The Hamiltonicity of k-Connected $[s,t]$-Graphs [1]

Jianglu Wang[2], Lei Mou[3]

School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China

Abstract

A graph G is a $[s,t]$-graph, if there are at least t edges in every included subgraph by s vertices of G. The concept of $[s,t]$-graph is the extension of independent number. In this paper, we prove that

(1) if G is a k-connected $[k+2,2]$-graph, then G has a Hamilton cycle or G is isomorphic to Petersen graph or to $\overline{K_{k+1}} \vee G_k$,

(2) if G is a k-connected $[k+3,2]$-graph, then G has a Hamilton path or G is isomorphic to $\overline{K_{k+2}} \vee G_k$,

where G_k is an arbitrary graph of order k.

This two results include the following known results obtained by *Chvatal-Erdös* and *Bondy*, respectively.

For any graph G of order $n \geq 3$,

(a) if $\alpha(G) \leq \kappa(G)$, then G has a Hamilton cycle.

(b) if $\alpha(G) - 1 \leq \kappa(G)$, then G has a Hamilton path.

Keywords: $[s,t]$-graphs; Hamilton paths(cycles); $k-$connected graphs

1. Introduction and notation

In this paper, we will consider only finite undirected graphs without loops and multiple edges. For notations and terminology not defined here we refer to [?]. Throughout this paper, let G be a graph and $V(G), E(G)$ denote the vertex set and the edge set of G, respectively. For any $a \in V(G)$, $S, T \subset V(G)$ and any subgraph H of G, we put

$$N_T(a) = \{u \in V(T) : ua \in E(G)\},$$

$$N_H(a) = N_{V(H)}(a), \quad N(a) = N_G(a),$$

$$N_T(S) = \bigcup_{v \in S} N_T(v), \quad N_T(H) = N_T(V(H)), \quad N_H(S) = N_{V(H)}(S),$$

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[2]E-mail address: yzhfzh@sina.com

[3]E-mail address: mouleili8@126.com

$$|H| = |V(H)|, \quad E(S,T) = \{uv : u \in S, v \in T\}.$$

The subgraph induced in G on S will be denoted by $G[S]$. If $E(G[S]) = \phi$, S is called an independent set of G. The number of vertices in a maximum independent set of G is called independent number of G and denoted by $\alpha(G)$. If $G - S$ is not connected, we call S a vertex cut. The number of vertices in a minimum vertex cut is denoted by $\kappa(G)$. If $\kappa(G) \geq k$ (k is a positive integer), we call G $k-$connected. A graph G is a $[s,t]$-graph, if there are at least t edges in every included subgraph by s vertices of G, where s, t are positive integers.

Lemma 1. Every $[s,t]$-graph is a $[s+1, t+1]$-graph.

Proof. Let G be a $[s,t]$-graph. If G is not a $[s+1,t+1]$-graph, there exists $S \subset V(G)$ with $|S| = s+1$ such that $|E(G[S])| \leq t$. Taking an edge $e = xy \in E(G[S])$, We have $|S - x| = s$ and $E(G[S-x]) \subset E(G[S]) - e$. Hence, $|E(G[S-x])| \leq |E(G[S]) - e| \leq t-1$. This contradicts that G is a $[s,t]$-graph. □

Lemma 2. For any graph G, $\alpha(G) \leq k$ if and only if G is a $[k+1, 1]$-graph.

Proof. Clearly, $\alpha(G) \leq k$ if and only if I is not an independent set for any $k+1$-vertex set I of G, i.e. G is a $[k+1,1]$-graph. □

From lemma 2, the concept of $[s,t]$-graph is the extension of independent number. Every graph G with $E(G) \neq \phi$ is some kind of $[s,t]-$graph. So the research on $[s,t]-$graph is of general significance. In addition, lots of practical problems can be study from $[s,t]$-graphs.

The next two known results are due to *Chvatal-Erdös* and *Bondy*, respectively.

Theorem 1. (Chvatal and Erdös [2]) Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)$, then G has a Hamilton cycle.

Theorem 2. (Bondy [3]) Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)+1$, then G has a Hamilton path.

This two theorems can be described as the following forms.

(a) Every $[\kappa(G)+1, 1]$-graph G of order $n \geq 3$ has a Hamilton cycle.

(b) Every $[\kappa(G)+2, 1]$-graph G of order $n \geq 3$ has a Hamilton path.

Theorem 3. (Liu [4]) Let G be a [4,2]-graph. Then G is 2-connected if and only if G has a Hamilton cycle or G is isomorphic to $K_{2,3}$ or $K_{1,1,3}$.

Theorem 4. (Li [5]) Let G be a 2-connected $[5,3]$-graph with $|G| \geq 8$ and $\delta \geq 3$. Then G has a Hamilton cycle.

In this paper, we show the following results.

Theorem 5. Let G be a k-connected $[k+2,2]$-graph of order $n \geq 3$. Then G has a Hamilton cycle or G is isomorphic to Petersen graph or to $\overline{K_{k+1}} \vee G_k$.

Theorem 6. Let G be a k-connected $[k+3,2]$-graph of order $n \geq 3$. Then G has a Hamilton path or G is isomorphic to $\overline{K_{k+2}} \vee G_k$.

