The minimal size of a graph with given generalized 3-edge-connectivity*

Xueliang Li, Yaping Mao
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
lxl@nankai.edu.cn; maoyaping@ymail.com

Abstract

For $S \subseteq V(G)$ and $|S| \geq 2$, $\lambda(S)$ is the maximum number of edge-disjoint trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is nothing new but the standard edge-connectivity $\lambda(G)$ of G. In this paper, graphs of order n such that $\lambda_3(G) = n-3$ are characterized. Furthermore, we determine the minimal number of edges of a graph G of order n with $\lambda_3(G) = 1, n-3, n-2$ and give a sharp lower bound for $2 \leq \lambda_3(G) \leq n-4$.

Keywords: edge-connectivity, Steiner tree, edge-disjoint trees, generalized edge-connectivity.

AMS subject classification 2010: 05C40, 05C05, 05C75.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let mH be the disjoint union of m copies of a graph H. For $X, Y \subseteq V(G)$, let $E_G[X, Y]$ denote the set of edges of G with one end in X and the other end in Y.

The generalized connectivity of a graph G, introduced by Chartrand et al. in [2], is a natural and nice generalization of the concept of (vertex-)connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint Steiner trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* $\kappa_k(G)$ of G is defined as

*Supported by NSFC No.11071130 and 11371205.

$\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G) \textit{ and } |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized k-connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. Results on the generalized connectivity can be found in [2, 3, 4, 5, 6, 7, 8, 9, 11, 10, 12].

As a natural counterpart of the generalized connectivity, we introduced the concept of generalized edge-connectivity in [11]. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \textit{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is nothing new but the standard edge-connectivity $\lambda(G)$ of G, that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized edge-connectivity of G. Also set $\lambda_k(G) = 0$ when G is disconnected.

In addition to being a natural combinatorial measure, the generalized connectivity and generalized edge-connectivity can be motivated by its interesting interpretation in practice. Suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them unless the vertices of S lie on a common path. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of Very Large Scale Integration (see [13]). For a set S of vertices, usually the number of totally independent ways to connect S is a local measure for the reliability of a network. Then the generalized k-connectivity and generalized k-edge-connectivity can serve for measuring the global capability of a network G to connect any k vertices in G.

The following two observations are easily seen.

Observation 1. *If G is a connected graph, then $\kappa_k(G) \leq \lambda_k(G) \leq \delta(G)$.*

Observation 2. *If H is a spanning subgraph of G, then $\kappa_k(H) \leq \kappa_k(G)$ and $\lambda_k(H) \leq \lambda_k(G)$.*

In [11], we obtained some results on the generalized edge-connectivity. The following results are restated, which will be used later.

Lemma 1. [11] *For every two integers n and k with $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

Lemma 2. [11] *For any connected graph G, $\lambda_k(G) \leq \lambda(G)$. Moreover, the upper bound is sharp.*

Lemma 3. [11] *Let k, n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa_k(G) \leq \lambda_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

In [11], we characterized the graphs attaining the above upper bound, namely, the graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$.

Lemma 4. [11] *Let k, n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ or $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

But it is not easy to characterize the graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil - 1$ or $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil - 1$. In [5], we focus on the case $k = 3$ and characterize the graphs with $\kappa_3(G) = n - 3$. Like [5], here we will consider the generalized 3-edge-connectivity. In Section 2, graphs of order n such that $\lambda_3(G) = n - 3$ are characterized.

Let $g(n, k, \ell)$ be the minimal number of edges of a graph G of order n with $\lambda_k(G) = \ell$ $(1 \leq \ell \leq n - \lceil \frac{k}{2} \rceil)$. From Lemma 4, we know that $g(n, k, n - \lceil \frac{k}{2} \rceil) = \binom{n}{2}$ for k even; $g(n, k, n - \lceil \frac{k}{2} \rceil) = \binom{n}{2} - \frac{k-1}{2}$ for k odd. It is not easy to determine the exact value of the parameter $g(n, k, \ell)$ for a general k $(3 \leq k \leq n)$ and a general ℓ $(1 \leq \ell \leq n - \lceil \frac{k}{2} \rceil)$. So we put our attention to the case $k = 3$. The exact value of $g(n, 3, \ell)$ for $\ell = n - 2, n - 3, 1$ is obtained in Section 3. We also give a sharp lower bound of $g(n, 3, \ell)$ for general ℓ $(2 \leq \ell \leq n - 4)$.

2 Graphs with $\lambda_3(G) = n - 3$

For the generalized 3-connectivity, we got the following result in [5].

