Noncrossing partitions with fixed points having specific properties*

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Abstract

The noncrossing partitions with fixed points were introduced and studied in the literature. In this paper, as their continuations, the expressions of $f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3})$ and $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4})$ are given, respectively. Moreover, the noncrossing partitions with fixed points having specific property $\mathscr{P}$ are introduced, and the number of such noncrossing partitions is described through a function of several variables: $f_m^{\mathscr{P}}(x_1, x_2, \dots, x_m)$. Besides, the counting formulas of $f_m^{\mathscr{P}}(x_1, 0^{m-1})$ and $f_m^{\mathscr{P}}(x_1, x_2, 0^{m-2})$ are obtained for various properties $\mathscr{P}$.

Keywords: Noncrossing partitions, Fixed points.

1 Introduction

A partition $\pi = B_1/B_2/\cdots/B_m$ of a totally ordered set X is called a *noncrossing partition* (n.c.p.) if and only if there do not exist four elements $a < b < c < d$ of X such that $a, c \in B_i$ and $b, d \in B_j$, where $i \neq j$. The set of all n.c.p. of X and the set of all n.c.p. of X that contain exactly m blocks $B_1, B_2, \dots, B_m$ are denoted by $NC(X)$ and $NC(X, m)$, respectively. If $|X| = n$, we can equivalently deal with $[n] = \{1, 2, \dots, n\}$ instead of X, and in this case we will use the notations NC_n and $NC_n(m)$ respectively.

Many authors have worked on n.c.p. (see e.g. [5, 6]). It is well known that $|NC_n|$ equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and $|NC_n(m)|$ equals the Narayana number $N(n, m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$. Furthermore, an interesting class of n.c.p. was introduced in [1]: $\pi \in NC(X)$ is called a *noncrossing partition with fixed points*

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the elements of a given set $A \subseteq X$ if and only if every block of π contains exactly one element of A. The set of all these n.c.p. is denoted by $NC(X, A)$.

Since the distribution of the elements of A in X determines the cardinality of $NC(X, A)$, it is more convenient to restrict the problem to the following equivalent case: let $A = [m]$ and $X = [m] \cup Y$, where the elements of Y are distributed in the intervals $(i, i+1)$ $(i = 1, 2, \dots, m-1)$ and $(m, +\infty)$, so that $X \cap (i, i+1) = X_i$ $(i = 1, 2, \dots, m-1)$ and $X \cap (m, +\infty) = X_m$. From now on, without loss of generality, we may assume that for every n.c.p. $\pi = B_1/B_2/\cdots/B_m \in NC(X, [m])$, we have $i \in B_i$ for every $i \in [m]$. A function f_m of m variables is defined by $f_m(x_1, x_2, \dots, x_m) = |NC(X, [m])|$, where $x_i = |X_i|$ $(\forall\, i \in [m])$.

For convenience, let $f_m(y_1^{l_1}, \dots, y_k^{l_k}) = f_m(\underbrace{y_1, \dots, y_1}_{l_1}, \dots, \underbrace{y_k, \dots, y_k}_{l_k})$, where $1 \le k \le m$, $\sum_{j=1}^{k} l_j = m$ and $0 \le l_j \le m$ $(j = 1, 2, \dots, k)$. For example,

$$f_6(1^2, 6^1, 3^0, 4^3) = f_6(1^2, 6^1, 4^3) = f_6(1, 1, 6, 4, 4, 4).$$

In [1], Sapounakis and Tsikouras claimed that it is difficult to obtain an explicit formula of $f_m(x_1, x_2, \dots, x_m)$. But for some special cases, the expressions were obtained in [1, 2]: for every m, $x_1, x_2 \in N$ with $m \ge 2$,

$$f_m(x_1, 0^{m-1}) = \binom{x_1 + m - 1}{m - 1},$$

$$f_m(x_1, x_2, 0^{m-2}) = \binom{x_1 + x_2 + m}{m} - \binom{x_1 + m - 1}{m} - \binom{x_2 + m - 1}{m} \text{ ([1, 2]).}$$

For every m, $x_1, x_{\rho+2} \in N$ with $m \ge 4$ and every $\rho \in N^*$ with $2\rho \le m - 2$,

$$\begin{aligned} f_m(x_1, 0^\rho, x_{\rho+2}, 0^{m-\rho-2}) &= \binom{x_1 + x_{\rho+2} + m}{m} - \binom{x_1 + m - 1}{m} - \binom{x_{\rho+2} + m - 1}{m} \\ &+ \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-\delta} \binom{x_1 + k - 1}{k} \binom{x_{\rho+2} + m - k - 1}{m - k} \text{ ([2]).} \end{aligned}$$

In addition, Weng and Liu [3] gave the expressions of $f_m(x_1, x_2, x_3, 0^{m-3})$ and $f_m(x_1, x_2, 0^\rho, x_{\rho+3}, 0^{m-\rho-3})$ $(2\rho \le m - 3)$. As their continuations, the recurrences of $f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3})$ $(1 \le \mu \le \rho \le m - \rho - \mu - 3)$ and $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4})$ $(1 \le \mu \le \rho \le m - \rho - \mu - 4)$ are obtained in Section 2.

