VULNERABIILITY OF MYCIELSKI GRAPHS VIA RESIDUAL CLOSENESS

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ABSTRACT. The vulnerability value of a communication network is the resistance of this communication network until some certain stations or communication links between these stations are disrupted and, thus communication interrupts. A communication network is modeled by a graph to measure the vulnerability as stations corresponding to the vertices and communication links corresponding to the edges. There are several types of vulnerability parameters depending upon the distance for each pair of two vertices. In this paper, closeness, vertex residual closeness (VRC) and normalized vertex residual closeness ($NVRC$) of some Mycielski graphs are calculated, furthermore upper and lower bounds are obtained.

1. Introduction

Networks are important structures and appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks [14]. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. As the network begins losing connection lines or centers, eventually, there is a loss of efficiency. In a communication network, the measures of vulnerability are essential to guide the designers in choosing a suitable network topology. They have an impact on solving difficult optimization problems for networks [14].

There are several types of theoretical parameters do not depending upon distance such as connectivity [11], toughness [16], integrity [4], bondage number [2], average lower independence number [1] and scattering number [10]. On the contrary, many graph theoretical parameters depending upon the distance such as vertex and edge betweenness, average vertex and edge betweenness, normalized average vertex and edge betweenness [12], closeness, vertex residual closeness, normalized vertex residual closeness [3, 5, 6, 18].

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Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions using throughout this paper. For any vertex $v \in V(G)$, the open neighborhood of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of vertex v in G denoted by $d_G(v)$, that is the size of its open neighborhood [7]. The distance $d_G(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The diameter of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The complement G' of a graph G has $V(G)$ as its vertex sets, but two vertex are adjacent in G' if only if they are not adjacent in G. A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$ [15].

The concept of VRC and $NVRC$ were introduced on 2006 by Chavdar Dangalchev [5] and has been further studied by Aytaç and Odabaş[3, 18]. The aim of residual closeness is to measure the vulnerability even when the actions (removal of the vertices) do not disconnect the graph. In [3] and [5], they are explained that Residual closeness is considered to be more sensitive for the vulnerability of graphs than the other known vulnerability measures.

The closeness of a graph G is defined as: $C(G) = \sum\limits_{v_i} C(v_i)$, where $C(v_i)$ is the closeness of a vertex v_i, and it is defined as: $C(v_i) = \sum\limits_{v_j \neq v_i} \frac{1}{2^{d_G(v_i, v_j)}}$ [5]. Let $d_{v_k}(v_i, v_j)$ be the distance between vertices v_i and v_j in the graph G, received from the original graph where all links of vertex v_k are deleted. Then the closeness after removing vertex v_k is defined as: $C(v_k) = \sum\limits_{v_i} \sum\limits_{v_j \neq v_i} \frac{1}{2^{d_{v_k}(v_i, v_j)}}$ [5]. The vertex residual closeness (VRC) of the graph G is defined as: $R(G) = \min\limits_{v_k}\{C_{v_k}\}$[5]. The normalized vertex residual closeness ($NVRC$) of the graph G is defined as dividing the residual closeness by the closeness $C(G)$: $R'(G) = R(G)/C(G)$ [5].

Our aim in this paper is to consider the computing the closeness, vertex residual closeness and normalized vertex residual closeness of Mycielski networks that are modeled by Mycielski graphs. Mycielski graphs that may be used to encoding use the adjacency relations between vertices of graph G and copy graphs G'. In section 2, well-known basic results are given for closeness, VRC and $NVRC$, respectively. In section 3, definitions of Mycielski graphs and known basic results for them are given. furthermore; closeness, VRC and $NVRC$ of some Mycielski graphs are computed. Finally, upper and lower bound are determined in section 4.

2. Basic Results

In this section well known basic results are given.