Theorem 1. [5] *Let G be a connected graph of order n $(n \geq 3)$. Then $\kappa_3(G) = n - 3$ if and only if G is a graph satisfying one of the following conditions.*

- $\overline{G} = P_4 \cup (n - 4)K_1$;
- $\overline{G} = P_3 \cup rP_2 \cup (n - 2r - 3)K_1$ $(r = 0, 1)$;
- $\overline{G} = C_3 \cup rP_2 \cup (n - 2r - 3)K_1$ $(r = 0, 1)$;
- $\overline{G} = sP_2 \cup (n - 2s)K_1$ $(2 \leq s \leq \lfloor \frac{n}{2} \rfloor)$.

But, for the edge case, we will show that the statement is different. Before giving our main result, we need some preparations. Choose $S \subseteq V(G)$. Then let $\mathscr{T}$ be a maximum set of edge-disjoint trees connecting S in G. Let $\mathscr{T}_1$ be the set of trees in $\mathscr{T}$ whose edges belong to $E(G[S])$, and let $\mathscr{T}_2$ be the set of trees containing at least one edge of $E_G[S, \bar{S}]$, where $\bar{S} = V(G) \setminus S$. Thus, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$.

In [11], we obtained the following useful lemma.

Lemma 5. [11] *Let $S \subseteq V(G)$, $|S| = k$ and T be a tree connecting S. If $T \in \mathscr{T}_1$, then T uses $k - 1$ edges of $E(G[S]) \cup E_G[S, \bar{S}]$; If $T \in \mathscr{T}_2$, then T uses at least k edges of $E(G[S]) \cup E_G[S, \bar{S}]$.*

By Lemma 5, we can derive the following result.

Lemma 6. *Let G be a connected graph of order n $(n \geq 3)$, and ℓ be a positive integer. If we can find a vertex subset $S \subseteq V(G)$ with $|S| = 3$ satisfying one of the following conditions, then $\lambda_3(G) \leq n - \ell$:*

(1) $\overline{G}[S] = 3K_1$ *and* $|E_{\overline{G}}[S, \bar{S}] \cup \overline{G}[S]| \geq 3\ell - 7$;
(2) $\overline{G}[S] = P_2 \cup K_1$ *and* $|E_{\overline{G}}[S, \bar{S}] \cup \overline{G}[S]| \geq 3\ell - 7$;
(3) $\overline{G}[S] = P_3$ *and* $|E_{\overline{G}}[S, \bar{S}] \cup \overline{G}[S]| \geq 3\ell - 8$;
(4) $\overline{G}[S] = K_3$ *and* $|E_{\overline{G}}[S, \bar{S}] \cup \overline{G}[S]| \geq 3\ell - 8$.

Proof. We only show that (1) and (3) hold, (2) and (4) can be proved similarly.

(1) Since $|E_{\overline{G}}[S,\bar{S}] \cup \overline{G}[S]| \geq 3\ell - 7$, we have $|E(G[S]) \cup E_G[S,\bar{S}]| \leq 3+3(n-3)-(3\ell-7) = 3n-3\ell+1$. Since $\overline{G}[S] = 3K_1$, we have $G[S] = K_3$. Therefore, $|E(G[S])| = 3$, and so there exists at most one tree belonging to $\mathscr{T}_1$ in G. If there exists one tree belonging to $\mathscr{T}_1$, namely $|\mathscr{T}_1| = 1$, then the other trees connecting S must belong to $\mathscr{T}_2$. From Lemma 5, each tree belonging to $\mathscr{T}_2$ uses at least 3 edges in $E(G[S]) \cup E_G[S,\bar{S}]$. So the remaining at most $(3n-3\ell+1)-2$ edges of $E(G[S]) \cup E_G[S,\bar{S}]$ can form at most $\frac{3n-3\ell-1}{3}$ trees. Thus $\lambda_3(G) \leq \lambda(S) = |\mathscr{T}| = |\mathscr{T}_1|+|\mathscr{T}_2| = 1+|\mathscr{T}_2| \leq n-\ell+\frac{2}{3}$, which results in $\lambda_3(G) \leq n-\ell$ since $\lambda_3(G)$ is an integer. Suppose that all trees connecting S belong to $\mathscr{T}_2$. Then $\lambda(S) = |\mathscr{T}| = |\mathscr{T}_2| \leq \frac{3n-3\ell+1}{3}$, which implies that $\lambda_3(G) \leq \lambda(S) \leq n-\ell$.