It is well known that there are various kinds of partitions having specific properties in classic combinatorics. It is natural to consider the noncrossing partitions with fixed points having specific properties. Let $NC(X, [m], \mathscr{P})$ denote the set of all n.c.p. $\pi = B_1/B_2/\cdots/B_m$ in $NC(X, [m])$ such that $B_1, \dots, B_m$ satisfy property $\mathscr{P}$. Analogously, we define $f_m^{\mathscr{P}}(x_1, x_2, \dots, x_m) = |NC(X, [m], \mathscr{P})|$, where $x_i = |X_i|$ $(i \in [m])$. For instance, the noncrossing matching with fixed points introduced in [4] is one kind of noncrossing partition with fixed points having a specific property $\mathscr{P}$, where $\mathscr{P}$ prescribes that $|B_i| = 2$ for every $i \in [m]$.

In Section 3, the counting formulas of $f_m^{\mathscr{P}}(x_1, 0^{m-1})$ and $f_m^{\mathscr{P}}(x_1, x_2, 0^{m-2})$ are obtained for various properties $\mathscr{P}$, such as $|B_i| \equiv l \ (mod\ k)$ $(\forall\, i \in [m])$, $|B_i| \neq |B_j|$ $(\forall\, i, j \in [m], i \neq j)$, $|B_i| \geq k$ $(\forall\, i \in [m])$, $|B_i| \leq k$ $(\forall\, i \in [m])$, $|B_i| \neq k$ $(\forall\, i \in [m])$ etc., where l, k are nonnegative integers.

2 The expressions of $f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3})$ and $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4})$

In this section, the recurrences of $f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3})$ and $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4})$ are obtained, respectively.

Note that f_m is well defined and $f_m(x_1, \dots, x_m) = f_m(y_1, \dots, y_m)$, whenever the sequence $(y_1, \dots, y_m)$ or its reverse is a cyclic permutation of $(x_1, \dots, x_m)$. Therefore, for $f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3})$, the variants μ, ρ can be restricted as $1 \leq \mu \leq \rho \leq m-\rho-\mu-3$. Similarly, suppose that $1 \leq \mu \leq \rho \leq m-\rho-\mu-4$ for $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4})$.

Theorem 2.1 *For every $m, \rho, \mu \in N$ with $1 \leq \mu \leq \rho \leq m-\rho-\mu-3$,*

$$
\begin{aligned}
& f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3}) \\
= & \sum_{\alpha=0}^{x_1} f_{m-1}(\alpha, 0^{\mu-1}\, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3}) \\
+ & \sum_{\alpha=0}^{x_1}\sum_{\beta=0}^{x_{\mu+2}-1}\sum_{\gamma=0}^{\beta} f_{m-\mu-1}(\alpha+\gamma, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3}) \cdot f_\mu(\beta-\gamma, 0^{\mu-1}) \\
+ & \sum_{\alpha=0}^{x_1}\sum_{\varphi=0}^{x_{\mu+\rho+3}-1}\sum_{\theta=0}^{\varphi} f_{\rho+\mu+1}(\varphi-\theta, 0^{\mu-1}, x_{\mu+2}, 0^\rho) \cdot f_{m-\rho-\mu-2}(\alpha+\theta, 0^{m-\rho-\mu-3}) \\
+ & \sum_{\alpha=0}^{x_1}\sum_{\beta=0}^{x_{\mu+2}-1}\sum_{\varphi=0}^{x_{\mu+\rho+3}-1}\sum_{\gamma=0}^{\beta}\sum_{\theta=0}^{\varphi} f_{\rho+1}(\varphi-\theta+\gamma, 0^\rho) \\
& \cdot f_{m-\rho-\mu-2}(\alpha+\theta, 0^{m-\mu-\rho-3}) \cdot f_\mu(\beta-\gamma, 0^{\mu-1}).
\end{aligned}
$$

Proof. Clearly, for every $m, \rho, \mu \in N$ with $1 \leq \mu \leq \rho \leq m-\rho-\mu-3$,

$$f_m(x_1, 0^\mu, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3}) = |NC(X, [m])|$$

with $X = [m] \cup (\cup_{i\in\{1, \mu+2, \rho+\mu+3\}} X_i)$, $X_i = X \cap (i, i+1)$ and $x_i = |X_i|$, where $i \in \{1, \mu+2, \rho+\mu+3\}$.

