompositions. We recall that by a decomposition of an integer n we mean an ordered number partition, for example $2+3$ and $3+2$ are two distinct decomposition of an integer 5 into parts 2 and 3.

We say that a sum $n_1+n_2+\ldots+n_t$ is *a* $(2,k)-$*decomposition of the number* n if $n_i \in \{2,k\}$ for $i=1,\ldots,t$. The number of all $(2,k)-$decompositions we will denote by $\sigma_0(k,n)$. A sum $1+n_1+n_2+\ldots+n_t$ (or $n_1+n_2+\ldots+n_t+1$) is *a* $(2,k)_{1^-}-$*decomposition*, $(2,k)_{1^+}-$*decomposition*, respectively, *of the number* n if $n_i \in \{2,k\}$ for $i=1,\ldots,t$. The number of all $(2,k)_{1^-}-$decompositions ($(2,k)_{1^+}-$decompositions) we will denote by $\sigma_{1^-}(k,n)$ ($\sigma_{1^+}(k,n)$, respectively). Clearly,

$$\sigma_{1^+}(k,n) = \sigma_{1^-}(k,n). \tag{2}$$

Let $\sigma_{\geq 1^-}(k,n)$ be the number of all $(2,k)-$decompositions and $(2,k)_{1^-}-$decompositions of the number n. In the other words

$$\sigma_{\geq 1^-}(k,n) = \sigma_{1^-}(k,n) + \sigma_0(k,n) \tag{3}$$

and analogously

$$\sigma_{\geq 1^+}(k,n) = \sigma_{1^+}(k,n) + \sigma_0(k,n). \tag{4}$$

Consequently,

$$\sigma_{\geq 1^-}(k,n) = \sigma_{\geq 1^+}(k,n). \tag{5}$$

We will use the following notation

$$\sigma(k,n) = \sigma_{1^-}(k,n) + \sigma_0(k,n) + \sigma_{1^+}(k,n). \tag{6}$$

It has been proved.

Theorem 1 [10] *Let $k \geq 3$, $n \geq 2$ be integers. Then $\sigma_{\geq 1^+}(k,n) = F_2^{(1)}(k,n)$.*

In this paper we give more general result for all three kinds of $(2,k)$-distance Fibonacci numbers.

Theorem 2 *Let $k \geq 3$, $n \geq 2$ be integers. Then*

$$\begin{aligned} \sigma_{\geq 1^-}(k,n) &= \sigma_{\geq 1^+}(k,n) = F_2^{(1)}(k,n), & (7)\\ \sigma_0(k,n) &= F_2^{(2)}(k,n), & (8)\\ \sigma(k,n) &= F_2^{(3)}(k,n). & (9) \end{aligned}$$

Proof. The equality (7) follows immediately from (5) and by Theorem 1. We shall show that $\sigma_0(k,n) = F_2^{(2)}(k,n)$. Let $n = 1,2,\ldots,k-1$. If $n = 2p$, $p \geq 1$, then there is a unique $(2,k)$-decomposition of the number n of the form $n = n_1 + n_2 + \ldots + n_p$, where $n_i = 2$ for $i = 1,\ldots,p$. Hence $\sigma_0(k,n) = 1$. If $n = 2p+1$ for $p \geq 0$, then there is no a $(2,k)$-decomposition of the number n on terms 2 and k, thus $\sigma_0(k,n) = 0$. For $n = k$ we have to distinguish two possibilities. If k is even then either $n = k$ or $n = n_1 + n_2 + \ldots + n_t$, where $n_i = 2$ for $i = 1,\ldots,t$. Hence $\sigma_0(k,k) = 2$. If k is odd then there is the unique $(2,k)$-decomposition of the number n.

Let $n \geq k+1$. Assume that the equality (8) is true for an arbitrary number n. We shall show that $\sigma_0(k,n+1) = F_2^{(2)}(k,n+1)$. Let $n+1 = n_1 + n_2 + \ldots + n_p$ be a $(2,k)$-decomposition of the number $n+1$. We consider two possibilities either $n_p = k$ or $n_p = 2$. In each case we obtain a $(2,k)$-decomposition either of the number $n-1$ or $n+1-k$. Consequently $\sigma_0(k,n+1) = \sigma_0(k,n-1) + \sigma_0(k,n+1-k)$. Using the induction hypothesis and the definition of $F_2^{(2)}(k,n)$, we obtain that

$$\sigma_0(k,n+1) = F_2^{(2)}(k,n-1) + F_2^{(2)}(k,n+1-k) = F_2^{(2)}(k,n+1),$$

which ends the proof of (8).