(3) Since $|E_{\overline{G}}[S,\bar{S}] \cup \overline{G}[S]| \geq 3\ell-8$, it follows that $|E(G[S]) \cup E_G[S,\bar{S}]| \leq 3+3(n-3)-(3\ell-8) = 3n-3\ell+2$. Since $\overline{G}[S] = P_3$, we have $G[S] = P_2 \cup K_1$. Clearly, $|E(G[S])| = 1$ and hence there exists no tree belonging to $\mathscr{T}_1$. So each tree connecting S must belong to $\mathscr{T}_2$. From Lemma 5, $\lambda(S) \leq |\mathscr{T}| = |\mathscr{T}_2| \leq \frac{3n-3\ell+2}{3}$, which implies that $\lambda_3(G) \leq \lambda(S) \leq n-\ell$ since $\lambda_3(G)$ is an integer. □

Lemma 7. *Let G be a connected graph with minimum degree δ. If there are two adjacent vertices of degree δ, then $\lambda_k(G) \leq \delta(G) - 1$.*

Proof. From Observation 1, $\lambda_k(G) \leq \delta(G)$. Suppose that there are two adjacent vertices of degree δ, say u_1 and u_2. Besides u_1 and u_2, we choose some vertices in $V(G \setminus \{u_1,u_2\})$ to get a k-subset S containing u_1,u_2. Pick up a vertex $u_3 \in S \setminus \{u_1,u_2\}$. Suppose that $T_1,T_2,\cdots,T_\delta$ are δ pairwise edge-disjoint trees connecting S. Since G is simple graph, obviously the δ edges incident to u_1 must be contained in $T_1,T_2,\cdots,T_\delta$, respectively, and so are the δ edges incident to u_2. Without loss of generality, we may assume that the edge u_1u_2 is contained in T_1. But, since T_1 is a tree connecting S, it must contain another edge incident with u_1 or u_2, a contradiction. Thus $\lambda_k(G) \leq \delta(G) - 1$. □

A subset M of $E(G)$ is called a *matching* of G if the edges of M satisfy that no two of them are adjacent in G. A matching M saturates a vertex v, or v is said to be *M-saturated*, if some edge of M is incident with v; otherwise, v is *M-unsaturated.* M is a *maximum matching* if G has no matching M' with $|M'| > |M|$.

Theorem 2. *Let G be a connected graph of order n ($n \geq 3$). Then $\lambda_3(G) = n-3$ if and only if G is a graph satisfying one of the following conditions.*

- $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor\frac{n}{2}\rfloor)$;
- $\overline{G} = P_4 \cup sP_2 \cup (n-2s-4)K_1$ $(0 \leq s \leq \lfloor\frac{n-4}{2}\rfloor)$;
- $\overline{G} = P_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor\frac{n-3}{2}\rfloor)$;
- $\overline{G} = C_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor\frac{n-3}{2}\rfloor)$.

Proof. Necessity: Assume that $\lambda_3(G) = n-3$. From Lemma 4, for a connected graph H, $\lambda_3(H) = n-2$ if and only if $0 \leq |E(\overline{H})| \leq 1$. Since $\lambda_3(G) = n-3$, it follows that $|E(\overline{G})| \geq 2$. We claim that $\delta(\overline{G}) \leq 2$. Assume, to the contrary, that $\delta(\overline{G}) \geq 3$. Then $\lambda_3(G) \leq \delta(G) = n - 1 - \delta(\overline{G}) \leq n - 4$, a contradiction. Since $\delta(\overline{G}) \leq 2$, it follows that each component of $\overline{G}$ is a path or a cycle (note that an isolated vertex in $\overline{G}$ is a trivial path). We will show that the following two claims hold.

Claim 1. $\overline{G}$ has at most one component of order larger than 2.

Suppose, to the contrary, that $\overline{G}$ has two components of order larger than 2, denoted by H_1 and H_2 (see Figure 1 (a)).

Let $x, y \in V(H_1)$ and $z \in V(H_2)$ such that $d_{H_1}(y) = d_{H_2}(z) = 2$ and x is adjacent to y in H_1. Thus $d_G(y) = n - 1 - d_{\overline{G}}(y) = n - 1 - d_{H_1}(y) = n - 3$. The same is true for z, that is, $d_G(z) = n - 3$. Pick $S = \{x, y, z\}$. This implies that $\delta(G) \leq d_G(z) \leq n - 3$. Since all other components of $\overline{G}$ are paths or cycles, $\delta(G) \geq n - 3$. So $\delta(G) = n - 3$ and hence $d_G(y) = d_G(z) = \delta(G) = n - 3$. Since $yz \in E(G)$, by Lemma 7 it follows that $\lambda_3(G) \leq \delta(G) - 1 = n - 4$, a contradiction.

Claim 2. If H is the component of $\overline{G}$ of order larger than 3, then H is a 4-path.

Assume, to the contrary, that H is a path or a cycle of order larger than 4, or a cycle of order 4.