We partition the set $NC(X, [m])$ into sets A_α, $B_{\alpha,\beta,\gamma}$, $C_{\alpha,\varphi,\theta}$ and $D_{\alpha,\beta,\gamma,\varphi,\theta}$ (with $\alpha, \beta, \gamma, \varphi, \theta \in N$), which are defined as follows:

Each set A_α consists of all $\pi \in NC(X, [m])$ with the property that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_{\mu+2}| = 0$, $|B_2 \cap X_{\rho+\mu+3}| = 0$. Therefore, $0 \leq \alpha \leq x_1$ and

$$|A_\alpha| = f_{m-1}(\alpha, 0^{\mu-1}\, x_{\mu+2}, 0^\rho, x_{\rho+\mu+3}, 0^{m-\mu-\rho-3}).$$

Each set $B_{\alpha,\beta,\gamma}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_{\mu+2}| = x_{\mu+2} - \beta$, $|\{y \in X_{\mu+2} \mid y > max\{x \in B_2\}\}| = \gamma$, $|B_2 \cap X_{\rho+\mu+3}| = 0$. Hence $0 \le \alpha \le x_1$, $0 \le \gamma \le \beta < x_{\mu+2}$ and

$$|B_{\alpha,\beta,\gamma}| = f_{m-\mu-1}(\alpha+\gamma,\ 0^{\rho},\ x_{\rho+\mu+3},\ 0^{m-\mu-\rho-3}) \cdot f_{\mu}(\beta-\gamma,\ 0^{\mu-1}).$$

Each set $C_{\alpha,\varphi,\theta}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_{\mu+2}| = 0$, $|B_2 \cap X_{\rho+\mu+3}| = x_{\rho+\mu+3} - \varphi$, $|\{y \in X_{\rho+\mu+3} \mid y > max\{x \in B_2\}\}| = \theta$. Hence $0 \le \alpha \le x_1$, $0 \le \theta \le \varphi < x_{\rho+\mu+3}$ and

$$|C_{\alpha,\varphi,\theta}| = f_{\rho+\mu+1}(\varphi-\theta,\ 0^{\mu-1},\ x_{\mu+2},\ 0^{\rho}) \cdot f_{m-\rho-\mu-2}(\alpha+\theta,\ 0^{m-\rho-\mu-3}).$$

Each set $D_{\alpha,\beta,\gamma,\varphi,\theta}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_{\mu+2}| = x_{\mu+2} - \beta$, $|\{y \in X_{\mu+2} \mid y > max\{x \in B_2\}\}| = \gamma$, $|B_2 \cap X_{\rho+\mu+3}| = x_{\rho+\mu+3} - \varphi$, $|\{y \in X_{\rho+\mu+3} \mid y > max\{x \in B_2\}\}| = \theta$. Hence $0 \le \alpha \le x_1$, $0 \le \gamma \le \beta < x_{\mu+2}$, $0 \le \theta \le \varphi < x_{\rho+\mu+3}$ and

$$|D_{\alpha,\beta,\gamma,\varphi,\theta}| = f_{\rho+1}(\varphi-\theta+\gamma,\ 0^{\rho}) \cdot f_{m-\rho-\mu-2}(\alpha+\theta,\ 0^{m-\mu-\rho-3}) \cdot f_{\mu}(\beta-\gamma,\ 0^{\mu-1}).$$

By combining the above cases, we have

$$f_m(x_1,\ 0,\ x_{\mu+2},\ 0^{\rho},\ x_{\rho+\mu+3},\ 0^{m-\rho-4}) = \sum_{\alpha=0}^{x_1} |A_\alpha| + \sum_{\alpha=0}^{x_1} \sum_{\beta=0}^{x_{\mu+2}-1} \sum_{\gamma=0}^{\beta} |B_{\alpha,\beta,\gamma}|$$

$$+ \sum_{\alpha=0}^{x_1} \sum_{\varphi=0}^{x_{\rho+\mu+3}-1} \sum_{\theta=0}^{\varphi} |C_{\alpha,\varphi,\theta}| + \sum_{\alpha=0}^{x_1} \sum_{\beta=0}^{x_{\mu+2}-1} \sum_{\varphi=0}^{x_{\rho+\mu+3}-1} \sum_{\gamma=0}^{\beta} \sum_{\theta=0}^{\varphi} |D_{\alpha,\beta,\gamma,\varphi,\theta}|.$$

Consequently, the desired result is obtained. □

Theorem 2.2 *For every $m, \rho, \mu \in N$ with $1 \le \mu \le \rho \le m - \rho - \mu - 4$,*