Now we shall show that $\sigma(k,n) = F_2^{(3)}(k,n)$. By (2), we have that

$$\sigma(k,n) = \sigma_{1^-}(k,n) + \sigma_0(k,n) + \sigma_{1^+}(k,n)$$

and next by (2) and the above point of this theorem we get

$$\begin{aligned} \sigma(k,n) &= \sigma_{\geq 1^-}(k,n) - \sigma_0(k,n) + \sigma_0(k,n) + \sigma_{\geq 1^+}(k,n) - \sigma_0(k,n) = \\ &= 2F_2^{(1)}(k,n) - F_2^{(2)}(k,n). \end{aligned}$$

We need to prove
Claim.

$$2F_2^{(1)}(k,n) - F_2^{(2)}(k,n) = F_2^{(3)}(k,n). \tag{10}$$

Proof (by induction on n). If $n = k$ then the result is obvious. Let $n > k$. Assume that the formula (10) is true for an arbitrary n. We will prove it for $n+1$. By the recurrence definitions of the numbers $F_2^{(j)}(k,n)$, $j = 1,2,3$ and by the induction hypothesis, we obtain

$$\begin{aligned} & F_2^{(3)}(k,n+1) = F_2^{(3)}(k,n-1) + F_2^{(3)}(k,n+1-k) = \\ = \; & 2F_2^{(1)}(k,n-1) - F_2^{(2)}(k,n-1) + 2F_2^{(1)}(k,n+1-k) + \\ - \; & F_2^{(2)}(k,n+1-k) = 2F_2^{(1)}(k,n+1) - F_2^{(2)}(k,n+1), \end{aligned}$$

which completes the proof. □

Corollary 3 *Let $k \geq 1$, $n \geq 0$ be integers. Then*

$$F_2^{(2)}(k,n) + F_2^{(3)}(k,n) = 2F_2^{(1)}(k,n).$$

Let $\sigma_{\pm 1}(k,n)$ be the number of all decompositions of the number $n = a_1 + a_2 + \ldots + a_p$, where $a_i \in \{2,k\}$ for $i = 2,\ldots,p-1$ and $a_1, a_p \in \{1,2,k\}$.

Theorem 4 *Let $k \geq 3$ and $n \geq k$ be integers. Then*

$$\sigma_{\pm 1}(k,n) = F_2^{(1)}(k,n-1) + F_2^{(1)}(k,n-2) + F_2^{(1)}(k,n-k).$$

Proof. Let $k \geq 3$ and $n \geq k$ be integers and let $n = a_1 + a_2 + \ldots + a_p$ be the decomposition of the number n such that $a_i \in \{2,k\}$ for $i = 2,\ldots,p-1$ and $a_1, a_p \in \{1,2,k\}$. Then we can write

$$n - a_p = a_1 + a_2 + \ldots + a_{p-1}.$$

It is clear that $a_1 + a_2 + \ldots + a_{p-1}$ is either $(2,k)_1$-−decomposition or $(2,k)$−decomposition of the number $n - a_p$. By the assumption that $a_p \in \{1,2,k\}$ and by (7), we obtain that

$$\sigma_{\pm 1}(k,n) = F_2^{(1)}(k,n-1) + F_2^{(1)}(k,n-2) + F_2^{(1)}(k,n-k)$$

and the proof is complete. □

Consider another type of the decomposition of the number n. For $k \geq 3$ and $n \geq k$ let $n = a_1 + a_2 + \ldots + a_p$ be the decomposition of n such that $a_1 = a_p = 1$ and $a_i \in \{2,k\}$ for $i = 2,\ldots,p-1$. Denote by $\overline{\sigma}_{\pm 1}(k,n)$ the number of all such decompositions of n.