Here G_k is an arbitrary graph of order k.

We can give the examples showing that the result of theorem 5 fails in k-connected $[k+3,2]$-graphs and theorem 6 fails in k-connected $[k+4,2]$-graphs.

2. Proof of Theorem 5

Suppose that the graph G satisfies the conditions of Theorem 5 and G contains no Hamilton cycle.

When $k=1$, G is a $[3,2]$-graph and there exists $v \in V(G)$ such that $G-v$ is not connected. If $n \geq 4$, take w, w', w'' not all from the same components of $G-v$. We have $|E(G[\{w,w',w''\}])| \leq 1$, which contradicts the fact that G is a $[3,2]$-graph. This contradiction shows that $n=3$ if $k=1$. Hence, G is isomorphic to $\overline{K_2} \vee G_1$. So the result of Theorem 5 is true when $k=1$. Next, we assume $k \geq 2$.

Let $C = v_1 v_2 \cdots v_m v_1$ be a longest cycle of G. In this section, for $v_i, v_j \in V(C)$, we put

$$v_i \overrightarrow{C} v_j = v_i v_{i+1} \cdots v_{j-1} v_j, \quad v_i \overleftarrow{C} v_j = v_i v_{i-1} \cdots v_{j+1} v_j,$$

$$v_i^{-l} = v_{i-l}, \quad v_i^{+l} = v_{i+l}, \quad v_i^- = v_i^{-1}, \quad v_i^+ = v_i^{+1},$$

where the indices are taken modulo m. For $x \in G$ and a component H of $G - V(C)$, we put

$$N_C^+(x) = \{w^+ : w \in N_C(x)\}, \quad N_C^-(x) = \{w^- : w \in N_C(x)\},$$

$$N_C^+(H) = \{w^+ : w \in N_C(H)\}, \quad N_C^-(H) = \{w^- : w \in N_C(H)\}.$$

We have that $|N_C(H)| \geq k \geq 2$ since G is k-connected and the following claims hold.

Claim 2.1 If $y, z \in N_C(H) (y \neq z)$, then

(a) $y \notin \{z^-, z^+\}$ and hence $|y^+ \overrightarrow{C} z^-| \geq 1$ or $|z^+ \overrightarrow{C} y^-| \geq 1$.

(b) $y^+ z^+, y^- z^- \notin E(G)$.

(c) $N^-_{y^+ \overrightarrow{C} z}(y^+) \cap N_C(z^+) = \emptyset$, $N^+_{z^+ \overrightarrow{C} y}(y^+) \cap N_C(z^+) = \emptyset$,
$N^-_{y \overrightarrow{C} z^-}(y^-) \cap N_C(z^-) = \emptyset$, $N^+_{z \overrightarrow{C} y^-}(y^-) \cap N_C(z^-) = \emptyset$.

(d) $N^-_{z \overrightarrow{C} y}(y^+) \cap N_C(z^-) = \emptyset$, $N^+_{z \overrightarrow{C} y^-}(y^+) \cap N_C(z^-) = \emptyset$,
$N^-_{y^+ \overrightarrow{C} z}(y^-) \cap N_C(z^+) = \emptyset$, $N^+_{y \overrightarrow{C} z}(y^-) \cap N_C(z^+) = \emptyset$.

Proof. Otherwise, it is easy to get a cycle longer than C. $\square$

Claim 2.2 $|G-V(C)|=1$.

Proof. Take a vertex $x \in V(H)$. If $|G-V(C)| \geq 2$, there exist $x' \in V(G-V(C))$ such that $x' \neq x$. Considering $S_1 = \{x, x'\} \cup N_C^+(H)$, we have $|S_1| \geq k+2$.

When $x' \notin V(H)$, x' is adjacent to at most one vertex in $N_C^+(H)$ (Otherwise, it is easy to get a cycle longer than C). When $x' \in V(H)$, x' is not adjacent to any vertex in S_1 expect x. Combining Claim 2.1(a) and (b), we have $|E(G[S_1])| \leq 1$. This contradicts Lemma 1. $\square$

Next, for a longest cycle C of G, the only vertex of $G-V(C)$ is denoted by x_C.

Claim 2.3 $|N_C(x_C)| = k$.

Proof. Obviously, $|N_C(x_C)| \geq k$. If $|N_C(x_C)| \geq k+1$, considering $S_2 = \{x_C\} \cup N_C^+(x_C)$, we have $|S_2| \geq k+2$. By Claim 2.1(a) and (b), $|E(G[S_2])| = 0$. This contradicts Lemma 1. $\square$

Put $N_C(x_C) = \{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\}$ (where $1 \leq i_1 < i_2 < \cdots < i_k \leq m$).

Claim 2.4 Let $v_{i_j} \in N_C(x_C)$.

(a) If $w \in N_C(v_{i_j}^-)$, then $w^+ \notin N_C(v_{i_j}^-)$.

(b) If $u \in N_C(v_{i_j}^+)$, then $u^- \notin N_C(v_{i_j}^+)$.