$$\begin{aligned}
& f_m(x_1,\ x_2,\ 0^{\mu},\ x_{\mu+3},\ 0^{\rho},\ x_{\rho+\mu+4},\ 0^{m-\rho-\mu-4}) \\
= & \sum_{\alpha=0}^{x_1} \sum_{\eta=0}^{x_2} f_{m-1}(\alpha+\eta,\ 0^{\mu},\ x_{\mu+3},\ 0^{\rho},\ x_{\rho+\mu+4},\ 0^{m-\mu-\rho-4}) \\
+ & \sum_{\alpha=0}^{x_1} \sum_{\eta=0}^{x_2} \sum_{\beta=0}^{x_{\mu+3}-1} \sum_{\gamma=0}^{\beta} f_{\mu+1}(\eta+\beta-\gamma,\ 0^{\mu}) \cdot f_{m-\mu-2}(\alpha+\gamma,\ 0^{\rho},\ x_{\rho+\mu+4},\ 0^{m-\mu-\rho-4}) \\
+ & \sum_{\alpha=0}^{x_1} \sum_{\eta=0}^{x_2} \sum_{\varphi=0}^{x_{\mu+\rho+4}-1} \sum_{\theta=0}^{\varphi} f_{\rho+\mu+2}(\eta+\varphi-\theta,\ 0^{\mu},\ x_{\mu+3},\ 0^{\rho}) \\
& \cdot f_{m-\mu-\rho-3}(\alpha+\theta,\ 0^{m-\mu-\rho-4}) \\
+ & \sum_{\alpha=0}^{x_1} \sum_{\eta=0}^{x_2} \sum_{\beta=0}^{x_{\mu+3}-1} \sum_{\varphi=0}^{x_{\mu+\rho+4}-1} \sum_{\gamma=0}^{\beta} \sum_{\theta=0}^{\varphi} f_{\mu+1}(\eta+\beta-\gamma,\ 0^{\mu}) \\
& \cdot f_{\rho+1}(\gamma+\varphi-\theta,\ 0^{\rho}) \cdot f_{m-\mu-\rho-3}(\alpha+\theta,\ 0^{m-\mu-\rho-4}).
\end{aligned}$$

Proof. Note that $f_m(x_1, x_2, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\rho-\mu-4}) = |NC(X, [m])|$ with $X = [m] \cup (\cup_{i\in\{1, 2, \mu+3, \rho+\mu+4\}} X_i)$, $X_i = X \cap (i, i+1)$ and $x_i = |X_i|$, where $i \in \{1, 2, \mu+3, \rho+\mu+4\}$. Hence the set $NC(X, [m])$ can be partitioned into sets $A_{\alpha,\eta}$, $B_{\alpha,\eta,\beta,\gamma}$, $C_{\alpha,\eta,\varphi,\theta}$ and $D_{\alpha,\eta,\beta,\gamma,\varphi,\theta}$ (with $\alpha, \eta, \beta, \gamma, \varphi, \theta \in N$), which are defined as follows:

Each set $A_{\alpha,\eta}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_2| = x_2 - \eta$, $|B_2 \cap X_{\mu+3}| = |B_2 \cap X_{\mu+\rho+4}| = 0$. Thus $0 \le \alpha \le x_1$, $0 \le \eta \le x_2$ and $|A_{\alpha,\eta}| = f_{m-1}(\alpha+\eta, 0^\mu, x_{\mu+3}, 0^\rho, x_{\rho+\mu+4}, 0^{m-\mu-\rho-4})$.

Each set $B_{\alpha,\eta,\beta,\gamma}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_2| = x_2 - \eta$, $|B_2 \cap X_{\mu+3}| = x_{\mu+3} - \beta$, $|\{y \in X_{\mu+3} \mid y > max\{x \in B_2\}\}| = \gamma$, $|B_2 \cap X_{\mu+\rho+4}| = 0$. Hence $0 \le \alpha \le x_1$, $0 \le \eta \le x_2$, $0 \le \gamma \le \beta < x_{\mu+3}$ and $|B_{\alpha,\eta,\beta,\gamma}| = f_{\mu+1}(\eta+\beta-\gamma, 0^\mu) \cdot f_{m-\mu-2}(\alpha+\gamma, 0^\rho, x_{\rho+\mu+4}, 0^{m-\mu-\rho-4})$.

Each set $C_{\alpha,\eta,\varphi,\theta}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1 - \alpha$, $|B_2 \cap X_2| = x_2 - \eta$, $|B_2 \cap X_{\mu+3}| = 0$, $|B_2 \cap X_{\mu+\rho+4}| = x_{\mu+\rho+4} - \varphi$ and $|\{y \in X_{\mu+\rho+4} \mid y > max\{x \in B_2\}\}| = \theta$. Consequently, $0 \le \alpha \le x_1$, $0 \le \eta \le x_2$, $0 \le \theta \le \varphi < x_{\mu+\rho+4}$, and $|C_{\alpha,\eta,\varphi,\theta}| = f_{\rho+\mu+2}(\eta+\varphi-\theta, 0^\mu, x_{\mu+3}, 0^\rho) \cdot f_{m-\mu-\rho-3}(\alpha+\theta, 0^{m-\mu-\rho-4})$.