Theorem 5 *Let $k \geq 3$, $n \geq k$ be integers. Then*

$$\overline{\sigma}_{\pm 1}(k,n) = F_2^{(2)}(k,n-2). \tag{11}$$

Proof. Let $k \geq 3$ and $n \geq k$ be integers and let $n = 1 + a_2 + \ldots + a_{p-1} + 1$, where $a_i \in \{2,k\}$ for $i = 2,\ldots,p-1$. Hence we can write

$$n - 2 = a_1 + a_2 + \ldots + a_{p-1}.$$

It is easily seen that it is a $(2,k)$-decomposition of the number $n-2$. By Theorem 2 we obtain that

$$\overline{\sigma}_{\pm 1}(k,n) = F_2^{(2)}(k,n-2),$$

which ends the proof. □

Corollary 6 *Let $k \geq 3$, $n \geq k$ be integers. Then*

$$F_2^{(2)}(k,n-2) = F_2^{(1)}(k,n-1) - F_2^{(2)}(k,n-1).$$

Proof. By the decomposition of n we obtain

$$n - 1 = 1 + a_2 + \ldots + a_{p-1}$$

and it is a $(2,k)_{1^-}$-decomposition of the number $n-1$. Hence by (4) and Theorem 2 we obtain that

$$\overline{\sigma}_{\pm 1}(k,n) = F_2^{(1)}(k,n-1) - F_2^{(2)}(k,n-1). \tag{12}$$

Using both formulas (11) and (12), we obtain that

$$F_2^{(2)}(k,n-2) = F_2^{(1)}(k,n-1) - F_2^{(2)}(k,n-1).$$

□

Analogously as classical Fibonacci numbers F_n, the $(2,k)$-distance Fibonacci numbers $F_2^{(j)}(k,n)$ can be extended to negative integers n. Let $k \geq 3$, $n \geq 1$ be integers. Then for $j = 1,2,3$

$$F_2^{(j)}(k,-n) = F_2^{(j)}(k,k-n) - F_2^{(j)}(k,k-2-n) \text{ for } n \geq 1$$

with initial conditions

$$\begin{aligned} F_2^{(j)}(k,0) &= 1 \text{ for } j = 1,2,3, \\ F_2^{(1)}(k,1) &= 1, \\ F_2^{(2)}(k,1) &= 0, \\ F_2^{(3)}(k,1) &= 2. \end{aligned}$$

Moreover, for $n \geq 1$

$$\begin{aligned} F_2^{(2)}(1,-n) &= (-1)^n F_2^{(2)}(1,n+2), \\ F_2^{(3)}(1,-n) &= (-1)^{n+1} F_2^{(3)}(1,n-4). \end{aligned}$$

Tables 4,5 includes initial $(2,k)$-distance Fibonacci numbers of the second and of the third kind for special k and some negative n.

Tab.4. $(2,k)$-distance Fibonacci numbers of the second kind for negative n

n	-7	-6	-5	-4	-3	-2	-1	0	1
$F_2^{(2)}(1,n)$	-21	13	-8	5	-3	2	-1	1	0
$F_2^{(2)}(3,n)$	-1	0	1	-1	1	0	0	1	0
$F_2^{(2)}(4,n)$	0	-1	0	1	0	0	0	1	0
$F_2^{(2)}(5,n)$	0	0	1	0	0	0	0	1	0

Tab.5. $(2,k)$-distance Fibonacci numbers of the third kind for negative n

n	-7	-6	-5	-4	-3	-2	-1	0	1
$F_2^{(3)}(1,n)$	5	-3	2	-1	1	0	1	1	2
$F_2^{(3)}(3,n)$	3	-2	1	1	-1	2	0	1	2
$F_2^{(3)}(4,n)$	4	-1	-2	1	2	0	0	1	2
$F_2^{(3)}(5,n)$	-2	0	1	2	0	0	0	1	2

3 Identities and relations between $(2,k)$-distance Fibonacci numbers

In this section we present some identities for $(2,k)$-distance Fibonacci numbers of the second and the third kind. Moreover, we show some relations between $(2,k)$-distance Fibonacci numbers of three kinds.

Theorem 7 *Let $n \geq 0$, $k \geq 3$ be integers. Then for $j = 2,3$*

$$\sum_{i=0}^{n} F_2^{(j)}(k,ki+m) = F_2^{(j)}(k,nk+m+2) \text{ for } 0 \leq m \leq k-3. \qquad (13)$$

Proof (by induction on n). We give the proof for $j = 3$ and $m = 0$. Analogously we can prove formula (13) for $j = 2$ and remaining values of m.
For $n = 0$ we have $F_2^{(3)}(k,0) = 1 = F_2^{(3)}(k,2)$. Assume that equality (13) holds for an arbitrary $n \geq 0$. We will prove it for $n+1$. By the induction hypothesis and the definition of $F_2^{(3)}(k,n)$, we get

$$\begin{aligned} &\sum_{i=1}^{n+1} F_2^{(3)}(k,ki) = \sum_{i=0}^{n} F_2^{(3)}(k,ki) + F_2^{(3)}(k,nk+k) = \\ = \;& F_2^{(3)}(k,nk+2) + F_2^{(3)}(k,nk+k) = F_2^{(3)}(k,nk+k+2), \end{aligned}$$