THEOREM 1. [3, 5] *The closeness of*
(a) *If* $G = K_n$, *where* K_n *is a complete graph with order* n, *then*

$$C(G) = (n(n-1))/2,$$

(*b*) If $G = K_{1,n}$, where $K_{1,n}$ is a star graph with order $(n+1)$, then

$$C(G) = (n(n+3))/4$$

(*c*) If $G = C_n$, where C_n is a cycle graph with order n, then

$$C(G) = \begin{cases} 2n(1 - 1/2^{(n-1)/2}) & , \textit{if } n \textit{ is odd;} \\ n(2 - 3/2^{n/2}) & , \textit{if } n \textit{ is even.} \end{cases}$$

THEOREM 2. [5] *The VRC of*
(*a*) *If* $G = K_n$, *then* $R(G) = ((n-1)(n-2))/2$,
(*b*) *If* $G = K_{1,n}$, *then* $R(G) = 0$.

THEOREM 3. [5] *The NVRC of*
(*a*) *If* $G = K_n$, *then* $R'(G) = (n-2)/n$,
(*b*) *If* $G = K_{1,n}$, *then* $R'(G) = 0$.

THEOREM 4. [5] *For a graph* G, $0 \leq R'(G) < 1$.

3. Residual Closeness of Some Mycielski Graphs

DEFINITION 5. [9, 13] *For a graph* G *on vertices* $V(G) = V = \{v_1, v_2, ..., v_n\}$ *and edges* $E(G) = E$, *let mycielski graph* $\mu(G)$ *be the graph on vertices and edges* $V \cup V' \cup \{u\} = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n, u\}$ *and* $E \cup \{v_i v'_j | v_i v_j \in E\} \cup \{ v'_i u | v'_i \in V', i = \overline{1,n}\}$, *respectively. In Figure1, we display Mycielski graph* $\mu(C_5)$.

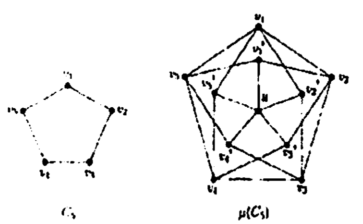

FIGURE 1. Graphs C_5 *and* $\mu(C_5)$

LEMMA 6. [8] *For a graph* G, $diam(\mu(G)) = min(max(2, diam(G)), 4)$.

LEMMA 7. [17] *If is a regular caterpillar, then the closeness of the regular caterpillar* $T_{n,m}$ *is* $C(T_{n,m}) = C(P_n)((m+2)^2/4) + nm(m+3)/4$.

LEMMA 8. [17] *For any graph* G, *if* $diam(G) \leq 2$, *then*

$$C(G) = (|V(G)|(|V(G)| - 1) + 2|E(G)|)/4 .$$

LEMMA 9. [6] *If a vertex* k *does not belong to any unique geodesic (shortest path) of graph* G *then* $C(G \backslash k) = C(G) - 2C(k)$.

THEOREM 10.

(a) *For $n \geq 3$; If $G = K_{1,n}$, the closeness (C) of $\mu(G)$ with order $(2n+3)$ is defined as:* $C(\mu(G)) = \left(2n^2+9n+4\right)/2$.

(b) *For $n \geq 3$; If $G = K_n$, the closeness (C) of $\mu(G)$ with order $(2n+1)$ is defined as:* $C(\mu(G)) = \left(7n^2+n\right)/4$.

(c) *For $n \geq 8$; If $G = C_n$, the closeness (C) of $\mu(G)$ with order $(2n+1)$ is defined as:* $C(\mu(G)) = \left(9n^2+77n\right)/16$.

PROOF. *For(a)*: Since $diam(K_{1,n}) = 2$, by Lemma 6, we have that $diam(\mu(K_{1,n})) = 2$. Thus, by Lemma 8, we obtain $C(\mu(G)) = \left(2n^2+9n+4\right)/2$.

For(b): Proof of (b) is similar to (a), and we omit it.

