A kind of identities for generalized q-harmonic numbers and reciprocals of q-binomial coefficients

Jizhen Yang[1] and Yunpeng Wang[2]
1 Department of Mathematics, Luoyang normal College,
1 Luoyang 471022, P. R. China
2 Department of Mathematics and Physical, Luoyang Institute of Science and Technology,
2 Luoyang 471022, P. R. China
email:yangjizhen116@163.com

Abstract

The purpose of this paper is to establish q-analogue of some identities and then generalize the result to give identities for finite sums for products of generalized q-harmonic numbers and reciprocals of q-binomial coefficients.

Keywords and phrases: q-analogue, q-harmonic numbers, reciprocals of q-binomial coefficients, finite sums

1 Introduction and Preliminaries

In [1], Bhatnagar referred to the following identity.

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\dots(k+m)} = \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)\cdots(n+m)}\right). \tag{1}$$

In [9], Sofo gave the generalization of (1) and obtained its integral representation as follows.

$$\sum_{n=1}^{p} \frac{1}{n\binom{n+z}{z}} = \int_0^1 (1-x)^{z-1}(1-x^p)dx = \frac{1}{z} - \frac{1}{z\binom{p+z}{z}}. \tag{2}$$

Further more, by differentiating (2) , Sofo obtained the following result.

Theorem 1.1 *Let* $z \in \mathbb{R}\backslash\mathbb{Z}_0^-$ *and* $p, q \in \mathbb{N}$*. Then*

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} = -Q^{(q)}(p,z), \tag{3}$$

and when $z = 0$*,*

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,0)}{n} = -Q^{(q)}(p,0), \tag{4}$$

where $Q(n,z) = \binom{n+z}{z}^{-1}$ *and* $Q(n,0) = 1$*,*

$$Q^{(\lambda)}(n,z) = \frac{d^\lambda Q}{dz^\lambda} = -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(n,z)P^{(\lambda-1-\rho)}(n,z), \tag{5}$$

and

$$P^{(i)}(n,z) = \frac{d^iP}{dz^i} = \frac{d^i}{dz^i}\left(\sum_{r=1}^{n} \frac{1}{r+z}\right) = (-1)^i i! \sum_{r=1}^{n} \frac{1}{(r+z)^{i+1}},$$

The purpose of this paper is to give the q-analogue of (1) and Theorem1.1. By this method we shall give some identities involving finite sums for products of generalized q-harmonic numbers and reciprocals of q-binomial coefficients.

First, we introduce the following definitions.

The q-Gamma function:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (0 < q < 1). \tag{6}$$

The q-binomial coefficients:

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}. \tag{7}$$

The q-Beta function:

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{8}$$

The q-integral:

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n. \tag{9}$$

The generalized q-harmonic numbers:

$$H_n^{(m)}(q) = \sum_{r=1}^{n} \frac{1}{(1-q^r)^m}.$$

Specially, q-harmonic number:

$$H_n^{(1)}(q) = \sum_{r=1}^{n} \frac{1}{1-q^r}.$$

The following results which will be useful throughout this paper.

$$B_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}d_qt \quad (Re(x)>0, y\neq 0,-1,-2,\cdots), ([4]). \tag{10}$$

$$_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \ ([4]). \tag{11}$$

$$_2\phi_1(q^a,q^b;q^c;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b-1}\frac{(tzq^a,tq;q)_\infty}{(tz,tq^{c-b};q)_\infty}d_qt, \ ([10]). \tag{12}$$

The Stirling numbers of the second kind $S(n,k)$ satisfy the triangular recurrence relation:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), n,k \geq 1;$$

$$S(n,0) = S(0,k) = 0, \text{ except } S(0,0) = 1, \text{ ([2])}.$$

2 Main results

Theorem 2.1 *Let* $0 < q < 1$, $z \in \mathbb{R}\backslash\mathbb{Z}_0^-$ *and* $p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{t^n}{(1-q^n)\left[{n+z \atop z}\right]} = \frac{t}{1-q}\int_0^1 \frac{(xq;q)_\infty(1-x^pt^p)}{(xq^{z+1};q)_\infty(1-xt)}d_qx \tag{13}$$