Proof. (a) If $w^+ \in N_C(v_{i_j}^-)$, by Claim 2.1(b), $w, w^+ \notin N_C^-(x_C)$. Considering $S_3 = \{x_C, v_{i_j}^{-2}\} \cup N_C^-(x_C)$, we have $|S_3| = k+2$. Hence $|E(G[S_3])| \geq 2$. By Claim 2.1(a) and (b), $\{x_C\} \cup N_C^-(x_C)$ is an independent set. Furthermore, $v_{i_j}^{-2} x_C \notin E(G)$(otherwise, the cycle $x_C v_{i_j} \overrightarrow{C} w v_{i_j}^- w^+ \overrightarrow{C} v_{i_j}^{-2} x_C$ is longer than C). Hence, there exists $v_{i_l}^-(l \neq j)$ such that $v_{i_j}^{-2} v_{i_l}^- \in E(G)$. We can get one of the following cycles longer than C:

$$x_C v_{i_j} \overrightarrow{C} v_{i_l}^- v_{i_j}^{-2} \overleftarrow{C} w^+ v_{i_j}^- w \overleftarrow{C} v_{i_l} x_C \quad (if\ v_{i_l} \in V(v_{i_j} \overrightarrow{C} w)),$$

$$x_C v_{i_j} \overrightarrow{C} w v_{i_j}^- w^+ \overrightarrow{C} v_{i_l}^- v_{i_j}^{-2} \overleftarrow{C} v_{i_l} x_C \quad (if\ v_{i_l} \in V(w^+ \overrightarrow{C} v_{i_j})).$$

This is a contraction.

(b) In a way similarly to (a), (b) can be proved. $\square$

By Claim 2.4,

$$v_{i_l}^+ v_{i_l}^- \notin E(G), \ l = 1, 2, \cdots, k. \tag{1}$$

Claim 2.5 If $|v_{i_j}^+\overrightarrow{C}v_{i_{j+1}}^-| = 1$, then $|v_{i_{j+1}}^+\overrightarrow{C}v_{i_{j+2}}^-| = 1$, where the indices are taken modulo m. (Next, we will no longer indicate it when indices need to be taken modulo m.)

Proof. First, we show that $N(v_{i_j}^+) \subseteq N_C(x_C)$.

By Claim 2.2 and Claim 2.1(a), $N(v_{i_j}^+) \subset V(C)$. If $N(v_{i_j}^+) \nsubseteq N_C(x_C)$, take $u \in N(v_{i_j}^+) - N_C(x_C)$. Combining $|v_{i_j}^+\overrightarrow{C}v_{i_{j+1}}^-| = 1$ and Claim 2.1(b) gives $u \notin N_C^-(x_C) \cup N_C^+(x_C)$. Considering $S_4 = \{x_C, u^+\} \cup N_C^+(x_C)$, we have $|S_4| = k+2$. By Claim 2.1(a) and (b), $\{x_C\} \cup N_C^+(x_C)$ is an independent set and $x_Cu^+ \notin E(G)$. By Claim 2.1(C),(d) and Claim2.4, $E(\{u^+\}, N_C^+(x_C)) = \phi$. Hence, $|E(G[S_4])| = 0$. This contradiction shows that $N(v_{i_j}^+) \subseteq N_C(x_C)$.

Since G is k-connected, $d(v_{i_j}^+) \geq k$. By Claim 2.3, $N(v_{i_j}^+) = N_C(x_C)$. Therefore

$$v_{i_j}^+ v_{i_l} \in E(G), \quad l = 1, 2, \cdots, k. \tag{2}$$

If $|v_{i_{j+1}}^+\overrightarrow{C}v_{i_{j+2}}^-| \geq 2$, considering $S_5 = \{x_C, v_{i_{j+2}}^-\} \cup N_C^+(x_C)$, we have $|S_5| = k+2$. Since $|E(G[S_5])| \geq 2$, by Claim 2.1(a),(b), there exist $v_{i_s}^+, v_{i_t}^+ \in N_C^+(x_C)(s \neq j \neq t)$ such that

$$v_{i_{j+2}}^- v_{i_s}^+, v_{i_{j+2}}^- v_{i_t}^+ \in E(G).$$

One of $v_{i_s}^+, v_{i_t}^+$ is not $v_{i_{j+1}}^+$. Suppose $v_{i_t}^+ \neq v_{i_{j+1}}^+$. Since $v_{i_j}^+ = v_{i_{j+1}}^-$, by (2), $v_{i_{j+1}}^- v_{i_t} \in E(G)$, which is contrary to Claim 2.1(c). □

Claim 2.6 If $|v_{i_j}^+\overrightarrow{C}v_{i_{j+1}}^-| = 1$ for some $v_{i_j} \in N_C(x_C)$, then G is isomorphic to $\overline{K_{k+1}} \vee G_k$ (where G_k is an arbitrary graph of order k).