For(c): Let the vertex set $\mu(G)$ be $V(\mu(G)) = V_1 \cup V_2 \cup \{u\}$, where: $V_1 = \{v_i \in V(G), 1 \leq i \leq n\}$ and $V_2 = \{v_i' \in V(G'), 1 \leq i \leq n\}$. We have three cases depending on the vertices of the graph $\mu(G)$.

Case1. For any vertex of $v_i \in V_1$ in the graph $\mu(G)$. Clearly, we have $\left|N_{\mu(G)}(v_i)\right| = 4$. By the structure of the graph $\mu(G)$ and $n \geq 8$, there are six paths of length 2, $(n-3)$-paths of length 3 and $(n-7)$-paths of length 4. Thus,

$$(n)C(v_i) = (n)\left(4\left(2^{-1}\right) + 6\left(2^{-2}\right) + (n-3)\left(2^{-3}\right) + (n-7)\left(2^{-4}\right)\right)$$

$$= \frac{3n^2+43n}{16} \tag{1}$$

Case2. For any vertex of $v_i' \in V_2$ in the graph $\mu(G)$. Due to $d_{\mu(G)}(v_i') = 3$, number of paths of length 1 is 3. It is not difficult to see that there are $(n+2)$-paths of length 2 and $(n-5)$-paths of length 3. Thus,

$$(n)C(v_i') = (n)\left(3\left(2^{-1}\right) + (n+2)\left(2^{-2}\right) + (n-5)\left(2^{-3}\right)\right)$$

$$= \frac{3n^2+11n}{8} \tag{2}$$

Case3. For the vertex u in the graph $\mu(G)$. Clearly, $d_{\mu(G)}(u) = n$. So, number of paths of length 1 is n. Then, distance from the vertex u to remaining n-vertices is 2. Thus,

$$C(u) = (n)\left(2^{-1}\right) + (n)\left(2^{-2}\right) = \frac{3n}{4} \tag{3}$$

As a result, by summing (1), (2) and (3), we obtain $C(\mu(G)) = \left(9n^2+77n\right)/16$. The proof is completed. □

THEOREM 11.

(a) *For $n \geq 3$; If $G = K_{1,n}$, the vertex residual closeness (VRC) of $\mu(G)$ with order $(2n+3)$ is defined as:* $R(\mu(G)) = \left(3n^2+10n+4\right)/4$.

(b) *For $n \geq 3$; If $G = K_n$, the vertex residual closeness (VRC) of $\mu(G)$ with order $(2n+1)$ is defined as:* $R(\mu(G)) = \left(7n^2-7n+4\right)/4$.

(c) *For $n \geq 8$; If $G = C_n$, the vertex residual closeness (VRC) of $\mu(G)$ with order $(2n+1)$ is defined as:* $R(\mu(G)) = \begin{cases} n(\frac{31}{4} - 2^{\frac{7-n}{2}}) & \text{, if } n \text{ is odd;} \\ n(\frac{31}{4} + 2^{\frac{4-n}{2}} - 2^{\frac{8-n}{2}}) & \text{, if } n \text{ is even.} \end{cases}$

PROOF. *For(a)*: Let the vertex set $\mu(G)$ be $V(\mu(G)) = \{v_c\} \cup V_1 \cup \{v'_c\} \cup V_2 \cup \{u\}$, where: let v_c, v'_c and u be center vertex of G, the vertex v_c in the copy G' and the vertex in the definition of Mycielski graph, respectively. Moreover, let $V_1 = \{v_i \in V(G)\backslash\{v_c\}, 1 \leq i \leq n\}$ and $V_2 = \{v'_i \in V(G')\backslash\{v'_c\}, 1 \leq i \leq n\}$. We have five cases depending on the vertices of the graph $\mu(G)$.