Proof. By (6)-(10), we have

$$\sum_{n=1}^{p} \frac{t^n}{(1-q^n)\left[{n+z \atop z}\right]} = \sum_{n=1}^{p} \frac{t^n\Gamma_q(n+1)\Gamma_q(z+1)}{(1-q^n)\Gamma_q(n+z+1)}$$

$$= \sum_{n=1}^{p} \frac{t^n}{1-q}B_q(n,z+1)$$

$$= \sum_{n=1}^{p} \frac{t^n}{1-q}\int_0^1 x^{n-1}\frac{(xq;q)_\infty}{(xq^{z+1};q)_\infty}d_qx$$

$$= \frac{t}{1-q}\int_0^1 \frac{(xq;q)_\infty}{(xq^{z+1};q)_\infty}\sum_{n=1}^{p}(xt)^{n-1}d_qx$$

$$= \frac{t}{1-q}\int_0^1 \frac{(xq;q)_\infty(1-x^pt^p)}{(xq^{z+1};q)_\infty(1-xt)}d_qx.$$

□

Corollary 2.2 *Let* $0 < q < 1$, $z \in \mathbb{R}\backslash\mathbb{Z}_0^-$ *and* $p \in \mathbb{N}$.*Then*

$$\sum_{n=1}^{p} \frac{q^{zn}}{(1-q^n)\left[{n+z \atop z}\right]} = \frac{q^z}{1-q^z}\left(1 - \frac{q^{pz}}{\left[{p+z \atop z}\right]}\right).$$

Proof. We rewrite Theorem 2.1 as follows

$$\begin{aligned}
&\frac{t}{1-q}\int_0^1 \frac{(xq,xtq;q)_\infty}{(xq^{z+1},xt;q)_\infty} d_qt - \frac{t}{1-q}\int_0^1 \frac{(xq,xtq;q)_\infty}{(xq^{z+1},xt;q)_\infty}(xt)^p d_qt \\
= \quad &\frac{t}{1-q^{z+1}}\ {}_2\phi_1(q,q;q^{z+2};q,t) - \frac{t(q;q)_p}{(q^{z+1};q)_{p+1}}\ {}_2\phi_1(q,q^{p+1};q^{z+p+2};q,t).
\end{aligned} \tag{14}$$

Taking $t = q^z$ in (14) and by (11), the right hand side of (14) equals

$$\begin{aligned}
&\frac{q^z}{1-q^{z+1}}\frac{(q^{z+1},q^{z+1};q)_\infty}{(q^{z+2},q^z;q)_\infty} - \frac{q^z(q;q)_p}{(q^{z+1};q)_{p+1}}\frac{(q^{z+p+1},q^{z+1};q)_\infty}{(q^{z+p+2},q^z;q)_\infty} \\
= \quad &\frac{q^z}{1-q^z} - \frac{q^z(q;q)_p}{(1-q^z)(q^{z+1};q)_p} \\
= \quad &\frac{q^z}{1-q^z}\left(1-\frac{q^{pz}}{\begin{bmatrix} p+z \\ z\end{bmatrix}}\right).
\end{aligned}$$

□

Corollary 2.3 *Let* $0 < q < 1$ *and* $m, p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{q^{mn}}{(q^n;q)_{m+1}} = \frac{q^m}{1-q^m}\left(\frac{1}{(q;q)_m} - \frac{q^{pm}}{(q^{p+1};q)_m}\right).$$

Proof. Take $z = m$ in Corollary 2.2. □

This identity is the q-analogue of (1).

Corollary 2.4 *([9, (8)]) Let* $z \in \mathbb{R}\backslash\mathbb{Z}_0^-$ *and* $p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{1}{n\binom{n+z}{z}} = \frac{1}{z}\left(1-\frac{1}{\binom{p+z}{z}}\right).$$

Proof. Take $q \to 1$ in Corollary 2.2. □

Corollary 2.5 *Let* $0 < q < 1, z > 0$ *and* $z \neq 1$, $p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{z^n (q;q)_{n-1}}{(zq;q)_n} = \frac{z}{1-z}\left(1 - \frac{z^p (q;q)_p}{(zq;q)_p}\right).$$

Proof. Take $q^z \to z$ in Corollary 2.2. □

Lemma 2.6 *Let* $0 < q < 1$, $z \neq -r$, $N \in \mathbb{Z}_0^+$, $y(z) = \frac{1}{q^{z+r}-1}$. *Then*

$$y^{(N)}(z) = (-\ln q)^N \sum_{k=0}^{N} \frac{k! S(N+1,k+1)}{(q^{z+r}-1)^{k+1}}.$$