Proof. By Claim 2.5 and (2),

$$|v_{i_l}^+\overrightarrow{C}v_{i_{l+1}}^-| = 1, \quad l = 1, 2, \cdots, k,$$

$$v_{i_j}^+ v_{i_l} \in E(G), \quad j, l = 1, 2, \cdots, k.$$

Hence, $|C| = 2k$. We obtain $C = v_1v_2\cdots v_{2k}$, $V(G) = V(C) \cup \{x_C\}$ and $|G| = 2k+1$. Without loss of generality, we assume that $N_C(x_C) = \{v_2, v_4, \cdots v_{2k}\}$. Then, we have $S = \{x_C, v_1, v_3, \cdots, v_{2k-1}\}$ is an independent set (by Claim 2.1) and , for any $a \in S$ and any $b \in N_C(x_C)$, $ab \in E(G)$. In addition, no matter what the edges among vertices of $N_C(x_C) = \{v_2, v_4, \cdots v_{2k}\}$ are, the graph G is k-connected $[k+2, 2]$-graph and has no Hamilton cycle. Therefore G is isomorphic to $\overline{K_{k+1}} \vee G_k$(where G_k is an arbitrary graph of order k). □

Next, we suppose $|v_{i_l}^+\overrightarrow{C}v_{i_{l+1}}^-| \geq 2$, $l = 1, 2, \cdots, k$.

For a longest cycle C of G, we put

$$\rho(C) = max\{|v_{i_j}^+\overrightarrow{C}v_{i_{j+1}}^-| : j = 1, 2, \cdots, k\}.$$

Choose a longest cycle C' such that $\rho(C')$ is as large as possible. Then the above Claims is true for this cycle C'. Next, for convenience, we use $C = v_1v_2\cdots v_mv_1$ instead of the cycle C'. We still suppose $N_C(x_C) = N_{C'}(x_{C'}) = \{v_{i_1}, v_{i_2}\dots v_{i_k}\}$(where $1 \le i_1 < i_2 < \cdots < i_k \le m$). Without loss of generality, we assume that $\rho(C) = |v_{i_1}^+\overrightarrow{C}v_{i_2}^-|$.

Claim 2.7 $N_{v_{i_1}^+\overrightarrow{C}v_{i_2}}(v_{i_1}^-) = \emptyset$, $N_{v_{i_1}\overrightarrow{C}v_{i_2}^-}(v_{i_2}^+) = \emptyset$.

Proof. First, we show that $N_{v_{i_1}^+\overrightarrow{C}v_{i_2}^-}(v_{i_1}^-) = \emptyset$.

If $N_{v_{i_1}^+\overrightarrow{C}v_{i_2}^-}(v_{i_1}^-) \neq \emptyset$, take $u \in N_{v_{i_1}^+\overrightarrow{C}v_{i_2}^-}(v_{i_1}^-)$. By (1), $u \neq v_{i_1}^+$. Considering $S_6 = \{x_C, u^-\} \cup N_C^-(x_C)$, we have $|S_6| = k+2$. By Claim 2.1 and Claim 2.4, $|E(G[S_6])| = 0$, which contradicts the fact that G is a $[k+2, 2]$-graph.

Now we show that $v_{i_1}^- v_{i_2} \notin E(G)$.

If $v_{i_1}^- v_{i_2} \in E(G)$, considering $S_7 = \{x_C, v_{i_2}^-\} \cup N_C^+(x_C)$, we have $|S_7| = k+2$. By Claim 2.1 and Claim 2.4, $E(G[S_7]) \subset \{v_{i_1}^+ v_{i_2}^-\}$ and hence $|E(G[S_7])| \le 1$, a contradiction.

Therefore $N_{v_{i_1}^+\overrightarrow{C}v_{i_2}}(v_{i_1}^-) = \emptyset$.

Similarly, we can proved that $N_{v_{i_1}\overrightarrow{C}v_{i_2}^-}(v_{i_2}^+) = \emptyset$. □

Claim 2.8 (a) $v_{i_1}^{-2}v_{i_2}^- \in E(G)$ and $v_{i_1}^{-2}v_{i_j}^- \notin E(G)$ $(j = 3, 4, \cdots, k)$;
(b) $v_{i_2}^{+2}v_{i_1}^+ \in E(G)$ and $v_{i_2}^{+2}v_{i_j}^+ \notin E(G)$ $(j = 3, 4, \cdots, k)$.

Proof. (a) First, we show that

$$v_{i_1}^{-2}v_{i_j}^- \notin E(G),\ j = 3, 4, \cdots, k. \tag{3}$$

Otherwise, there is $v_{i_r}^-(3 \le r \le k)$ such that $v_{i_1}^{-2}v_{i_r}^- \in E(G)$. Put

$$C_1 = x_C v_{i_1}\overrightarrow{C}v_{i_r}^- v_{i_1}^{-2}\overleftarrow{C}v_{i_r}x_C.$$

Then $|C_1| = |C|$ and hence C_1 is also a longest cycle of G. Obviously, $x_{C_1} = v_{i_1}^-$, $v_{i_1} \in N_{C_1}(x_{C_1})$. By Claim 2.7,

$$\rho(C_1) \ge |v_{i_1}^+\overrightarrow{C_1}v_{i_2}| = |v_{i_1}^+\overrightarrow{C}v_{i_2}| > |v_{i_1}^+\overrightarrow{C}v_{i_2}^-| = \rho(C),$$

which contradicts the fact that $\rho(C)$ is largest.