Each set $D_{\alpha,\eta,\beta,\gamma,\varphi,\theta}$ consists of all $\pi \in NC(X, [m])$ such that $|B_2 \cap X_1| = x_1-\alpha$, $|B_2 \cap X_2| = x_2-\eta$, $|B_2 \cap X_{\mu+3}| = x_{\mu+3}-\beta$, $|\{y \in X_{\mu+3} \mid y > max\{x \in B_2\}\}| = \gamma$, $|B_2 \cap X_{\mu+\rho+4}| = x_{\mu+\rho+4} - \varphi$ and $|\{y \in X_{\mu+\rho+4} \mid y > max\{x \in B_2\}\}| = \theta$. Thus $0 \le \alpha \le x_1$, $0 \le \eta \le x_2$, $0 \le \gamma \le \beta < x_{\mu+1}$, $0 \le \theta \le \varphi < x_{\mu+\rho+1}$, $|D_{\alpha,\eta,\beta,\gamma,\varphi,\theta}| = f_{\mu+1}(\eta+\beta-\gamma, 0^\mu) \cdot f_{\rho+1}(\gamma+\varphi-\theta, 0^\rho) \cdot f_{m-\mu-\rho-3}(\alpha+\theta, 0^{m-\mu-\rho-4})$.

Similarly as in Theorem 2.1, the result follows. □

3 Noncrossing partitions with fixed points having property $\mathscr{P}$

Let $P[m; n; \mathscr{P}]$ denote the number of positive integer solutions of the equation $t_1 + t_2 + \cdots + t_m = n$, where $t_1, \ldots, t_m$ satisfy property $\mathscr{P}$ ([7, 8]). To begin with, a useful lemma for characterizing $f_m^{\mathscr{P}}(x_1, 0^{m-1})$ is obtained.

Lemma 3.1 *For every* $m \in N^*$, $f_m^{\mathscr{P}}(x_1, 0^{m-1}) = P[m; x_1+m; \mathscr{P}]$.

Proof. We deal with the set $NC(X, [m], \mathscr{P})$ with $X = [m] \cup Y$, where $Y \subseteq (1, 2)$ and $|Y| = x_1$. Then $f_m^{\mathscr{P}}(x_1, 0^{m-1}) = |NC(X, [m], \mathscr{P})|$. Consequently, the proof of Lemma 3.1 is similar to that of Proposition 3.2 of [1]. □

Now for various properties $\mathscr{P}$, the counting formulas of $f_m^{\mathscr{P}}(x_1, 0^{m-1})$ and $f_m^{\mathscr{P}}(x_1, x_2, 0^{m-2})$ are investigated.

Let $f_m^{j\,(mod\,k)}(x_1, x_2, \ldots, x_m) = |NC(X, [m], j\,(mod\,k))|$, where $\pi = B_1/\cdots/B_m$ in $NC(X, [m], j\,(mod\,k))$ ($j, k \in N$ with $k > j \ge 0$) satisfies $|B_i| \equiv j\,(mod\,k)$ ($\forall\, i \in [m]$). In particular, $f_m(x_1, x_2, \ldots, x_m) = f_m^{0\,(mod\,1)}(x_1, x_2, \ldots, x_m)$.

Theorem 3.2 *(1) For every* $m, k \in N^*$, *and* $x_1 \equiv m(j-1)\,(mod\,k)$,

$$f_m^{j\,(mod\,k)}(x_1, 0^{m-1}) = \begin{cases} \binom{\frac{x_1+m}{k}-1}{m-1}, & if\ j = 0; \\ \binom{\frac{x_1+m(k-j+1)}{k}-1}{m-1}, & if\ j \neq 0. \end{cases}$$

(2) For every $m, k \in N^*$ with $m \geq 2$, and $x_1 + x_2 \equiv m(j-1) \pmod k$,

$$f_m^{j \pmod k}(x_1, x_2, 0^{m-2}) = \begin{cases} \sum_{u=0}^{x_1} \sum_v \binom{\frac{u+v+m-1}{k}-1}{m-2}, & if\ j = 0; \\ \sum_{u=0}^{x_1} \sum_v \binom{\frac{u+v+(m-1)(k-j+1)}{k}-1}{m-2}, & if\ j \neq 0, \end{cases}$$

where the sum $\sum_v$ is over all integers $0 \leq v \leq x_2$ with $v \equiv x_1+x_2-u-j+1 \pmod k$.

Proof. (1) By Lemma 3.1, for every $m \in N^*$ and $x_1 \equiv m(j-1) \pmod k$,

$$f_m^{j \pmod k}(x_1, 0^{m-1}) = P[m;\ x_1 + m;\ t_i \equiv j \pmod k\ (\forall\ i \in [m])].$$

When $j = 0$, let $t_i = ks_i$ ($\forall\ i \in [m]$); it follows that

$$f_m^{0 \pmod k}(x_1, 0^{m-1}) = P[m;\ \frac{x_1+m}{k}] = \binom{\frac{x_1+m}{k}-1}{m-1}.$$

When $j \neq 0$, let $t_i = k(s_i - 1) + j$ ($\forall\ i \in [m]$); then we have

$$f_m^{j \pmod k}(x_1, 0^{m-1}) = P[m;\ \frac{x_1 + m(k-j+1)}{k}] = \binom{\frac{x_1+m(k-j+1)}{k}-1}{m-1}.$$

