Lattices generated by orbits of subspaces in t-singular linear spaces

Haixia Guo[1,2,*], Shufang Zhao [3]

1. College of Science, Tianjin University of Technology and Education, Tianjin, 300222, P.R. China
2. Dept. of Applied Math., Dalian University of Technology, Dalian, 116024, P.R. China
3. Science and Educational Department, Hebei First People's Hospital, Shijiazhuang, 050051, P. R. China

Abstract

For non-negative integers $n_1, n_2, \ldots, n_t$, let $GL_{n_1,n_2,\ldots,n_t}(\mathbb{F}_q)$ denote the t-singular general linear group of degree $n_1+n_2+\cdots+n_t$ and $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ denote the $(n_1+n_2+\cdots+n_t)$-dimensional t-singular linear space over the finite field $\mathbb{F}_q$. Let $\mathcal{M}$ be any orbit of subspaces under $GL_{n_1,n_2,\ldots,n_t}(\mathbb{F}_q)$. Denote by $\mathcal{L}$ the set of all intersections of subspaces in $\mathcal{M}$. Ordered $\mathcal{L}$ by ordinary or reverse inclusion, two posets are obtained. This paper discusses their geometricity and computes their characteristic polynomials.
Key words: Lattice; t-singular linear space; Atomic lattice; Geometric lattice; Characteristic polynomial.

1 Introduction

Let P be a partially ordered set (or poset) with a binary relation $\leq$. We use the obvious notation $a < b$ to mean $a \leq b$ and $a \neq b$. We say that two elements a and b of P are comparable if $a \leq b$ or $b \leq a$, otherwise a and b are incomparable. If $a, b \in P$, then we say that b covers a or a is covered by b, denoted $a \lessdot b$, if $a < b$ and no element $c \in P$ satisfies $a < c < b$. An element $m \in P$ is called a minimal (resp. maximal) element if there exists no $a \in P$ such that $a < m$ (resp. $m < a$). If P has the unique minimal (resp. maximal) element, then we say that P has the minimum (resp. maximum) element, denoted by 0 (resp. 1). From now on we suppose P is a poset with 0 (resp. 1). If the least upper bound of a and b exists, then it is clearly unique and is denoted $a \vee b$. Dually one can define the greatest lower bound $a \wedge b$, when it exists. A lattice is a poset L for which every pair of elements has the least upper bound and the greatest lower bound. Let L be a finite lattice with 0. An atom of L is an element covering 0, and L is said to be *atomic* if every element of L is the least upper bound of

*E-mail address: ghx626@126.com.

some atoms. Let P be a finite poset. By a *rank function* on P, we mean a function r from P to the set of all the nonnegative integers such that

(i) $r(0) = 0$,

(ii) $r(b) = r(a) + 1$ whenever $a \lessdot b$.

A finite atomic lattice L with 0 is said to be *geometric* if L admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$$

for any two elements $a, b \in L$.

Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, t) = \sum_{a \in P} \mu(0, a) t^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P, where r is the rank function on P.

There have been many interesting results for lattices generated by subspaces, see Huo, Liu and Wan ([7]-[9]), Huo and Wan ([11]), Guo et al. ([1]-[5],[10]) and Wang et al. ([12]-[14]). These research stimulates us to consider lattices generated by orbits of subspaces in t-singular linear spaces.

This paper is organized as follows. In Section 2, t-singular linear spaces are introduced. In Section 3, two families of finite atomic lattices are obtained and their geometricity are discussed. In Section 4, we compute their characteristic polynomials.

2 t-singular linear spaces

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. Let $n_1, n_2, \ldots, n_t$ be non-negative integers and $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ be the $(n_1 + n_2 + \cdots + n_t)$-dimensional row vector space over $\mathbb{F}_q$. The set of all $(n_1 + n_2 + \cdots + n_t) \times (n_1 + n_2 + \cdots + n_t)$ nonsingular matrix over $\mathbb{F}_q$

$$T = \begin{array}{c} \begin{matrix} n_1 & n_2 & \cdots & n_t \end{matrix} \\ \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1t} \\ & T_{22} & \cdots & T_{2t} \\ & & \ddots & \vdots \\ & & & T_{tt} \end{pmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_t \end{matrix} \end{array}$$

forms a group under matrix multiplication, called *t-singular general linear group* of degree $n_1 + n_2 + \cdots + n_t$ over $\mathbb{F}_q$ and denoted by $GL_{n_1, n_2, \ldots, n_t}(\mathbb{F}_q)$.