Next, we show that $v_{i_1}^{-2}v_{i_2}^- \in E(G)$.

Considering $S_8 = \{x_C, v_{i_1}^{-2}\} \cup N_C^-(x_C)$, we have $|S_8| = k+2$. Since G is a $[k+2, 2]$-graph, by Claim 2.1 and (3), there exist $v_{i_s}^-, v_{i_t}^- \in N_C^-(x_C)$ $(s \neq t, 1 \le s, t \le 2)$ such that

$$v_{i_1}^{-2}v_{i_s}^-, v_{i_1}^{-2}v_{i_t}^- \in E(G).$$

Hence, $v_{i_1}^{-2}v_{i_2}^- \in E(G)$.

(b) In a way similarly to (a), (b) can be proved. □

Claim 2.9 There exists q $(1 \neq q \neq 2)$, such that $v_{i_2}^{-2}v_{i_q}^-$, $v_{i_1}^-v_{i_q}^{-2} \in E(G)$.

Proof. Considering $S_9 = \{x_C, v_{i_2}^{-2}\} \cup N_C^-(x_C)$, we have $|S_9| = k + 2$. Since G is a $[k+2, 2]$-graph, by Claim 2.1, there exist $v_{i_p}^-, v_{i_q}^- \in N_C^-(x_C)$ such that $v_{i_2}^{-2}v_{i_p}^-, v_{i_2}^{-2}v_{i_q}^- \in E(G)$. By Claim 2.7, $p \neq 1 \neq q$ and hence one of $v_{i_p}^-, v_{i_q}^-$ is not $v_{i_2}^-$. We assume that $v_{i_q}^- \neq v_{i_2}^-$.

Put $C_2 = x_C v_{i_1} \overrightarrow{C} v_{i_2}^{-2} v_{i_q}^- \overleftarrow{C} v_{i_2}^- v_{i_1}^{-2} \overleftarrow{C} v_{i_q} x_C$. Then C_2 is a longest cycle of G and $x_{C_2} = v_{i_1}^-$, $v_{i_1} \in N_{C_2}(x_{C_2})$. By Claim 2.1(b), $v_{i_1}^-v_{i_q}^- \notin E(G)$. If $v_{i_1}^-v_{i_q}^{-2} \notin E(G)$, by Claim 2.7,

$$\rho(C_2) \geq |v_{i_1}^+ \overrightarrow{C_2} v_{i_q}^{-2}| = |v_{i_1}^+ \overrightarrow{C} v_{i_2}^{-2}| + |\{v_{i_q}^-, v_{i_q}^{-2}\}| > |v_{i_1}^+ \overrightarrow{C} v_{i_2}^-| = \rho(C),$$

which contradicts the fact that $\rho(C)$ is largest. □

Claim 2.10 $\rho(C) = 2$.

Proof. If $\rho(C) = |v_{i_1}^+ \overrightarrow{C} v_{i_2}^-| \geq 3$, considering $S_{10} = \{x_C, v_{i_2}^{-3}\} \cup N_C^-(x_C)$, we have $|S_{10}| = k + 2$. Since G is a $[k+2, 2]$-graph, by Claim 2.1, there exist $v_{i_h}^-, v_{i_g}^- \in N_C^-(x_C)$ such that $v_{i_h}^-v_{i_2}^{-3}, v_{i_g}^-v_{i_2}^{-3} \in E(G)$. One of $v_{i_h}^-, v_{i_g}^-$ is not $v_{i_1}^-$. Suppose $v_{i_g}^- \neq v_{i_1}^-$. Since $v_{i_2}^{-2}v_{i_q}^- \in E(G)$, we can get a contradiction to Claim 2.1(c) if $i_g < i_q$ and a contradiction to Claim 2.4 if $i_g = i_q$. Thus $i_g > i_q$. Noting Claim 2.9, we get the follow cycle which is longer than C:

$$C_3 = x_C v_{i_1} \overrightarrow{C} v_{i_2}^{-3} v_{i_g}^- \overleftarrow{C} v_{i_q}^- v_{i_2}^{-2} \overrightarrow{C} v_{i_q}^{-2} v_{i_1}^- \overleftarrow{C} v_{i_g} x_C,$$

a contradiction. □

By Claim 2.10,

$$|v_{i_l}^+ \overrightarrow{C} v_{i_{l+1}}^-| = \rho(C) = 2 \ \ (l = 1, 2, \cdots, k), \ |G| = 3k + 1.$$

Claim 2.11 $k = 3$.