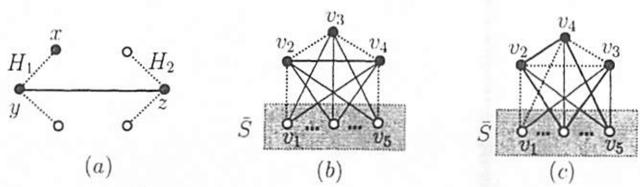

Figure 1: Graphs for Claims 1 and 2.

First, we consider the former. We can pick a P_5 in H. Without loss of generality, let $P_5 = v_1, v_2, v_3, v_4, v_5$. Choose $S = \{v_2, v_3, v_4\}$. Then $\bar{S} = G \setminus \{v_2, v_3, v_4\}$ (see Figure 1 (b)). Clearly, $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]| \geq 4$. Since $v_2v_3, v_3v_4 \in E(\overline{G}[S])$, it follows that $\overline{G}[S] = P_3$. From (3) of Lemma 6, $\lambda_3(G) \leq n - 4$ (Note that if $3\ell - 8 = 4$, then $\ell = 4$). This contradicts to $\lambda_3(G) = n - 3$.

Now we consider the latter. Let $H = v_1, v_2, v_3, v_4$ be a cycle. Choose $S = \{v_2, v_3, v_4\}$ (see Figure 1 (c)). Then $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]| \geq 4$. Since $v_2v_3, v_3v_4 \in E(\overline{G}[S])$, it follows that $\overline{G}[S] = P_3$. From (3) of Lemma 6, $\lambda_3(G) \leq n - 4$ (Note that if $3\ell - 8 = 4$, then $\ell = 4$), which also contradicts to $\lambda_3(G) = n - 3$.

From the above two claims, we know that if $\overline{G}$ has a component P_4, then it is the only component of order larger than 3 and the other components must be independent edges. Let s be the number of such independent edges. $\overline{G}$ can have as many as such independent edges, which implies that $s \leq \lfloor \frac{n-4}{2} \rfloor$. From Lemma 4, $s \geq 0$. Thus $0 \leq s \leq \lfloor \frac{n-4}{2} \rfloor$.

By the similar analysis, we conclude that $\overline{G} = rP_2 \cup (n - 2r)K_1$ $(2 \leq r \leq$

$\lfloor \frac{n}{2} \rfloor$) or $\overline{G} = P_4 \cup sP_2 \cup (n-2s-4)K_1$ $(0 \leq s \leq \lfloor \frac{n-4}{2} \rfloor)$ or $\overline{G} = P_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor)$ or $\overline{G} = C_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor)$.

Sufficiency: We will show that $\lambda_3(G) \geq n-3$ if G is a graph satisfying one of the conditions of this theorem. We have the following cases to consider.

Case 1. $\overline{G} = P_3 \cup tP_2 \cup (n-2t-3)K_1$ or $\overline{G} = C_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor)$.

We only need to show that $\lambda_3(G) \geq n-3$ for $t = \lfloor \frac{n-3}{2} \rfloor$. If $\lambda_3(G) \geq n-3$ for $\overline{G} = C_3 \cup tP_2 \cup (n-2t-3)K_1$, then $\lambda_3(G) \geq n-3$ for $\overline{G} = P_3 \cup tP_2 \cup (n-2t-3)K_1$. It suffices to check that $\lambda_3(G) \geq n-3$ for $\overline{G} = C_3 \cup \lfloor \frac{n-3}{2} \rfloor P_2 \cup (n - 2\lfloor \frac{n-3}{2} \rfloor - 3)K_1$.

Let $C_3 = v_1, v_2, v_3$ and $S = \{x, y, z\}$ be a 3-subset of G, and $M = \lfloor \frac{n-3}{2} \rfloor P_2$. It is clear that M is a maximum matching of $\overline{G} \setminus V(C_3)$. Then $\overline{G} \setminus V(C_3)$ has at most one M-unsaturated vertex.

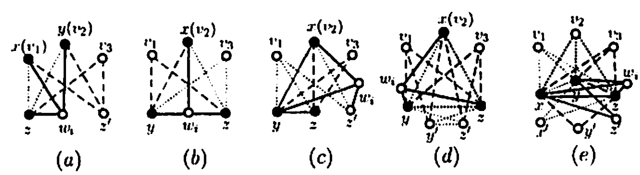

Figure 2: Graphs for Case 1.