which ends the proof. □

Theorem 8 *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$(i) \sum_{i=0}^{n} F_2^{(2)}(k, ki+m) = F_2^{(2)}(k, nk+m+2) \text{ for } m = k-1,$$

$$(ii) \sum_{i=0}^{n} F_2^{(3)}(k, ki+m) = F_2^{(3)}(k, nk+m+2) - 2 \text{ for } m = k-1,$$

$$(iii) \sum_{i=0}^{n} F_2^{(2)}(k, ki+m) = F_2^{(j)}(k, nk+m+2) - 1 \text{ for } m = k-2 \text{ or } m = k.$$

This theorem we can prove analogously as Theorem 7, so we omit the proof.

Theorem 9 *For $k \geq 1$, $n \geq 2k-2$ and $j = 2, 3$*

$$F_2^{(j)}(k, n) = F_2^{(j)}(k, n-2) + F_2^{(j)}(k, n-k+2) - F_2^{(j)}(k, n-2k+2).$$

Proof. We give the proof for $j = 3$. By the definition of numbers $F_2^{(3)}(k, n)$, we have

$$F_2^{(3)}(k, n-2) + F_2^{(3)}(k, n-k+2) - F_2^{(3)}(k, n-2k+2) =$$
$$= F_2^{(3)}(k, n-2) + F_2^{(3)}(k, n-k) + F_2^{(3)}(k, n-2k+2)$$
$$-F_2^{(3)}(k, n-2k+2) = F_2^{(3)}(k, n-2) + F_2^{(3)}(k, n-k) = F_2^{(3)}(k, n).$$

For $j = 2$ we prove analogously. □

Theorem 10 *For $n \geq 0$ and odd $k \geq 1$*

(i) $\sum_{i=0}^{n} F_2^{(2)}(k, 2i) = F_2^{(2)}(k, k+2n)$,

(ii) $\sum_{i=0}^{n} F_2^{(2)}(k, 2i+1) = F_2^{(2)}(k, k+2n+1)$.

Corollary 11 *For $n \geq 0$ and odd $k \geq 1$*

(i) $\sum_{i=0}^{2n} F_2^{(2)}(k, i) = F_2^{(2)}(k, k+2n-1) + F_2^{(2)}(k, k+2n) - 1$,

(ii) $\sum_{i=0}^{2n+1} F_2^{(2)}(k, i) = F_2^{(2)}(k, k+2n) + F_2^{(2)}(k, k+2n+1) - 1$.

Theorem 12 *For $n \geq 0$ and even $k \geq 1$*

(i) $\sum_{i=0}^{n} F_2^{(2)}(k, 2i) = F_2^{(2)}(k, k+2n) - 1$,

(ii) $\sum_{i=0}^{n} F_2^{(2)}(k, 2i+1) = 0$.

Analogously as it was for positive numbers, we can prove by induction the following identities for $(2,k)$-distance Fibonacci numbers for negative integers.

Theorem 13 *Let $n \geq 0$. Then*

(i) $\sum_{i=1}^{n} F_2^{(j)}(k,-ki+m) = -F_2^{(j)}(k,-nk-k+2+m)$ *for* $j=2,3$ *and* $0 \leq m \leq k-3$,

(ii) $\sum_{i=1}^{n} F_2^{(j)}(k,-ki+m) = -F_2^{(j)}(k,-nk-k+2+m)+1$ *for* $j=2,3$ *and* $m=k-2$ *or* $m=k$,

(iii) $\sum_{i=1}^{n} F_2^{(3)}(k,-ki+m) = -F_2^{(3)}(k,-nk-k+2+m)+2$ *for* $m=k-1$,

(iv) $\sum_{i=1}^{n} F_2^{(2)}(k,-ki+m) = -F_2^{(2)}(k,-nk-k+2+m)$ *for* $m=k-1$.

