A combinatorial proof of a general two-term recurrence

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Abstract

We give a short combinatorial proof for the solution of a general two-term recurrence $u(n,k) = u(n-1,k-1) + (a_{n-1} + b_k)u(n-1,k)$, which was discovered by Mansour et al. [4].

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1 Introduction

Let $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ be sequences of complex numbers where the b_i's are distinct. Mansour et al. [4] discovered a two-term recurrence

$$u(n,k) = u(n-1,k-1) + (a_{n-1} + b_k)u(n-1,k), \; n,k \geq 1, \tag{1.1}$$

with boundary conditions $u(n,0) = \prod_{i=0}^{n-1}(a_i + b_0)$ and $u(0,k) = \delta_{0,k}$, where $\delta_{i,j}$ is the Kronecker delta function. This recurrence relation is a generalization of the recurrence formulas for the Stirling numbers and the Lah numbers [1,3,5,8,9]. Using generating functions, the authors [4] derived the following formula.

Theorem 1.1.

$$u(n,k) = \sum_{j=0}^{k} \frac{\prod_{i=0}^{n-1}(b_j + a_i)}{\prod_{i=0,i\neq j}^{k}(b_j - b_i)}. \tag{1.2}$$

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Recently, Simpson [6] also presented an inductive proof of formula (1.2). In this note, we will give a short combinatorial proof of Theorem 1.1 showing its equivalence to a basic result from the theory of Schur functions.

2 A combinatorial proof of Theorem 1.1

Let $S = \{a_i\}_{i\geq 0} \cup \{b_i\}_{i\geq 0} \cup \{1\}$. To construct a combinatorial structure for the recurrence (1.1), we consider the set of words $w = w_1 w_2 \cdots w_n$ on S satisfying the following conditions:

- There are exactly i letters to the left of each a_i;
- There are exactly i 1's to the left of each b_i.

Denote by $\mathcal{U}_{n,k}$, the set of such words of length n containing exactly k 1's, and by $P(w)$, the product of all the letters within a word w. For example, $w = 11b_2a_3b_21b_3a_7$ is a word in $\mathcal{U}_{8,3}$ and $P(w) = a_3a_7b_2^2b_3$. Define $p(n,k) = \sum_{w\in\mathcal{U}_{n,k}} P(w)$. For a word $w = w_1w_2\cdots w_n$ in $\mathcal{U}_{n,k}$, it is possible that $w_n = 1$ or $w_n = a_{n-1}$ or $w_n = b_k$. This implies $p(n,k)$ satisfies the same recurrence relation as (1.1) along with the same boundary conditions. It follows that $p(n,k) = u(n,k)$.

Based on the above combinatorial interpretation of $u(n,k)$, we proceed to give an expression for $u(n,k)$. Consider the terms of $u(n,k)$ with exactly t a_i's, that is, $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ where $0 \leq i_1 < i_2 < \cdots < i_t \leq n-1$. These terms correspond to the words in $\mathcal{U}_{n,k}$ with exactly t a_i's whose positions are determined by the index sequence $i_1, i_2, \ldots, i_t$. To obtain such words, we need to fill the remaining $n-t$ positions by $n-k-t$ b_i's and k 1's. Each filling can be determined by the index sequence $j_1, j_2, \ldots, j_{n-k-t}$ of b_i's with $0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k-t} \leq k$. Therefore, the coefficient of the terms in $u(n,k)$ with exactly t a_i's is

$$h_{n-k-t}(b_0, b_1, \ldots, b_k) = \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k-t}\leq k} b_{j_1}b_{j_2}\cdots b_{j_{n-k-t}}, \tag{2.1}$$

where $h_{n-k-t}(b_0, b_1, \ldots, b_k)$ is the complete homogeneous symmetric function, see [7].

Proof of Theorem 1.1. We only need to prove that the coefficient of the terms with exactly t a_i's in (1.2) is the same as (2.1).

It is clear that the coefficient of the terms with exactly t a_i's in (1.2) equals

$$\sum_{j=0}^{k} \frac{b_j^{n-t}}{\prod_{i=0,i\neq j}^{k}(b_j - b_i)} = \frac{\sum_{j=0}^{k}(-1)^j b_j^{n-t} \prod_{\substack{0\leq i<m\leq k\\ i,m\neq j}}(b_i - b_m)}{\prod_{0\leq i<j\leq k}(b_i - b_j)} = \frac{\det(b_i^{\lambda_j+k-j})_{i,j=0}^{k}}{\det(b_i^{k-j})_{i,j=0}^{k}}. \tag{2.2}$$

with $\lambda_0 = n - k - t, \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$, by cofactor expansion along the first column and the formula for Vandermonde's determinant. (For a combinatorial proof of Vandermonde's formula, see [2].) The expression (2.2) equals (2.1) by applying [7, Theorem 7.15.1]. This completes the proof. □

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