If $S = V(C_3)$, then there exist $n-3$ pairwise edge-disjoint trees connecting S since each vertex in S is adjacent to every vertex in $G \setminus S$. Suppose $S \neq V(C_3)$. If $|S \cap V(C_3)| = 2$, then one element of S belongs to $\in V(G) \setminus V(C_3)$, denoted by z. Since $d_G(v_1) = d_G(v_2) = d_G(v_3) = n-3$, we can assume that $x = v_1$, $y = v_2$. When z is M-unsaturated, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = xz \cup yz$ form $n-3$ pairwise edge-disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-4}\} = V(G) \setminus \{x, y, z, v_3\}$. When z is M-saturated, we let z' be the adjacent vertex of z under M. Then the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = xz \cup yz$ and $T_2 = xz' \cup yz' \cup z'v_3 \cup zv_3$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 2 (a)), where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, z', v_3\}$. If $|S \cap V(C_3)| = 1$, then two elements of S belong to $\in V(G) \setminus V(C_3)$, denoted by y and z. Without loss of generality, let $x = v_2$. When y and z are adjacent under M, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = xy \cup yv_1 \cup v_1 z$ and $T_2 = xz \cup zv_3 \cup v_3 y$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 2 (b)), where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, v_1, v_3\}$. When y and z are nonadjacent under M, we consider whether y and z are M-saturated. If one of $\{y, z\}$ is M-unsaturated, without loss of generality, we assume that y is M-unsaturated. Since $G \setminus V(C_3)$ has at most one M-unsaturated vertex, z is M-saturated. Let z' be the adjacent vertex of z under M. Then the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = xy \cup yz$ and $T_2 = v_1 y \cup v_1 z \cup z'v_1 \cup z'x$ and $T_3 = xz \cup zv_3 \cup v_3 y$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 2 (c)), where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, z', v_1, v_3\}$. If both y and z are M-saturated, we let y', z' be the adjacent vertex of y, z under M, respectively. Then the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xz \cup yz$, $T_2 = xy \cup yz' \cup z'y' \cup y'z$, $T_3 = yv_3 \cup z'v_3 \cup zv_3 \cup xz'$ and $T_4 = yv_1 \cup y'v_1 \cup zv_1 \cup y'x$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 2 (d)), where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, y', z', v_1, v_3\}$. Otherwise, $S \subseteq G \setminus V(C_3)$. When one of $\{x, y, z\}$ is M-unsaturated, without loss of generality, we assume that x is M-unsaturated. Since $G \setminus V(C_3)$ has at most one M-unsaturated vertex, both y and z are M-saturated. Let y', z' be the adjacent vertex of y, z under M, respectively. We pick a vertex x' of $V(G) \setminus \{x, y, y', z, z', v_1, v_2, v_3\}$. When x, y, z are all M-saturated, we let x', y', z' be the adjacent vertex of x, y, z under M, respectively. Then the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_j = xv_j \cup yv_j \cup zv_j (1 \leq j \leq 3)$ and $T_4 = xy \cup yx' \cup x'z$ and $T_5 = xy' \cup zy' \cup zy$ and $T_6 = zx \cup xz' \cup z'y$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 2 (e)), where $\{w_1, w_2, \cdots, w_{n-9}\} = V(G) \setminus \{x, y, z, x', y', z', v_1, v_2, v_3\}$.

From the above discussion, we get that $\lambda(S) \geq n-3$ for $S \subseteq V(G)$, which implies $\lambda_3(G) \geq n-3$. So $\lambda_3(G) = n-3$.

Case 2. $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$ or $\overline{G} = P_4 \cup sP_2 \cup (n-2s-4)K_1$ $(0 \leq s \leq \lfloor \frac{n-4}{2} \rfloor)$.

We only need to show that $\lambda_3(G) \geq n-3$ for $r = \lfloor \frac{n}{2} \rfloor$ and $s = \lfloor \frac{n-4}{2} \rfloor$. If $\lambda_3(G) \geq n-3$ for $\overline{G} = P_4 \cup \lfloor \frac{n-4}{2} \rfloor P_2 \cup (n-2\lfloor \frac{n-4}{2} \rfloor - 4)K_1$, then $\lambda_3(G) \geq n-3$ for $\overline{G} = \lfloor \frac{n}{2} \rfloor P_2 \cup (n-2\lfloor \frac{n}{2} \rfloor)K_1$. So we only need to consider the former. Let $P_4 = v_1, v_2, v_3, v_4$, $S = \{x, y, z\}$ be a 3-subset of G, and $M = \overline{G} \setminus E(P_4)$. Clearly, M is a maximum matching of $\overline{G} \setminus V(P_4)$. It is easy to see that $\overline{G} \setminus V(P_4)$ has at most one M-unsaturated vertex. For any $S \subseteq V(G)$, we will show that there exist $n-3$ edge-disjoint trees connecting S in G.