Case1. Removing the central vertex v_c of the graph G from the graph $\mu(G)$. If vertex v_c is removed from the graph $\mu(G)$, then remaining subgraph is a regular caterpillar $T_{2,n}$. By Lemma 7, we directly obtain

$$C_{v_c} = \frac{3n^2 + 10n + 4}{4}$$

Case2. Removing a vertex $v_i \in V_1$ in the graph $\mu(G)$. Since the vertex v_i does not belong to any unique geodesic of $\mu(G)$, then by Lemma 9, $C_{v_i} = C(\mu(G)) - 2C(v_i)$. For $v_i \in V_1$, we have $N_{\mu(G)}(v_i) = \{v_c, v'_c\}$. Then, distance from the vertex v_i to remaining $2n$-vertices is 2. Thus, $C(v_i) = \frac{n+2}{2}$. Consequently, by Theorem 10.(a), we have

$$C_{v_i} = \frac{2n^2 + 9n + 4}{4} - 2\left(\frac{n+2}{2}\right) = \frac{2n^2 + 7n}{2}$$

Case3.Removing the vertex v'_c in the graph $\mu(G)\backslash\{v'_c\}$. We have four sub cases depending on the vertices of the survival subgraph $\mu(G)\backslash\{v'_c\}$.

SubCase1.For the central vertex of the graph G from the graph $\mu(G)\backslash\{v'_c\}$. Then, we have

$$C_{v'_c}(v_c) = (2n)(2^{-1}) + (2^{-2}) = \frac{4n+1}{4} \tag{4}$$

SubCase2.For a vertex $v_i \in V_1$ in the graph $\mu(G)\backslash\{v'_c\}$. The vertex v_i is adjacent to only vertex v_c in the survival subgraph $\mu(G)\backslash\{v'_c\}$. It is clear that $d_{v'_c}(v_i, u) = 3$ in the survival subgraph $\mu(G)\backslash\{v'_c\}$. Moreover, distance from the vertex v_i to remaining $(2n-1)$-vertices is 2. Thus,

$$(n)C_{v'_c}(v_i) = (n)(2^{-1} + (2n-1)(2^{-2}) + 2^{-3}) = \frac{4n^2 + 3n}{8} \tag{5}$$

SubCase3.For a vertex $v'_i \in V_2$ in the graph $\mu(G)\backslash\{v'_c\}$. The vertex v'_i is adjacent to vertices v_c and u. It is clear that distance from the vertex v'_i to remaining $(2n-1)$-vertices is 2. Thus,

$$(n)C_{v'_c}(v'_i) = (n)(2(2^{-1}) + (2n-1)(2^{-2})) = \frac{2n^2 + 3n}{4} \tag{6}$$

SubCase4.For a vertex $u \in \mu(G)\backslash\{v'_c\}$. It is not difficult to see that,

$$C_{v'_c}(u) = (n)(2^{-1}) + 2^{-2} + (n)(2^{-3}) = \frac{5n+2}{8} \tag{7}$$

By summing (4), (5), (6) and (7), we have $C_{v'_c} = \frac{4n^2+11n+2}{4}$.

Case4.Removing a vertex $v'_i \in V_2$ from the graph $\mu(G)$. We have five sub cases depending on the vertices of the survival subgraph $\mu(G)\backslash\{v_i'\}$. Proof of this case is similar to Case2. Then, we obtain $C_{v_i'} = \frac{2n^2+7n}{2}$.

Case5. Removing the vertex u from the graph $\mu(G)$. We have four sub cases depending on the vertices of the survival subgraph $\mu(G)\backslash\{u\}$.

SubCase1. For the central vertex v_c of the graph G in the graph $\mu(G)\backslash\{u\}$. It is clear that $|N_{\mu(G)\backslash\{u\}}(v_c)| = 2n$, and then $d_u(v_c, v_c') = 2$ in the survival subgraph $\mu(G)\backslash\{u\}$. Thus,

$$C_u(v_c) = (2n)(2^{-1}) + (2^{-2}) = \frac{4n+1}{4} \tag{8}$$

SubCase2. For a vertex $v_i \in V_1$ in the survival subgraph $\mu(G)\backslash\{u\}$. It is easily seen that, we obtain,