Proof. Since $v_{i_1}^{-2} = v_{i_k}^+$, by Claim 2.8,

$$v_{i_k}^+ v_{i_2}^- \in E(G), \ v_{i_k}^+ v_{i_j}^- \notin E(G) \ \ (j = 3, 4, \cdots, k).$$

By Claim 2.4,

$$v_{i_k}^+ v_{i_1}, v_{i_k}^+ v_{i_2} \notin E(G).$$

By Claim 2.1(a) and (b),

$$v_{i_k}^+ x_C \notin E(G), \quad v_{i_k}^+ v_{i_l}^+ \notin E(G) \quad (l = 1, 2, \cdots, k).$$

Therefore $v_{i_k}^+$ is not adjacent to these $2k+1$ vertices of G. Since G is k-connected, $|N(v_{i_k}^+)| \geq k$. Hence

$$N(v_{i_k}^+) = \{v_{i_1}^-, v_{i_2}^-, v_{i_3}, \cdots, v_{i_{k-1}}, v_{i_k}\}.$$

If $k > 3$, $v_{i_{k-1}} \in N(v_{i_k}^+)$ and hence $v_{i_k}^+ v_{i_{k-1}} \in E(G)$. Considering $S_{11} = \{x_C, v_{i_{k-1}}^+\} \cup N_C^-(x_C)$, we have $|S_{11}| = k+2$. By Claim 2.1(d), $v_{i_{k-1}}^+ v_{i_l}^- \notin E(G)$ $(l = 1, 2, \cdots, k-1)$. By Claim 2.1(a) and (b), $|E(G[S_{11}])| = 1$, which contradicts the fact that G is a $[k+2, 2]$-graph. Thus $k = 3$. □

By Claim 2.11, $|G| = 10$. Similarly to the proof of Claim 2.8, we can show

$$v_{i_j}^{-2} v_{i_{j+1}}^- \in E(G), \ j = 1, 2, 3.$$

Furthermore, we have

$$E(G) = E(C) \cup \{x_C v_{i_j} : j = 1, 2, 3\} \cup \{v_{i_j}^{-2} v_{i_{j+1}}^- : j = 1, 2, 3\}$$

(Otherwise, it is easy to get the Hamilton cycles of G), where indices are taken modulo 3. Therefore, G is isomorphic to Petersen graph.

The proof of Theorem 5 is complete.

Corollary 2.1 If G is a k-connected $[k+2, 2]$-graph with $|G| \geq 2k+2$(where $k \geq 4$), then G contains a Hamilton cycle.

Proof. Since $|G| \geq 2k+2$, G is not isomorphic to $\overline{K_{k+1}} \vee G_k$. Since $k \geq 4$, G is not isomorphic to F. By Theorem 5, G has a Hamilton cycle. □

Corollary 2.2 Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)$, then G has a Hamilton cycle.

Proof. Since $\alpha(G) \leq \kappa(G)$, G is a $\kappa(G)$-connected $[\kappa(G)+1, 1]$-graph (by Lemma 2). Hence, G is a $\kappa(G)$-connected $[\kappa(G)+2, 2]$-graph (by Lemma 1). Obviously, G is isomorphic to neither $\overline{K_{\kappa(G)+1}} \vee G_{\kappa(G)}$ nor Petersen graph (since neither one of them is $[\kappa(G)+1, 1]$-graph). By Theorem 5, G has a Hamilton cycle. □

3. Proof of Theorem 6

Suppose that the graph G satisfies the conditions of theorem 6 and G contains no Hamilton path.

Now, we prove that the result of Theorem 6 is ture when $k = 1$.

When $k = 1$, G is a $[4,2]$-graph and there exists $z \in V(G)$ such that $G - z$ is not connected.

Case 1. $n = 3$

Obviously, G has a Hamilton Path.

Case 2. $n = 4$

Obviously, G has a Hamilton path when there are two components in $G - z$ and G is isomorphic to $\overline{K_3} \vee G_1$ when there are three components in $G - z$.

Case 3. $n \geq 5$

There are two components R_1, R_2 in $G - z$(Otherwise, it is easy to take $w_1, w_2, w_3, w_4 \in V(G - z)$ such that $|E(G[\{w_1, w_2, w_3, w_4\}])| \leq 1$, which contradicts that G is a $[4,2]$-graph.). If R_1, R_2 are both complete graphs, G has a Hamilton path. Without loss of generality, suppose that R_1 is not complete graph. Then, $|R_1| > 2$ and there exist $z_1, z_2 \in V(R_1)$ such that $z_1z_2 \notin E(G)$. If $|R_2| \geq 2$, taking $z_3, z_4 \in V(R_2)$, we have $|E(G[\{z_1, z_2, z_3, z_4\}])| \leq 1$. This contradiction shows that $V(R_2) = \{z'\}$. Because G is a $[4,2]$-graph, R_1 is a 1-connected $[3,2]$-graph(Otherwise, there exist $x_1, x_2, x_3 \in V(R_1)$ such that $|E(G[\{x_1, x_2, x_3\}])| \leq 1$, and hence $|E(G[\{x_1, x_2, x_3\} \cup V(R_2)])| \leq 1$, a contradiction.). By Theorem 5, R_1 has a Hamilton cycle or R_1 is isomorphic to Petersen graph or $\overline{K_2} \vee G_1$. Obviously, G has a Hamilton path when R_1 has a Hamilton cycle or R_1 is isomorphic to Petersen graph. If R_1 is isomorphic to $\overline{K_2} \vee G_1$, i.e. $V(R_1) = \{z_1, z_2, z''\}$ and $z''z_1, z''z_2 \in E(R_1)$. Considering $\{z, z_1, z_2\}$, because $z_1z_2 \notin E(G)$, we have $zz_1 \in E(G)$ or $zz_2 \in E(G)$. It is easy to see that G has a Hamilton path.