Theorem 14 *Let $k \geq 3$, $n \geq k$ be integers. Then*

$$\sum_{j=1}^{3} a_j F_2^{(j)}(k,n) = \sum_{i=0}^{p} \binom{p}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,n-ik-2(p-i)) \right). \qquad (14)$$

Proof (by induction on n). Let $n=k$. Then $p=1$ and by the definition of numbers $F_2^{(j)}(k,n)$ we have

$$\sum_{j=1}^{3} a_j F_2^{(j)}(k,k) = \sum_{j=1}^{3} a_j (F_2^{(j)}(k,k-2) + F_2^{(j)}(k,0)).$$

On the other hand

$$\sum_{i=0}^{1} \binom{1}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,k-ik-2(1-i)) \right) = \sum_{j=1}^{3} a_j \left(F_2^{(j)}(k,k-2) + F_2^{(j)}(k,0) \right)$$

Assume that the equality (14) holds for an arbitrary n. We will prove that

$$\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1) = \sum_{i=0}^{p} \binom{p}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1-ik-2(p-i)) \right).$$

Using the definition of the numbers $F_2^{(j)}(k,n)$, we have

$$\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1) = \sum_{j=1}^{3} a_j F_2^{(j)}(k,n-1) + \sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1-k).$$

Using the recurrence relation and the induction hypothesis we obtain

$$\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1) = \sum_{i=0}^{p} \binom{p}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,n-1-ik-2(p-i)) \right) +$$

$$+\sum_{i=0}^{p} \binom{p}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1-k-ik-2(p-i)) \right) =$$

$$= \sum_{i=0}^{p} \binom{p}{i} \left(\sum_{j=1}^{3} a_j F_2^{(j)}(k,n+1-ik-2(p-i)) \right),$$

which ends the proof. □

Corollary 15 *For* $j = 1,2,3$

$$F_2^{(j)}(k,n) = \sum_{i=0}^{p} \binom{p}{i} F_2^{(j)}(k,n-ik-2(p-i)). \tag{15}$$

Proof. Putting $a_1 = 1$ and $a_2 = a_3 = 0$ in formula (14), we obtain equality (15). □

4 $(2,k)$-distance Lucas numbers

In this section we introduce a new generalization of Lucas numbers which is closely related to the $(2,k)$-distance Fibonacci numbers. This is a sequel of the paper [1]. Let $k \geq 1$, $n \geq 0$ be integers, $j = 1,2,3$. Then $(2,k)$-distance Lucas numbers $L_2^{(j)}(k,n)$ of the jth kind we define by the recurrence relation

$$L_2^{(j)}(k,n) = L_2^{(j)}(k,n-2) + L_2^{(j)}(k,n-k) \text{ for } n \geq k+2 \tag{16}$$

with initial conditions

$$\begin{aligned} L_2^{(1)}(k,0) &= 2 \text{ for } k = 1,2,3, \\ L_2^{(1)}(k,0) &= k \text{ for } k \geq 4, \\ L_2^{(1)}(k,1) &= k \text{ for } k \geq 1, \\ L_2^{(1)}(k,n) &= 2 \text{ for } n = 2,3,\ldots,k-1, \end{aligned}$$

$$L_2^{(1)}(k,k) = \begin{cases} k & \text{if } k = 1, \\ k+2 & \text{if } k \neq 1, \end{cases}$$

$$\text{for } k \geq 1 \quad L_2^{(1)}(k, k+1) = k+2,$$

$$\text{for } k \geq 2 \qquad L_2^{(2)}(k,0) = k, \quad L_2^{(2)}(k,1) = 0,$$
$$L_2^{(2)}(1,0) = 2,$$

$$L_2^{(2)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is odd} \quad \text{and } 3 \leq n \leq k-1, \\ 2 & \text{if } n \text{ is even} \quad \text{and } 2 \leq n \leq k-1, \end{cases}$$

$$L_2^{(2)}(k,k) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k+2 & \text{if } k \text{ is even}, \end{cases}$$

$$L_2^{(2)}(k,k+1) = \begin{cases} 0 & \text{if } k \text{ is even}, \\ 2 & \text{if } k \text{ is odd and } k \neq 1, \\ 3 & \text{if } k = 1, \end{cases}$$

$$\begin{aligned} L_2^{(3)}(k,0) &= 2 \quad \text{for } k \neq 3, \\ L_2^{(3)}(3,0) &= 1, \\ L_2^{(3)}(k,1) &= 2 \quad \text{for } k \neq 1. \\ L_2^{(3)}(k,n) &= 1 \quad \text{for } 2 < n \leq k-1, \end{aligned}$$

$$L_2^{(3)}(k,k) = \begin{cases} 1 & \text{if } k = 1, \\ 4 & \text{if } k = 2, \\ 3 & \text{if } k \geq 3, \end{cases}$$

$$L_2^{(3)}(k,k+1) = \begin{cases} 3 & \text{if } k \neq 2, \\ 4 & \text{if } k = 2. \end{cases}$$

In the paper [1] they were introduced numbers $L_2^{(1)}(k,n)$ and $L_2^{(3)}(k,n)$. Table 6 includes initial words of $(2,k)$-distance Lucas numbers of the second kind for special k and n.