$$(n)C_u(v_i) = (n)(2(2^{-1}) + (2n-1)(2^{-2})) = \frac{2n^2+3n}{4} \tag{9}$$

SubCase3. For the vertex v_c' of the graph G' in the graph $\mu(G)\backslash\{u\}$. So, we have $|N_{\mu(G)\backslash\{u\}}(v_c')| = n$. Moreover, $d_u(v_c, v_c') = 2$ in the survival graph $\mu(G)\backslash\{u\}$. Since the vertex u removing the graph $\mu(G)$, distance from the vertex v_c' to every vertex $v_i' \in V_2$ is not 2. So, it clear that distance from the vertex v_c' to every vertices of V_2 is 3. Thus,

$$C_u(v_c') = (n)(2^{-1}) + 2^{-2} + (n)(2^{-3}) = \frac{5n+2}{8} \tag{10}$$

SubCase4. For a vertex $v_i' \in V_2$ in the graph $\mu(G)\backslash\{u\}$. The vertex v_i' is adjacent to only the vertex v_c. Moreover, $d_u(v_i', v_c') = 3$ in the survival subgraph $\mu(G)\backslash\{u\}$. Then, it is clear that distance from the vertex v_i' to remaining $(2n-1)$-vertices is 2. Thus,

$$(n)C_u(v_i') = (n)(2^{-1} + (2n-1)(2^{-2}) + 2^{-3}) = \frac{4n^2+3n}{8} \tag{11}$$

By summing (8), (9), (10) and (11), we obtain $C_u = \frac{4n^2+11n+2}{4}$.
From the definition of the vertex residual closeness (VRC) of the graph as follows, $R(\mu(G)) = \min\{C_{v_c}, C_{v_i}, C_{v_c'}, C_{v_i'}, C_u\}$

$$= \min\{\tfrac{3n^2+10n+4}{4}, \tfrac{2n^2+7n}{2}, \tfrac{4n^2+11n+2}{4}\}$$

For $n \geq 3$, $R(\mu(G)) = (3n^2+10n+4)/4$ is obtained. Proof of (b) and (c) are similar to Theorem2.(a) obviously, and we omit them.
The proof is completed. □

COROLLARY 12.
(a) For $n \geq 3$; If $G = K_{1,n}$, the normalized vertex residual closeness $(NVRC)$ of $\mu(G)$ with order $(2n+3)$ is defined as: $R'(\mu(G)) = \frac{3n^2+10n+4}{4n^2+18n+8}$.
(b) For $n \geq 3$; If $G = K_n$, the normalized vertex residual closeness $(NVRC)$ of $\mu(G)$ with order $(2n+1)$ is defined as: $R'(\mu(G)) = 1 - \frac{8n-4}{7n^2+n}$.
(c) For $n \geq 8$; If $G = C_n$, the normalized vertex residual closeness $(NVRC)$ of

$\mu(G)$ with order $(2n+1)$ is defined as:

$$R'(\mu(G)) = \begin{cases} \frac{124-2^{\frac{15-n}{2}}}{9n+77} & \text{, if } n \text{ is odd;} \\ \frac{124+2^{\frac{12-n}{2}}-2^{\frac{16-n}{2}}}{9n+77} & \text{, if } n \text{ is even.} \end{cases}$$

4. Bounds on the Closeness of a Mycielski Graph

THEOREM 13. *Let G be any connected graph of order n. Then, $C(\mu(G)) \leq (4n^2-1)/2$.*

PROOF. It is not difficult show that, $C(v_i) \leq (2n-2)(2^{-1}) + 2(2^{-2}) = (2n-1)/2$. Because, we know that $d_G(v_i, v_i') = d_G(v_i, u) = 2$. So, $(2n-1)/2$ is upper bound for vertices of the graph $\mu(G)$. Furthermore, $C(\mu(G)) \leq (2n+1)((2n-1)/2) = (4n^2-1)/2$ is obtained. □

THEOREM 14. *Let G be any connected graph of order n. If $diam(G) > 4$, then $C(\mu(G)) \geq (6n^2+51n+24)/16$.*