If $S \subseteq V(P_4)$, then there exist $n-4$ pairwise edge-disjoint trees connecting S since each vertex in S is adjacent to every vertex in $G \setminus V(P_4)$. Since $d_G(v_1) = d_G(v_4) = n-2$ and $d_G(v_2) = d_G(v_3) = n-3$, we only need to consider $S = \{v_1, v_2, v_3\}$ and $S = \{v_1, v_2, v_4\}$. These trees together with $T = yv_4 \cup v_4x \cup v_4z$ for $S = \{v_1, v_2, v_3\}$, or $T = xy \cup yz$ for $S = \{v_1, v_2, v_3\}$ form $n-3$ pairwise edge-disjoint trees connecting S. Suppose $S \cap V(P_4) \neq 3$. If $|S \cap V(P_4)| = 2$, then one element of S belongs to $\in V(G) \setminus V(P_4)$, denoted by z. Since $d_G(v_1) = d_G(v_4) = n-2$ and $d_G(v_2) = d_G(v_3) = n-3$, we only need to consider $x = v_1, y = v_2$ or $x = v_2, y = v_3$ or $x = v_1, y = v_4$. When z is M-unsaturated, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xz \cup yz$, $T_2 = xv_4 \cup yv_4 \cup zv_4$ for $x = v_1, y = v_2$, or $T_2 = xv_4 \cup v_4v_1 \cup v_1y \cup v_4z$ for $x = v_2, y = v_3$, or $T_2 = xv_3 \cup yv_3 \cup zv_3$ for $x = v_1, y = v_4$ form $n-3$ pairwise edge-disjoint trees connecting S, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus (V(P_4) \cup \{z\})$. When z is M-unsaturated, we let z' be the adjacent vertex of z under M. For $x = v_2, y = v_3$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xz \cup yz$, $T_2 = xz' \cup yz' \cup z'v_4 \cup zv_4$ and $T_2 = yv_1 \cup v_1v_4 \cup zv_1 \cup xv_4$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 3 (a)), where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, z', v_1, v_4\}$. One can check that the same is true for $x = v_1, y = v_2$ and $x = v_1, y = v_4$ (see Figure 3 (b) and (c)). If $|S \cap V(P_4)| = 1$, then two elements of S belong to $\in V(G) \setminus V(P_4)$, denoted by y and z. We only need to consider $x = v_1$ or $x = v_2$. When y and z are adjacent under M, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xy \cup zv_1 \cup yv_1$, $T_2 = xz \cup zv_3 \cup yv_3$ and $T_3 = xv_4 \cup yv_4 \cup zv_4$ form $n-3$ pairwise edge-disjoint trees connecting S for $x = v_2$ (see Figure 3 (*d*)), where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, v_1, v_3, v_4\}$. The same is true for $x = v_1$ (see Figure 3 (*e*)). When y and z are nonadjacent under M, we consider whether y and z are M-saturated. If one of $\{y, z\}$ is M-unsaturated, without loss of generality, we assume that y is M-unsaturated. Since $G \setminus V(P_4)$ has at most one M-unsaturated vertex, z is M-saturated. Let z' be the adjacent vertex of z under M. For $x = v_2$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xz \cup yz$, $T_2 = v_4x \cup v_4y \cup v_4z$, $T_3 = v_1y \cup v_1z \cup zx$ and $T_4 = z'x \cup v_3y \cup z'v_3 \cup zv_3$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 3 (*f*)), where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, z', v_1, v_3, v_4\}$. The same is true for $x = v_1$ (see Figure 3 (*g*)). If both y and z are M-saturated, we let y', z' be the adjacent vertex of y, z under M, respectively. For $x = v_2$, the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_1 = xz \cup yz$, $T_2 = yv_3 \cup zv_3 \cup zx$, $T_3 = xv_4 \cup yv_4 \cup zv_4$, $T_4 = yv_1 \cup y'v_1 \cup zv_1 \cup xy'$ and $T_5 = xz' \cup z'y \cup z'y' \cup y'z$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 3 (*h*)), where $\{w_1, w_2, \cdots, w_{n-8}\} = V(G) \setminus \{x, y, z, y', z', v_1, v_3, v_4\}$. The same is true for $x = v_1$ (see Figure 3 (*i*)). If $S \subseteq G \setminus V(P_4)$, when one of $\{x, y, z\}$ is M-unsaturated, without loss of generality, we let x is M-unsaturated, then both y and z are M-saturated. Let y', z' be the adjacent vertex of y, z under M, respectively. We pick a vertex x' of $V(G) \setminus \{x, y, y', z, z', v_1, v_2, v_3\}$. When x, y, z are all M-saturated, we let x', y', z' be the adjacent vertex of x, y, z under M, respectively. Then the trees $T_i = w_ix \cup w_iy \cup w_iz$ together with $T_j = xv_j \cup yv_j \cup zv_j (1 \leq j \leq 4)$ and $T_5 = yx \cup xy' \cup y'z$ and $T_6 = yx' \cup zx' \cup zx$ and $T_7 = zy \cup yz' \cup z'x$ form $n-3$ pairwise edge-disjoint trees connecting S (see Figure 3 (*j*)), where $\{w_1, w_2, \cdots, w_{n-10}\} = V(G) \setminus \{x, y, z, x', y', z', v_1, v_2, v_3, v_4\}$.