(2) Notice that $f_m^{j \pmod k}(x_1, x_2, 0^{m-2}) = |NC(X, [m], j \pmod k)|$ with $X = [m] \cup Y$, $Y \subseteq (1, 2) \cup (2, 3)$, $|Y \cap (1, 2)| = x_1$ and $|Y \cap (2, 3)| = x_2$. Then we partition the set $NC(X, [m], j \pmod k)$ into sets $A_{u, v}$ ($u, v \in N$, $u \leq x_1$, $v \leq x_2$), where each set $A_{u, v}$ consists of all $\pi = B_1/B_2/\cdots/B_m \in NC(X, [m], j \pmod k)$ with the property that B_2 contains $x_1 - u$ elements of $(1, 2)$ and $x_2 - v$ elements of $(2, 3)$, $|B_i| \equiv j \pmod k$ ($\forall\ i \in [m]$). Thus, by (1), we get

$$|A_{u, v}| = f_{m-1}^{j \pmod k}(u + v, 0^{m-2}) = \begin{cases} \binom{\frac{u+v+m-1}{k}-1}{m-2}, & if\ j = 0; \\ \binom{\frac{u+v+(m-1)(k-j+1)}{k}-1}{m-2}, & if\ j \neq 0. \end{cases}$$

Moreover, notice that $(x_1 - u) + (x_2 - v) + 1 \equiv j \pmod k$, and hence $v \equiv x_1 + x_2 - u - j + 1 \pmod k$. This completes the proof. □

Let $f_m^{even}(x_1, x_2, \cdots, x_m) = |NC(X, [m], even)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X, [m], even)$ satisfies $|B_i|$ ($\forall\ i \in [m]$) is even. (Analogously, we can define $f_m^{odd}(x_1, x_2, \cdots, x_m) = |NC(X, [m], odd)|$.) Then $f_m^{even}(x_1, x_2, \ldots, x_m) = f_m^{0 \pmod 2}(x_1, x_2, \ldots, x_m)$, and $f_m^{odd}(x_1, x_2, \ldots, x_m) = f_m^{1 \pmod 2}(x_1, x_2, \ldots, x_m)$. From Theorem 3.2, for $k = 2$ we obtain the following results.

Corollary 3.3 *(1) For every $m \in N^*$ such that $x_1 + m$ even,*

$$f_m^{even}(x_1, 0^{m-1}) = \binom{\frac{x_1+m}{2}-1}{m-1}.$$

(2) For every $m \in N^*$ and x_1 even,

$$f_m^{odd}(x_1, 0^{m-1}) = \binom{\frac{x_1}{2}+m-1}{m-1}.$$

Let $f_m^{\neq}(x_1, x_2, \ldots, x_m) = |NC(X, [m], \neq)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X, [m], \neq)$ satisfies $|B_i| \neq |B_j|$ ($\forall\, i, j \in [m], i \neq j$). Let $P_m(n)$ (resp. $P_m^{\neq}(n)$) denote the number of partitions of n into m (resp. distinct) parts. Obviously, $P_1(n) = 1, P_m(m) = 1, P_m(n) = 0$ $(m > n)$, and $P_m^{\neq}(n) = P_m(n - \frac{m^2-m}{2})$ ([7]).

Theorem 3.4 *For every $m \in N^*$,*

$$f_m^{\neq}(x_1, 0^{m-1}) = m! \cdot P_m^{\neq}(x_1 + m) = m! \cdot P_m(x_1 - \frac{m(m-3)}{2}).$$

Proof. By Lemma 3.1, for every $m \in N^*$,

$$f_m^{\neq}(x_1, 0^{m-1}) = P[m;\ x_1 + m;\ t_i \neq t_j \ for\ i, j \in [m], i \neq j]$$

$$= m! \cdot P_m^{\neq}(x_1 + m) = m! \cdot P_m(x_1 - \frac{m(m-3)}{2}).$$

□

Remark 1 *In particular, since $P_2(n) = \lfloor \frac{n}{2} \rfloor$ and $P_3(n) = \lfloor \frac{n^2+3}{12} \rfloor$, we have*

$$f_2^{\neq}(x_1, 0) = 2 \cdot \lfloor \frac{x_1 + 1}{2} \rfloor, \ and\ f_3^{\neq}(x_1, 0, 0) = 6 \cdot \lfloor \frac{x_1^2 + 3}{12} \rfloor.$$

Let $f_m^{\geq k}(x_1, x_2, \ldots, x_m) = |NC(X, [m], \geq k)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X, [m], \geq k)$ satisfies $|B_i| \geq k$ ($\forall\, i \in [m]$).