Proof. We have

$$\begin{aligned}
y'(z) &= (-\ln q)\left(\frac{1}{q^{z+r}-1} + \frac{1}{(q^{z+r}-1)^2}\right),\\
y''(z) &= (-\ln q)^2\left(\frac{1}{q^{z+r}-1} + \frac{3}{(q^{z+r}-1)^2} + \frac{2}{(q^{z+r}-1)^3}\right),\\
y'''(z) &= (-\ln q)^3\left(\frac{1}{q^{z+r}-1} + \frac{7}{(q^{z+r}-1)^2} + \frac{12}{(q^{z+r}-1)^3}\right.\\
&\quad \left. + \frac{6}{(q^{z+r}-1)^4}\right).
\end{aligned}$$

Inspection of these three equations leads to the formation of the following sum which can be easily proved by induction.

$$y^{(N)}(z) = (-\ln q)^N \sum_{k=0}^{N} \frac{a_{N,k}}{(q^{z+r}-1)^{k+1}},$$

where $a_{k,k} = k!, a_{N,0} = 1$ for $N \geq 0, a_{N,k} = 0$ for $k > N$ and when $N \geq 1$

$$a_{N,k} = k a_{N-1,k-1} + (k+1) a_{N-1,k}. \tag{15}$$

Letting $a_{N,k} = k! b_{N+1,k+1}$ in (15), where $b_{N,0} = b_{0,k} = 0$ except $b_{0,0} =$

1, we obtain

$$k!b_{N+1,k+1} = k!b_{N,k} + (k+1)!b_{N,k+1},$$

then we have

$$b_{N+1,k+1} = b_{N,k} + (k+1)b_{N,k+1}.$$

By the recurrence relation of $S(n,k)$, we get

$$b_{N+1,k+1} = S(N+1,k+1),$$

and so

$$a_{N,k} = k!S(N+1,k+1).$$

□

Lemma 2.7 *Let $0 < q < 1$, $z \in \mathbb{R}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}$ and let $Q_q(n,z) = \left[{n+z \atop z}\right]^{-1}$ be an analytic function of z. Then*

$$Q_q^{(1)}(n,z) = -\ln q Q_q(n,z) P_q^{(0)}(n,z), \tag{16}$$

and

$$Q_q^{(\lambda)}(n,z) = \frac{d^\lambda Q}{dz^\lambda} = -\ln q \sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q_q^{(\rho)}(n,z) P_q^{(\lambda-1-\rho)}(n,z), \tag{17}$$

where $P_q^{(0)}(n,z) = n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1}$, and for $m \geq 1$

$$\begin{aligned} P_q^{(m)}(n,z) &= \frac{d^m}{dz^m}\left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1}\right) \\ &= (-\ln q)^m \sum_{r=1}^{n} \sum_{k=0}^{m} \frac{k!S(m+1,k+1)}{(q^{z+r}-1)^{k+1}}. \end{aligned}$$

Proof. For

$$Q_q(n,z) = \begin{bmatrix} n+z \\ z \end{bmatrix}^{-1} = \frac{(q;q)_n}{(q^{z+1};q)_n},$$

and

$$\begin{aligned}
\frac{dQ_q(n,z)}{dz} &= -\ln q Q_q(n,z) \sum_{r=1}^{n} \frac{q^{z+r}}{q^{z+r}-1} \\
&= -\ln q Q_q(n,z) \left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right) \\
&= -\ln q Q_q(n,z) P_q^{(0)}(n,z),
\end{aligned}$$

we obtain (16) and differentiating it with respect to the variable z for $\lambda - 1$ times, we get (17). □

Now we list some particular cases of Lemma 2.7, which will be used later.

$$\begin{aligned}
Q_q^{(1)}(n,z) &= -\ln q Q_q(n,z) \sum_{r=1}^{n} \frac{q^{z+r}}{q^{z+r}-1} \\
&= -\ln q Q_q(n,z) \left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right), \\
Q_q^{(2)}(n,z) &= (\ln q)^2 Q_q(n,z) \left[\left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right)^2 + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right. \\
&\quad \left. + \sum_{r=1}^{n} \frac{1}{(q^{z+r}-1)^2} \right], \\
Q_q^{(3)}(n,z) &= (-\ln q)^3 Q_q(n,z) \left[\left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right)^3 \right. \\
&\quad + 3 \left(n + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} \right) \left(\sum_{r=1}^{n} \frac{1}{q^{z+r}-1} + \sum_{r=1}^{n} \frac{1}{(q^{z+r}-1)^2} \right) \\
&\quad \left. + \sum_{r=1}^{n} \frac{1}{q^{z+r}-1} + \sum_{r=1}^{n} \frac{3}{(q^{z+r}-1)^2} + \sum_{r=1}^{n} \frac{2}{(q^{z+r}-1)^3} \right].
\end{aligned}$$