Tab.6. $(2,k)$-distance Lucas numbers of the second kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L_2^{(2)}(1,n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521
$L_2^{(2)}(2,n)$	2	0	4	0	8	0	16	0	32	0	64	0	128	0
$L_2^{(2)}(3,n)$	3	0	2	3	2	5	5	7	10	12	17	22	29	39
$L_2^{(2)}(4,n)$	4	0	2	0	6	0	8	0	14	0	22	0	36	0
$L_2^{(2)}(5,n)$	5	0	2	0	2	5	2	7	2	9	7	11	14	13
$L_2^{(2)}(6,n)$	6	0	2	0	2	0	8	0	10	0	12	0	20	0

Note that for $k=1$ we have $L_2^{(2)}(1,n)=L_n$.

By the definition of the numbers $L_2^{(1)}(k,n)$ and $L_2^{(2)}(k,n)$ we can observe the following relations.

Let $k \geq 2$, $n \geq 0$ be integers. Then for even k and even n

$$L_2^{(1)}(k,n)=L_2^{(2)}(k,n).$$

Let $k \geq 3$, $n \geq 0$ be integers. Then for odd k

$$L_2^{(1)}(k,n)=L_2^{(2)}(k,n+2).$$

Theorem 16 *Let $k \geq 2$, $n \geq k$ be integers. Then*

$$L_2^{(2)}(k,n)=2F_2^{(2)}(k,n-2)+kF_2^{(2)}(k,n-k). \tag{17}$$

Proof (by induction on n). Let $n=k$. Then by the definitions of $L_2^{(2)}(k,n)$ and $F_2^{(2)}(k,n)$, for odd k we have

$$L_2^{(2)}(k,k)=k=2F_2^{(2)}(k,k-2)+kF_2^{(2)}(k,0).$$

For even k we obtain

$$L_2^{(2)}(k,k)=2+k=2F_2^{(2)}(k,k-2)+kF_2^{(2)}(k,0).$$

Assume that the formula (17) holds for an arbitrary n. We will prove it for $n+1$. By the recurrence definitions of the numbers $L_2^{(2)}(k,n)$ and $F_2^{(2)}(k,n)$ and by the induction hypothesis, we have

$$\begin{aligned}
& L_2^{(2)}(k,n+1)=L_2^{(2)}(k,n-1)+L_2^{(2)}(k,n+1-k)= \\
= \; & 2F_2^{(2)}(k,n-3)+kF_2^{(2)}(k,n-k-1)+2F_2^{(2)}(k,n-1-k)+ \\
+ \; & kF_2^{(2)}(k,n+1-2k)=2F_2^{(2)}(k,n-1)+kF_2^{(2)}(k,n+1-k),
\end{aligned}$$

which ends the proof. □

Theorem 17 *Let $n \geq 0$. Then*

(*i*) $\sum\limits_{i=0}^{n} L_2^{(2)}(k,2i) = \begin{cases} L_2^{(2)}(k,2n+k)-2, & \text{if } k \text{ is even} \\ L_2^{(2)}(k,2n+k), & \text{if } k \text{ is odd}, k \geq 3, \end{cases}$

(*ii*) *for* $k \geq 3$

$$\sum_{i=1}^{n} L_2^{(2)}(k,ki) = L_2^{(2)}(k,kn+2)-2.$$

Proof. (*i*) (by induction on n). We prove formula (*i*) for even k. It is easy to check that (*i*) holds for $n=0$. Assume that equality (*i*) is true for an arbitrary n. We shall prove that it is true for $n+1$. By the induction hypothesis and the definition of numbers $L_2^{(2)}(k,n)$, we have

$$\begin{aligned} &\sum_{i=0}^{n+1} L_2^{(2)}(k,2i) = \sum_{i=0}^{n} L_2^{(2)}(k,2i) + L_2^{(2)}(k,2n+2) = \\ =\ & L_2^{(2)}(k,2n+k)-2+L_2^{(2)}(k,2n+2) = L_2^{(2)}(k,2n+2+k)-2, \end{aligned}$$

which ends the proof.
Similarly we can prove (*i*) for odd k, $k \geq 3$.
(*ii*) Analogously as in (*i*). □

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