Theorem 3.5 *(1) For every $m, k \in N^*$, and $x_1 \geq m(k-1)$,*

$$f_m^{\geq k}(x_1, 0^{m-1}) = \binom{x_1 - m(k-2) - 1}{m-1}.$$

(2) For every $m, k \in N^*$ with $m \geq 2$, and $x_1 + x_2 \geq m(k-1)$,

$$f_m^{\geq k}(x_1, x_2, 0^{m-2}) = \sum_{u=0}^{x_1} \sum_{v=0}^{x_1+x_2-u-k+1} \binom{u + v - (m-1)(k-2) - 1}{m-2}.$$

Proof. (1) By Lemma 3.1, for every $m, k \in N^*$, and $x_1 \geq m(k-1)$,

$$f_m^{\geq k}(x_1, 0^{m-1}) = P[m;\ x_1 + m;\ t_i \geq k\ (\forall\, i \in [m])].$$

Let $t_i = s_i + k - 1$ $(i \in [m])$. It follows that

$$f_m^{\geq k}(x_1, 0^{m-1}) = P[m;\ x_1 - m(k-2)] = \binom{x_1 - m(k-2) - 1}{m-1}.$$

(2) Note that $f_m^{\geq k}(x_1, x_2, 0^{m-2}) = |NC(X, [m], \geq k)|$ with $X = [m] \cup Y$, $Y \subseteq (1, 2) \cup (2, 3)$, $|Y \cap (1, 2)| = x_1$ and $|Y \cap (2, 3)| = x_2$. We partition the set $NC(X, [m], \geq k)$ into sets $A_{u,v}$ $(u, v \in N, u \leq x_1, v \leq x_2)$ defined as follows:

Each set $A_{u,\, v}$ consists of all $\pi = B_1/B_2/\cdots/B_m \in NC(X,\ [m], \geq k)$ with the property that B_2 contains $x_1 - u$ elements of $(1,\ 2)$ and $x_2 - v$ elements of $(2,\ 3)$, $|B_i| \geq k$ ($\forall\, i \in [m]$). Thus, using (1), we get

$$|A_{u,\, v}| = f_{m-1}^{\geq k}(u + v,\ 0^{m-2}) = \binom{u + v - (m-1)(k-2) - 1}{m-2}.$$

Besides, note that $(x_1 - u) + (x_2 - v) + 1 \geq k$ implies $v \leq x_1 + x_2 - u - k + 1$. The proof is finished. □

Let $f_m^{\leq k}(x_1,\ x_2,\ \ldots,\ x_m) = |NC(X,\ [m], \leq k)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X,\ [m], \leq k)$ satisfies $|B_i| \leq k$ ($\forall\, i \in [m]$).

Theorem 3.6 *(1) For every $m,\ k \in N^*$, and $x_1 \leq m(k-1)$,*

$$f_m^{\leq k}(x_1,\ 0^{m-1}) = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x_1 + m - ki - 1}{m-1}.$$

(2) For every $m,\ k \in N^*$ with $m \geq 2$, and $x_1 + x_2 \leq m(k-1)$,

$$f_m^{\leq k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1}\ \sum_{v=x_1+x_2-u-k+1}^{x_2}\ \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{u + v + m - ki - 2}{m-2}.$$

Proof. (1) By Lemma 3.1, for every $m,\ k \in N^*$ and $x_1 \leq m(k-1)$,

$$f_m^{\leq k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m;\ t_i \leq k\ (\forall\, i \in [m])].$$

According to the principle of Inclusion-Exclusion, we have

$$f_m^{\leq k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m] + \sum_{i=1}^{m}(-1)^i\binom{m}{i}\cdot P[m;\ x_1 + m;\ t_1,\ t_2,\ \ldots,\ t_i \geq k+1]$$

$$= \sum_{i=0}^{m}(-1)^i\binom{m}{i}P[m;\ x_1 + m - ki] = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x_1 + m - ki - 1}{m-1}.$$

(2) Similarly to the proof of Theorem 3.5 (2), combining with Theorem 3.6 (1), we obtain the desired result. □

Let $f_m^{\neq k}(x_1,\ x_2,\ \ldots,\ x_m) = |NC(X,\ [m],\ \neq k)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X,\ [m],\ \neq k)$ such that $|B_i| \neq k$ ($\forall\, i \in [m]$).

Theorem 3.7 *(1) For every $m,\ k \in N^*$,*

$$f_m^{\neq k}(x_1,\ 0^{m-1}) = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x_1 + m - ki - 1}{m-i-1}.$$

(2) For every $m,\ k \in N^*$ with $m \geq 2$,

$$f_m^{\neq k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1}\ \sum_{\substack{v=0\\ v\neq x_1+x_2-u-k+1}}^{x_2}\ \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{u + v + m - ki - 2}{m-i-2}.$$

Proof. (1) Applying Lemma 3.1, for every $m,\ k \in N^*$,

$$f_m^{\neq k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m;\ t_i \neq k\ (\forall\ i \in [m])].$$