From the above, we know that the result of Theorem 6 is ture when $k = 1$. Next, we assume $k \geq 2$.

Let $P = v_1v_2\cdots v_p$ is a longest path of G. In this section, for the vertices $v_i, v_j \in V(P)$ $(1 \leq i < j \leq p)$, we put

$$v_i\overrightarrow{P}v_j = v_iv_{i+1}\cdots v_j,\ v_j\overleftarrow{P}v_i = v_jv_{j-1}\cdots v_i,$$

$$v_i^{-l} = v_{i-l},\ v_i^{+l} = v_{i+l}\ (1 \leq i - l < i + l \leq p),$$

$$v_i^- = v_i^{-1},\ v_i^+ = v_i^{+1}.$$

For $x \in G$ and a component H of $G - V(P)$, we put

$$N_P^+(x) = \{w^+ : w \in N_P(x)\},\quad N_P^-(x) = \{w^- : w \in N_P(x)\},$$

$$N_P^+(H) = \{w^+ : w \in N_P(H)\},\quad N_P^-(H) = \{w^- : w \in N_P(H)\}.$$

Let $u = v_1$ and $v = v_p$. We have $|N_P(H)| \geq k \geq 2$ and the following claims hold.

Claim 3.1 Let $y, z \in N_P(H) (y \neq z)$.Then

(a) $N(u) \cup N(v) \subseteq V(P)$.

(b) $uv \notin E(G)$.

(c) $z \notin \{y^+, y^-\}$.

(d) $uy^+, vy^- \notin E(G)$.

(e) $y^+z^+, y^-z^- \notin E(G)$.
(f) If $y \in V(u\overrightarrow{P}z^-)$, $uz^-, vy^+ \notin E(G)$.

Proof. Otherwise, it is easy to get the path longer than P. □

Claim 3.2 $|G - V(P)| = 1$.

Proof. Take $x \in V(H)$. If $|G - V(P)| \geq 2$, there exists $x' \in V(G - V(P))$ such that $x' \neq x$. Considering $T_1 = \{x, x', u\} \cup N_P^+(H)$, since $|N_P^+(H)| = |N_P(H)| \geq k$ and $u \notin N_P^+(H)$, we have $|T_1| \geq k + 3$.

Noting $|N_{N_P^+(H)}(x')| \leq 1$ if $x' \notin V(H)$ and $|N_{N_P^+(H)}(x')| = 0$ if $x' \in V(H)$ (otherwise, it is easy to get paths longer than P), we have $|E(G[T_1])| \leq 1$ by Claim 3.1. This contradicts Lemma 1. □

Next, for a longest path P of G, the only vertex of $G - V(P)$ will be denoted by x_P.

Claim 3.3 (a) $N_P(x_P) = k$.
(b) $u \in N_P^-(x_P)$, $v \in N_P^+(x_P)$.

Proof. (a) Obviously, $|N_P(x_P)| \geq k$. If $|N_P(x_P)| \geq k + 1$, considering $T_2 = \{x_P, u\} \cup N_P^+(x_P)$, we have $|T_2| \geq k+3$ and $|E(G[T_2])| = 0$ by Claim 3.1. This contradicts the fact that G is a $[k + 3, 2]$-graph.

(b) If $u \notin N_P^-(x_P)$, considering $T_3 = \{x_P, u, v\} \cup N_P^-(x_P)$, we have $|T_3| = k + 3$ and $|E(G[T_3])| \leq 1$ by Claim 3.1. This contradicts the fact that G is a $[k + 3, 2]$-graph.

If $v \notin N_P^+(x_P)$, considering $T_4 = \{x_P, u, v\} \cup N_P^+(x_P)$, we can get a similar contradiction. □

Put $N_P(x_P) = \{v_{i_1}, v_{i_2} \cdots v_{i_k}\}$ $(i_1 < i_2 < \cdots < i_k)$.

Claim 3.4 $|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| = 1$, $j = 1, 2, \cdots, k - 1$.

Proof. By Claim 3.3(b), $v_{i_1} = v_2, v_{i_k} = v_{p-1}$. By Claim 3.1(c),

$$|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| \geq 1, j = 1, 2, \cdots, k - 1.$$

Let $P_s = v_{i_s}^- \overleftarrow{P} v_2 x_P v_{i_s} \overrightarrow{P} v_p$ $(s = 2, 3, \cdots, k)$. Then P_s is also a longest path of G and $x_{P_s} = v_1$. Using Claim 3.3(b) to P_s, we have

$$v_1 v_{i_k}, v_1 v_{i_s}^{-2} \in E(G), \quad s = 2, \cdots, k. \tag{4}$$

Let $Q_t = v_{i_t}^+ \overrightarrow{P} v_{p-1} x_P v_{i_t} \overleftarrow{P} v_1$ $(t = 1, 2, \cdots, k-1)$. Similarly, Q_t is a longest path of G, $x_{Q_t} = v_p$ and

$$v_p v_{i_1}, v_p v_{i_t}^{+2} \in E(G), \quad t = 1, \cdots, k - 1. \tag{5}$$

If there is $j \in \{1, 2, \cdots, k-1\}$ such that $|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| = 2$, then $v_{i_{j+1}}^{-2} = v_{i_j}^+$. By (4), $uv_{i_j}^+ = v_1 v_{i_{j+1}}^{-2} \in E(G)$, which contradicts Claim 3.1(d). Therefore,

$$|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| \neq 2, \quad j = 1, 2, \cdots, k-1. \tag{6}$$