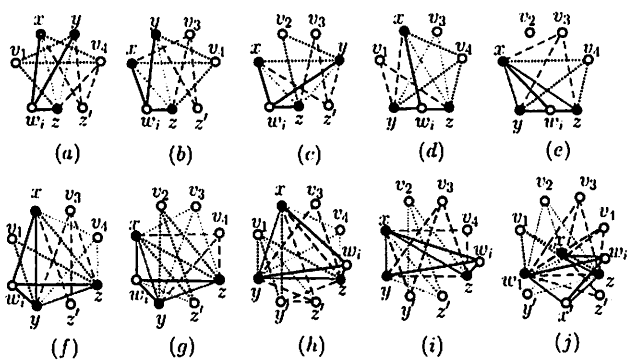

Figure 3: Graphs for S in Case 2.

From the above arguments, we conclude that for any $S \subseteq V(G)$ $\lambda(S) \geq n-3$. From the arbitrariness of S, we have $\lambda_3(G) \geq n-3$. The proof is now complete. □

3 The minimal size of a graph G with $\lambda_3(G) = \ell$

Recall that $g(n, k, \ell)$ is the minimal number of edges of a graph G of order n with $\lambda_k(G) = \ell$ $(1 \leq \ell \leq n - \lceil \frac{k}{2} \rceil)$. Let us focus on the case $k = 3$ and derive the following result.

Theorem 3. *Let n be an integer with $n \geq 3$. Then*
(1) $g(n, 3, n-2) = \binom{n}{2} - 1$;
(2) $g(n, 3, n-3) = \binom{n}{2} - \lfloor \frac{n+3}{2} \rfloor$;
(3) $g(n, 3, 1) = n - 1$;
(4) $g(n, 3, \ell) \geq \lceil \frac{\ell(\ell+1)}{2\ell+1} n \rceil$ *for $n \geq 11$ and $2 \leq \ell \leq n-4$. Moreover, the bound is sharp.*

Proof. (1) From Lemma 4, $\lambda_3(G) = n-2$ if and only if $G = K_n$ or $G = K_n \setminus e$ where $e \in E(K_n)$. So $g(n, 3, n-2) = \binom{n}{2} - 1$.

(2) From Theorem 2, $\lambda_3(G) = n-3$ if and only if $\overline{G} = rP_2 \cup (n-2r)K_1$ $(2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$ or $\overline{G} = P_4 \cup sP_2 \cup (n-2s-4)K_1$ $(0 \leq s \leq \lfloor \frac{n-4}{2} \rfloor)$ or $\overline{G} = P_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor)$ or $\overline{G} = C_3 \cup tP_2 \cup (n-2t-3)K_1$ $(0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor)$. If n is even, then $\max\{e(\overline{G})\} = \frac{n+2}{2}$, which implies that $g(n, 3, n-3) = \binom{n}{2} - \max\{e(\overline{G})\} = \binom{n}{2} - \frac{n+2}{2}$. If n is odd, then $\max\{e(\overline{G})\} = \frac{n+3}{2}$, which implies that $g(n, 3, n-3) = \binom{n}{2} - \max\{e(\overline{G})\} = \binom{n}{2} - \frac{n+3}{2}$. So $g(n, 3, n-3) = \binom{n}{2} - \lfloor \frac{n+3}{2} \rfloor$.

(3) It is clear that the tree T_n is the graph such that $\lambda_3(T_n) = 1$ with the minimal number of edges. So $g(n, 3, 1) = n - 1$.

(4) Since $\lambda_k(G) = \ell$ $(2 \leq \ell \leq n-4)$, by Lemma 7, we know that $\delta(G) \geq \ell$ and any two vertices of degree ℓ are not adjacent. Denote by X the set of vertices of degree ℓ. We have that X is an independent set. Put $Y = V(G) \setminus X$ and obviously there are $2|X|$ edges joining X to Y. Assume that m' is the number of edges joining two vertices belonging to Y. It is clear that $e = \ell|X| + m'$. Since every vertex of Y has degree at least $\ell+1$ in G, then $\sum_{v \in Y} d(v) = \ell|X| + 2m' \geq (\ell+1)|Y| = (\ell+1)(n-|X|)$, namely, $(2\ell+1)|X| + 2m' \geq (\ell+1)n$. Combining this with $e = \ell|X| + m'$, we have $\frac{2\ell+1}{\ell} e(G) = (2\ell+1)|X| + \frac{2\ell+1}{\ell} m' \geq (2\ell+1)|X| + 2m' \geq (\ell+1)n$ Therefore, $e(G) \geq \frac{\ell(\ell+1)}{2\ell+1} n$. Since the number of edges is an integer, it follows that $e(G) \geq \lceil \frac{\ell(\ell+1)}{2\ell+1} n \rceil$.