According to the principle of Inclusion-Exclusion, we have

$$f_m^{\neq k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m] + \sum_{i=1}^{m}(-1)^i\binom{m}{i}\cdot P[m;\ x_1 + m;\ t_1,\ t_2,\ \dots,\ t_i = k]$$

$$= \sum_{i=0}^{m}(-1)^i\binom{m}{i}P[m - i;\ x_1 + m - ki] = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x_1 + m - ki - 1}{m - i - 1}.$$

(2) Analogously to the proof of Theorem 3.5 (2), combining with Theorem 3.7 (1), the desired result is obtained. □

Finally, let $f_m^{max\{|B_i|\}=k}(x_1,\ x_2,\ \dots,\ x_m) = |NC(X,\ [m],\ max\{|B_i|\} = k)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X,\ [m],\ max\{|B_i|\} = k)$ satisfies $max\{|B_i| \mid i \in [m]\} = k$. Let $f_m^{min\{|B_i|\}=k}(x_1,\ x_2,\ \dots,\ x_m) = |NC(X,\ [m],\ min\{|B_i|\} = k)|$, where $\pi = B_1/B_2/\cdots/B_m$ in $NC(X,\ [m],\ min\{|B_i|\} = k)$ satisfies $min\{|B_i| \mid i \in [m]\} = k$.

Theorem 3.8 *(1) Let $m,\ k \in N^*$. For $x_1 < m(k-1)$,*

$$f_m^{max\{|B_i|\}=k}(x_1,\ 0^{m-1}) = \sum_{i=1}^{m-1}\sum_{j=0}^{m-i}(-1)^j\binom{m}{i}\binom{m-i}{j}\binom{x_1 + m - ki - (k-1)j - 1}{m - i - 1}.$$

For $x_1 > m(k-1)$,

$$f_m^{min\{|B_i|\}=k}(x_1,\ 0^{m-1}) = \sum_{i=1}^{m-1}\binom{m}{i}\binom{x_1 - m(k-1) - 1}{m - i - 1}.$$

(2) Let $m,\ k \in N^*$. For $x_1 + x_2 < m(k-1)$,

$$f_m^{max\{|B_i|\}=k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1}[\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{x_1 + x_2 + 2m - k(i+1) - 2}{m-2}$$

$$+ \sum_{v=x_1+x_2+2-u-k}^{x_2}\ \sum_{i=1}^{m-2}\sum_{j=0}^{m-i-1}(-1)^j\binom{m-1}{i}\binom{m-i-1}{j}\binom{u + v + 2m - ki - (k-1)j - 3}{m - i - 2}].$$

For $x_1 + x_2 > m(k-1)$,

$$f_m^{min\{|B_i|\}=k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1}[\binom{x_1 + x_2 - k - (m-1)(k-3)}{m-2}$$

$$+ \sum_{v=0}^{x_1+x_2-u-k}\ \sum_{i=1}^{m-2}\binom{m-1}{i}\binom{u + v - (m-1)(k-2) - 1}{m - i - 2}].$$

Proof. (1) By Lemma 3.1, Theorems 3.5 and 3.6, for every $m, k \in N^*$,

$$f_m^{max(|B_i|)=k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m;\ max_{i\in[m]}\{t_i\} = k]$$

$$= \sum_{i=1}^{m-1} \binom{m}{i} \cdot P[m-i;\ x_1 + m - ki;\ s_j \le k-1\ (j \in [m-i])]$$

$$= \sum_{i=1}^{m-1} \binom{m}{i} \cdot \sum_{j=0}^{m-i} (-1)^j \binom{m-i}{j} \binom{x_1 + m - ki - (k-1)j - 1}{m-i-1},\ where\ x_1 < m(k-1).$$

$$f_m^{min(|B_i|)=k}(x_1,\ 0^{m-1}) = P[m;\ x_1 + m;\ min_{i\in[m]}\{t_i\} = k]$$

$$= \sum_{i=1}^{m-1} \binom{m}{i} \cdot P[m-i;\ x_1 + m - ki; s_j \ge k+1\ (j \in [m-i])]$$

$$= \sum_{i=1}^{m-1} \binom{m}{i} \binom{x_1 - m(k-1) - 1}{m-i-1},\ where\ x_1 > m(k-1).$$

(2) Similarly to the proof of Theorem 3.5 (2), we have

$$f_m^{max(|B_i|)=k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1} [[f_{m-1}^{\le k}(x_1 + x_2 + m - k,\ 0^{m-2})] + \sum_{v=x_1+x_2+2-u-k}^{x_2} f_{m-1}^{max(|B_i|)=k}(u + v + m - 1,\ 0^{m-2}),$$

$$f_m^{min(|B_i|)=k}(x_1,\ x_2,\ 0^{m-2}) = \sum_{u=0}^{x_1} [[f_{m-1}^{\ge k}(x_1 + x_2 + m - k,\ 0^{m-2}) + \sum_{v=0}^{x_1+x_2-u-k} f_{m-1}^{min(|B_i|)=k}(u + v + m - 1,\ 0^{m-2}).$$

By Theorems 3.5, 3.6, 3.8 (1), the desired result is obtained. □

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