When $z = 0$, we have

$$Q_q^{(1)}(n,0) = -\ln q\left(n - H_n^{(1)}(q)\right),$$
$$Q_q^{(2)}(n,0) = (\ln q)^2\left[(n - H_n^{(1)}(q))^2 - H_n^{(1)}(q) + H_n^{(2)}(q)\right],$$
$$Q_q^{(3)}(n,0) = (-\ln q)^3\left[\left(n - H_n^{(1)}(q)\right)^3 + 3\left(n - H_n^{(1)}(q)\right)\right.$$
$$\left.\times\left(-H_n^{(1)}(q) + H_n^{(2)}(q)\right) - H_n^{(1)}(q) + 3H_n^{(2)}(q) - 2H_n^{(3)}(q)\right].$$

Lemma 2.8 *Let* $0 < q < 1$, $z \in \mathbb{R}\backslash\mathbb{Z}^-$ *and* $p \in \mathbb{N}$ *and let* $Q_q(p,z) = \begin{bmatrix} p+z \\ z \end{bmatrix}^{-1}$ *be an analytic function of* z. *Then*

$$Q_q^{(N)}(p,z) = (-\ln q)^N \sum_{r=1}^{p}\sum_{k=0}^{N}\begin{bmatrix} p \\ r \end{bmatrix}\frac{(-1)^r q^{\binom{r}{2}}(1-q^r)k!S(N+1,k+1)}{(q^{z+r}-1)^k}. \tag{18}$$

Proof. For $n \geq 0$ and $0 \leq l \leq n$, there holds

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(q/z;q)_k(zq^{-l};q)_{n-k}}{1-xq^k}z^k = \frac{(q;q)_n(xz;q)_{n-l}(zq^{-l};q)_l}{(x;q)_{n+1}}, \tag{19}$$

see [5]. By taking $n = p-1, l = 0, z \to 0, x = q^{z+1}$ and $k = r-1$ in (19), we have

$$\sum_{r=1}^{p}\frac{(-1)^r q^{\binom{r}{2}}}{q^{z+r}-1}\begin{bmatrix} p-1 \\ r-1 \end{bmatrix} = \frac{(q;q)_{p-1}}{(q^{z+1};q)_p}.$$

For

$$(1-q^r)\begin{bmatrix} p \\ r \end{bmatrix} = (1-q^p)\begin{bmatrix} p-1 \\ r-1 \end{bmatrix},$$

we have

$$Q_q(p,z) = \begin{bmatrix} p+z \\ z \end{bmatrix}^{-1} = \frac{(q;q)_p}{(q^{z+1};q)_p} = \sum_{r=1}^{p}\begin{bmatrix} p \\ r \end{bmatrix}\frac{(-1)^r q^{\binom{r}{2}}(1-q^r)}{q^{z+r}-1}.$$

Differentiating the above identity with respect to the variable z for N times and by Lemma 2.6, we can get the results. □

Theorem 2.9 *Let* $0 < q < 1$, $z \in \mathbb{R}\backslash\mathbb{Z}^-$, $Q_q(m,z) = \begin{bmatrix} m+z \\ z \end{bmatrix}^{-1}$ *be an analytic function of* z, *and* $Q_q^{(N)}(m,z)$ *be expressed by (17) or (18) for* $m \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p}\sum_{N_1=0}^{N} \frac{(\ln q)^{N_1}\binom{N}{N_1}\left[(n-p-1)^{N_1}q^{(n-p-1)z} - (n-p)^{N_1}q^{(n-p)z}\right]}{1-q^n}$$
$$\times Q_q^{(N-N_1)}(n,z) = (-p\ln q)^N q^{-pz} - Q_q^{(N)}(p,z). \tag{20}$$

Proof. By Corollary 2.2

$$\sum_{n=1}^{p} \frac{q^{zn}}{(1-q^n)\begin{bmatrix} n+z \\ z \end{bmatrix}} = \frac{q^z}{1-q^z}\left(1 - \frac{q^{pz}}{\begin{bmatrix} p+z \\ z \end{bmatrix}}\right),$$

we rewrite it as follows

$$\sum_{n=1}^{p} \frac{\left[q^{(n-p-1)z} - q^{(n-p)z}\right] Q_q(n,z)}{1-q^n} = q^{-pz} - Q_q(p,z).$$

Differentiating the above identity with respect to the variable z for N times, we can get the result.□

Note. Taking $q \to 1$ in Theorem 2.9, we obtain Theorem 1.1.