PROOF. If $diam(G) > 4$, then we know that $diam(\mu(G)) = 4$ from the Lemma 6. Let v_i be any vertex of $V(G)$ in the graph $\mu(G)$. By the structure of Mycielski graphs and Lemma 6, there are two paths of length 1, four paths of length 2, $(n-2)$-paths of length 3 and $(n-4)$-paths of length 4 for the minimum value of $C(v_i)$. So, we have $C(v_i) \geq 2(2^{-1}) + 4(2^{-2}) + (n-2)(2^{-3}) + (n-4)(2^{-4}) = (3n+24)/16$. Since the inequality holds for every other vertices of graph $\mu(G)$, we get

$$C(v_i) \geq (2n+1)((3n+24)/16) = (6n^2+51n+24)/16$$

□

THEOREM 15. *Let G be any connected graph of order n and size m. If domination number $\gamma(G) = 1$, then $C(\mu(G)) = (2n^2+2n+3m)/2$.*

PROOF. By the structure of the mycielski graph $\mu(G)$ and definition of the domination number, we get $C(v_i) = 2(d_G(v_i))(2^{-1}) + 2(n-d_G(v_i))(2^{-2})$ for every vertices of $v_i \in V(G)$, where $i = \overline{1,n}$. Similarly, we get $C(v_i') = (d_G(v_i)+1)(2^{-1}) + (2n-1-d_G(v_i))(2^{-2})$ for every vertices of $v_i' \in V'(G)$, where $i = \overline{1,n}$. Finally, $C(u) = n(2^{-1}) + n(2^{-2}) = 3n/4$ is obtained. Thus,

$$C(\mu(G)) = \sum_{i=1}^{n} C(v_i) + \sum_{i=1}^{n} C(v_i') + C(u)$$
$$= \sum_{i=1}^{n} d_G(v_i) + \frac{1}{2}\sum_{i=1}^{n}(n-d_G(v_i)) + \frac{1}{2}\sum_{i=1}^{n}(d_G(v_i)+1) + \frac{1}{4}\sum_{i=1}^{n}(2n-1-d_G(v_i)) + \frac{3n}{4}$$
$$= (2n^2+2n+3m)/2$$ □

DEFINITION 16. [10] *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be graphs. Let G be a join graph $G_1 + G_2$. Vertices and edges of join graph G are $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$, respectively.*

THEOREM 17. *Let G_1 and G_2 be two graphs on disjoint sets of n and s vertices, m and p edges, respectively. Then, $C(\mu(G_1+G_2)) = (n^2+s^2+n+s) + \frac{(6m+6p+14ns)}{4}$.*

PROOF. Let the vertex set $V(\mu(G_1+G_2)) = V(G_1)\cup V(G_2)\cup V'(G_1)\cup V'(G_2)\cup \{u\}$. When the $C(\mu(G_1+G_2))$ is calculated for all vertices in the graph $\mu(G_1+G_2)$, the vertices in five cases should be examined.

Case1. Let $x_i \in V(G_1)$, where $i=\overline{1,n}$. The vertex x_i is adjacent to $d_{G_1}(x_i)$-vertices of $V(G_1)$, similarly $d_{G_1}(x_i)$-vertices of $V'(G_1)$, and whole vertices of $V(G_2)$ and $V'(G_2)$. There are $2(n-d_{G_1}(x_i))$- vertices remaining in the graph $\mu(G_1+G_2)$, and then distance from vertex x_i to these vertices is 2. Thus, we get

$$(12)\qquad \sum_{i=1}^{n} C(x_i) = \sum_{i=1}^{n}(d_{G_1}(x_i)+s) + \frac{1}{2}\sum_{i=1}^{n}(n-d_{G_1}(x_i))$$