If there is $l \in \{1, 2, \cdots, k-1\}$ such that $|v_{i_l}^+ \overrightarrow{P} v_{i_{l+1}}^-| \geq 3$, considering $T_5 = \{v_{i_l}^+, x_P, v_p\} \cup N_P^-(x_P)$, we have $|T_5| = k+3$ and

$$v_{i_l}^+ v_{i_m}^- \notin E(G), \quad m = 1, 2, \cdots, k.$$

Since otherwise, by (4) and (5), we can get one of the following paths longer than P:

$$v_1 \overrightarrow{P} v_{i_m}^- v_{i_l}^+ \overleftarrow{P} v_{i_m} x_P v_{i_{l+1}} \overrightarrow{P} v_p v_{i_l}^{+2} \overrightarrow{P} v_{i_{l+1}}^- \quad (m \leq l),$$

$$v_p \overleftarrow{P} v_{i_m} x_P v_{i_l} \overleftarrow{P} v_1 v_{i_m}^{-2} \overleftarrow{P} v_{i_l}^+ v_{i_m}^- \quad (m \geq l+1).$$

We have $v_p v_{i_l}^+ \notin E(G)$ (otherwise, the path $u \overrightarrow{P} v_{i_l} x_P v_{i_{l+1}} \overrightarrow{P} v v_{i_l}^+ \overrightarrow{P} v_{i_{l+1}}^-$ is longer than P). By Claim 3.1, $|E(G[T_5])| = 0$. This contradiction shows that

$$|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| < 3, \quad j = 1, 2, \cdots, k-1.$$

By (6),

$$|v_{i_j}^+ \overrightarrow{P} v_{i_{j+1}}^-| = 1 \quad j = 1, 2, \cdots, k-1. \ \square$$

By Claim 3.4 and Claim 3.3(b), $|P| = 2k+1$. Suppose $P = v_1 v_2 \cdots v_{2k+1}$, then

$$N_P(x_P) = \{v_2, v_4, \cdots v_{2k}\}, V(G) = V(P) \cup \{x_P\}, |G| = 2k+2.$$

Let

$$S = \{x_P, v_1, v_3, \cdots, v_{2k+1}\},$$

then $|S| = k+2$. For any $x, y \in S$, by Claim 3.1, $xy \notin E(G)$. For any $x \in S$ and any $z \in N_P(x_P) = \{v_2, v_4, \cdots v_{2k}\}$, since $d(x) \geq k$, $xz \in E(G)$. We notice that G is k-connected $[k+3, 2]$-graph no matter how $E(G[N_P(x_P)])$ is. Hence, G is isomorphic to $\overline{K_{k+2}} \vee G_k$ (where G_k is an arbitrary graph of order k).

The proof of Theorem 6 is complete.

Corollary 3.1 If G is a k-connected $[k+3, 2]$-graph with $|G| \geq 2k+3$, then G contains a Hamilton path.

Proof. Since $|G| \geq 2k+3$, G is not isomorphic to $\overline{K_{k+2}} \vee G_k$. By Theorem 6,$G$ has a Hamilton path. $\square$

Corollary 3.2 Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G) + 1$, then G has a Hamilton path.

Proof. Since $\alpha(G) \leq \kappa(G) + 1$, G is a $\kappa(G)$-connected $[\kappa(G) + 2, 1]$-graph (by Lemma 2). Hence, G is a $\kappa(G)$-connected $[\kappa(G) + 3, 2]$-graph (by Lemma 1). Obviously, G is not isomorphic to $\overline{K_{\kappa(G)+2}} \vee G_{\kappa(G)}$ (since $\overline{K_{\kappa(G)+2}} \vee G_{\kappa(G)}$ is not $[\kappa(G) + 2, 1]$-graph). By Theorem 6, G has a Hamilton path. □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[2] V.Chvatal and P.Erdös, A note on hamiltonian circuits, Discrete Math.2(1972),111-113.

[3] J.A. Bondy, Longest Paths and Cycles in Graphs of High Degree, Research Report CORR 16-18, University of Waterloo, Waterloo, 1980.

[4] Chunfang Liu, Jianglu wang, $[s, t]$-Graphs and Their Hamiltonicity, Journal of Shandong Normal University (Natural Science),20(2005)1-2.

[5] Min Li, Jianglu Wang, Hamilton Cycles of 2-Connected $[5, 3]$-Graphs, Journal of Inner Mongolia Normal University (Natural Science),35(2006)285-287.

[6] Lei Mou, Jianglu Wang, The Hamilton Patn in k-Connected $[k + 3, k]$-Graphs, Journal of Shandong Normal University (Natural Science),24(2009)27-28.