Corollary 2.10 *Let* $0 < q < 1$, $p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{H_n^{(1)}(q)}{1-q^n} = \frac{(H_p^{(1)}(q))^2 + H_p^{(2)}(q)}{2},$$

$$\sum_{n=1}^{p} \frac{(H_n^{(1)}(q))^2 + H_n^{(2)}(q)}{1-q^n} = \frac{(H_p^{(1)}(q))^3 + 3H_p^{(1)}(q)H_p^{(2)}(q) + 2H_p^{(3)}(q)}{3},$$

$$\sum_{n=1}^{p} \frac{(H_n^{(1)}(q))^3 + 3H_n^{(1)}(q)H_n^{(2)}(q) + 2H_n^{(3)}(q)}{1-q^n} = \frac{(H_p^{(1)}(q))^4}{4}$$
$$+\frac{6(H_p^{(1)})^2(q)H_p^{(2)}(q) + 8H_p^{(1)}(q)H_p^{(3)}(q) + 3(H_p^{(2)}(q))^2 + 6H_p^{(4)}(q)}{4}.$$

Proof. Let $Q_q^{(N)}(p,z)$ and $Q_q^{(N-N_1)}(n,z)$ of (20) be expressed in (17).

Taking $z = 0$, we get the result.

Corollary 2.11 *Let* $0 < q < 1$, $p \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{p} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-q^n} \begin{bmatrix} p \\ n \end{bmatrix} = p - H_p^{(1)}(q), \tag{21}$$

$$\sum_{n=1}^{p} \frac{(-1)^n q^{\binom{n+1}{2}}}{(1-q^n)^2} \begin{bmatrix} p \\ n \end{bmatrix} = \frac{p^2 - p - 2pH_p^{(1)}(q) + \left(H_p^{(1)}(q)\right)^2 + H_p^{(2)}(q)}{2}, \tag{22}$$

$$\sum_{n=1}^{p} \frac{(-1)^n q^{\binom{n+1}{2}}}{(1-q^n)^3} \begin{bmatrix} p \\ n \end{bmatrix}$$
$$= \frac{p^3 - 3p^2 + 2p - 3p^2 H_p^{(1)}(q) + 3p\left[\left(H_p^{(1)}(q)\right)^2 + H_p^{(2)}(q)\right]}{6}$$
$$- \frac{\left(H_p^{(1)}(q)\right)^3 + 3H_p^{(1)}(q)H_p^{(2)}(q) + 2H_p^{(3)}(q)}{6}. \tag{23}$$

Proof. Let $Q_q^{(N-N_1)}(n,z)$ of (20) be expressed by (17) and $Q_q^{(N)}(p,z)$ of (20) be expressed by (18). Taking $z = 0$, we get the result.

Remark. The identity (21) can be written as

$$\sum_{n=1}^{p} \frac{(-1)^{n-1} q^{\binom{n+1}{2}}}{1-q^n} \begin{bmatrix} p \\ n \end{bmatrix} = \sum_{r=1}^{p} \frac{q^r}{1-q^r}, \tag{24}$$

which is due to Van Hamme [11]. Many generalizations of (24) have been given by different authors. See, for example [3] and [5]. The identity (22) is the $(m,r) = (2,1)$ case of [6, Theorem 3.12]:*For* $m, n \geq 1$ *and* $0 \leq r \leq m + n - 1$, *there holds*

$$\sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+k(m-r)}}{(1-q^k)^m} = \binom{r}{m} + \binom{r}{m-1} \sum_{k_1=1}^{n} \frac{q^{k_1}}{1-q^{k_1}}$$
$$+ \binom{r}{m-2} \sum_{1 \leq k_1 \leq k_2 \leq n} \frac{q^{k_1+k_2}}{(1-q^{k_1})(1-q^{k_2})}$$

$$+\cdots+\binom{r}{0}\sum_{1\le k_1\le\cdots\le k_m\le n}\frac{q^{k_1+\cdots+k_m}}{(1-q^{k_1})\cdots(1-q^{k_m})}. \tag{25}$$

Similarly, the identity (23) is the $(m,r)=(3,2)$ case of (25).

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