Case2. Let $y_j \in V(G_2)$, where $j=\overline{1,s}$. The proof is similar to Case1. So, we get

$$(13)\qquad \sum_{j=1}^{s} C(y_j) = \sum_{j=1}^{s}(d_{G_2}(y_j)+n) + \frac{1}{2}\sum_{j=1}^{s}(s-d_{G_2}(y_j))$$

Case3. Let $x'_i \in V'(G_1)$, where $i=\overline{1,n}$. The vertex x'_i is adjacent to whole vertices of $V(G_2)$, $d_{G_1}(x_i)$-vertices of $V(G_1)$ and vertex u. Moreover, distance from vertex x'_i to remaining vertices is 2. Then, we get

$$(14)\qquad \sum_{i=1}^{n} C(x'_i) = \frac{1}{2}\sum_{i=1}^{n}(d_{G_1}(x_i)+s+1) + \frac{1}{4}\sum_{i=1}^{n}(2n+s-d_{G_1}(x_i)-1)$$

Case4. Let $y'_j \in V'(G_2)$, where $j=\overline{1,s}$. The proof similar to Case3. So, we get

$$(15)\qquad \sum_{j=1}^{s} C(y'_j) = \frac{1}{2}\sum_{j=1}^{s}(d_{G_2}(y_j)+n+1) + \frac{1}{4}\sum_{j=1}^{s}(2s+n-d_{G_2}(y_j)-1)$$

Case5. Let $x_i \in V(G_1)$, $y_j \epsilon V(G_2)$, $x'_i \epsilon V'(G_1)$ and $y'_j \epsilon V'(G_2)$. By the structure of graph $\mu(G_1+G_2)$, we have $d_{\mu(G_1+G_2)}(u,x'_i) = d_{\mu(G_1+G_2)}(u,y'_j) = 1$ and $d_{\mu(G_1+G_2)}(u,x_i) = d_{\mu(G_1+G_2)}(u,y_j) = 2$ for $i=\overline{1,n}$ and $j=\overline{1,s}$. As a result, we get

$$(16)\qquad C(u) = (n+s)(2^{-1}) + (n+s)(2^{-1}) = \frac{3n+3s}{4}$$

By summing (12), (13), (14),(15) and (16), we have:

$$\begin{aligned} C(\mu(G_1+G_2)) &= \sum_{i=1}^{n} C(x_i) + \sum_{j=1}^{s} C(y_j) + \sum_{i=1}^{n} C(x'_i) + \sum_{j=1}^{s} C(y'_j) + C(u) \\ &= \sum_{i=1}^{n}(d_{G_1}(x_i)+s) + \tfrac{1}{2}\sum_{i=1}^{n}(n-d_{G_1}(x_i)) + \sum_{j=1}^{s}(d_{G_2}(y_j)+n) \\ &+\tfrac{1}{2}\sum_{j=1}^{s}(s-d_{G_2}(y_j)) + \tfrac{1}{2}\sum_{i=1}^{n}(d_{G_1}(x_i)+s+1) \\ &+\tfrac{1}{4}\sum_{i=1}^{n}(2n+s-d_{G_1}(x_i)-1) + \tfrac{1}{2}\sum_{j=1}^{s}(d_{G_2}(y_j)+n+1) \\ &+\tfrac{1}{4}\sum_{j=1}^{s}(2s+n-d_{G_2}(y_j)-1) + \tfrac{(3n+3s)}{4} \\ &= (n^2+s^2+n+s) + \tfrac{(6m+6p+14ns)}{4} \end{aligned}$$

□

5. Conclusion

Network design problems arise in many important fields such as telecommunication, transportation, distribution and logistics. Since this situation, vulnerability measures of networks are important increasingly. In this paper we investigate a new measure which is more sensitive than other vulnerability parameters for the reliability of a graph, vertex residual closeness, recently introduced by Dangalchev. Mycielski networks can be modelled by Mycielski graphs. They may be use encoding to transformation of a graph. Consequently, these considerations motivated us to investigate the vulnerability of some Mycielski networks by using the closeness, vertex residual closeness and normalized vertex residual closeness.

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