To show that the upper bound is sharp, we consider the complete bipartite graph $G = K_{\ell,\ell+1}$. Let $U = \{u_1, u_2, \cdots, u_\ell\}$ and $W = \{w_1, w_2, \cdots, w_{\ell+1}\}$ be the two parts of $K_{\ell,\ell+1}$. Choose $S \subseteq V(G)$. We will show that there are ℓ edge-disjoint trees connecting S.

If $|S \cap U| = 3$, without loss of generality, let $S = \{u_1, u_2, u_3\}$, then the trees $T_i = u_1w_i \cup u_2w_i \cup u_3w_i$ $(1 \leq i \leq \ell+1)$ are $\ell+1$ edge-disjoint trees connecting S. If $|S \cap U| = 2$, then $|S \cap W| = 1$. Without loss of generality, let $S = \{u_1, u_2, w_1\}$. Then the trees $T_i = u_1w_i \cup u_iw_i \cup u_iw_1$ $(4 \leq i \leq \ell+1)$ and $T_1 = u_1w_1 \cup u_1w_3 \cup u_2u_3$ and $T_2 = u_2w_1 \cup u_2w_2 \cup u_1w_2$ are ℓ edge-disjoint trees connecting S. If $|S \cap U| = 1$, then $|S \cap W| = 2$. Without loss of generality, let $S = \{u_1, w_1, w_2\}$. Then the trees $T_i = u_1v_{i+1} \cup u_iw_{i+1} \cup u_iw_1 \cup u_iw_2$ $(2 \leq i \leq \ell)$ and $T_1 = u_1w_1 \cup u_1w_2$ are ℓ edge-disjoint trees connecting S. Suppose $|S \cap W| = 3$. Without loss of generality, let $S = \{w_1, w_2, w_3\}$, then the trees $T_i = w_1u_i \cup w_2u_i \cup w_3u_i$ $(1 \leq i \leq \ell)$ are ℓ edge-disjoint trees connecting S.

From the above arguments, we conclude that, for any $S \subseteq V(G)$, $\lambda(S) \geq \ell$. So $\lambda_3(G) \geq \ell$. On the other hand, $\lambda_3(G) \leq \delta(G) = \ell$ and hence $\lambda_3(G) = \ell$. Clearly, $|V(G)| = 2\ell + 1$, $e(G) = \ell(\ell + 1) = \lceil \frac{\ell(\ell+1)}{2\ell+1} n \rceil$.

So the lower bound is sharp for $k = 3$ and $2 \leq \ell \leq n - 4$. □

Acknowledgement: The authors are very grateful to the referee's comments and suggestions, which helped us to improve this paper.

References

[1] J. A. Bondy, U. S. R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] G. Chartrand, S. F. Kappor, L. Lesniak, D. R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. 2(1984), 1-6.

[3] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(4)(2010), 360-367.

[4] H. Li, X. Li, Y. Sun, *The generalied 3-connectivity of Cartesian product graphs*, Discrete Math. Theor. Comput. Sci. 14(1)(2012), 43-54.

[5] H. Li, X. Li, Y. Mao, Y. Sun, *Note on the generalized connectivity*, Ars Combin. 114(2014).

[6] S. Li, W. Li, X. Li, *The generalized connectivity of complete bipartite graphs*, Ars Combin. 104(2012), 65-79.

[7] S. Li, W. Li, X. Li, *The generalized connectivity of complete equipartition 3-partite graphs*, Bull. Malays. Math. Sci. Soc.(2) 37(1)(2014), 103-121.

[8] S. Li, X. Li, *Note on the hardness of generalized connectivity*, J. Combin. Optimization 24(2012), 389-396.

[9] S. Li, X. Li, Y. Shi, *The minimal size of a graph with generalized connectivity* $\kappa_3(G) = 2$, Australasian J. Combin. 51(2011), 209-220.

[10] S. Li, X. Li, W. Zhou, *Sharp bounds for the generalized connectivity* $\kappa_3(G)$, Discrete Math. 310(2010), 2147-2163.

[11] X. Li, Y. Mao, Y. Sun, *On the generalized (edge-)connectivity of graphs*, Australasian J. Combin. 58(2)(2014), 304-319.

[12] F. Okamoto, P. Zhang, *The tree connectivity of regular complete bipartite graphs*, J. Combin. Math. Combin. Comput. 74(2010), 279-293.

[13] N.A. Sherwani, *Algorithms for VLSI Physical Design Automation*, 3rd Edition, Kluwer Acad. Pub., London, 1999.