There is an action of $GL_{n_1,n_2,\ldots,n_t}(\mathbb{F}_q)$ on $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ defined as follows:

$$\begin{array}{rcl}\mathbb{F}_q^{n_1+n_2+\cdots+n_t} \times GL_{n_1,n_2,\ldots,n_t}(\mathbb{F}_q) & \to & \mathbb{F}_q^{n_1+n_2+\cdots+n_t} \\ ((x_1,x_2,\ldots,x_{n_1+n_2+\cdots+n_t}),T) & \mapsto & (x_1,x_2,\ldots,x_{n_1+n_2+\cdots+n_t})T.\end{array}$$

The vector space $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ together with the above group action is called $(n_1+n_2+\cdots+n_t)$-dimensional t-singular linear space over $\mathbb{F}_q$.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$, denote also by P an $m\times(n_1+n_2+\cdots+n_t)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P. For $1\le j\le n_1+n_2+\cdots+n_t$, let e_j be the row vector in $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ whose j-th coordinate is 1 and all other coordinates are 0. For $2\le i\le t$, denote by E_i the $(n_i+n_{i+1}+\cdots+n_t)$-dimensional subspace of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ spanned by $e_{n_1+\cdots+n_{i-1}+1}, e_{n_1+\cdots+n_{i-1}+2}, \ldots, e_{n_1+\cdots+n_i+\cdots+n_t}$. A k_1-dimensional subspace P of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ is called a subspace of type $(k_1,k_2,\ldots,k_t)$ if $\dim(P\cap E_i)=k_i$ for each i with $2\le i\le t$.

Denoted by $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ the set of all the subspaces of type $(k_1,k_2,\ldots,k_t)$ of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ and denoted by $\mathcal{M}'(l_1,l_2,\ldots,l_t;k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ the set of all the subspaces of type $(k_1,k_2,\ldots,k_t)$ containing a given subspace of type $(l_1,l_2,\ldots,l_t)$.

Proposition 2.1. *([6, Proposition 2.2]) The set $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ is non-empty if and only if*

$$0\le k_i-k_{i+1}\le n_i\ (1\le i\le t-1)\ \textit{and}\ 0\le k_t\le n_t. \tag{1}$$

Moreover, if (1) holds, then $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ forms an orbit under $GL_{n_1,n_2,\ldots,n_t}(\mathbb{F}_q)$ and $|\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)| = \begin{bmatrix} n_t\\ k_t\end{bmatrix}_q \times \prod_{j=1}^{t-1} q^{(k_j-k_{j+1})(n_{j+1}+\cdots+n_t-k_{j+1})}\begin{bmatrix} n_j\\ k_j-k_{j+1}\end{bmatrix}_q$.

Proposition 2.2. *([6, Corollary 2.3]) The set $\mathcal{M}'(l_1,l_2,\ldots,l_t;k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ is non-empty if and only if*

$$0\le l_i-l_{i+1}\le k_i-k_{i+1}\le n_i\ (1\le i\le t-1)\ \textit{and}\ 0\le l_t\le k_t\le n_t. \tag{2}$$

Moreover, if (2) holds, then $|\mathcal{M}'(l_1,l_2,\ldots,l_t;k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)|$
$$= q^{\sum_{j=1}^{t-1}(k_j-k_{j+1}-l_j+l_{j+1})(n_{j+1}+\cdots+n_t-k_{j+1})}\begin{bmatrix} n_t-l_t\\ k_t-l_t\end{bmatrix}_q \prod_{j=1}^{t-1}\begin{bmatrix} n_j-l_j+l_{j+1}\\ k_j-k_{j+1}-l_j+l_{j+1}\end{bmatrix}_q.$$

3 Lattices generated by orbits of subspaces

Let $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ denote the $(n_1+n_2+\cdots+n_t)$-dimensional t-singular linear space over $\mathbb{F}_q$, and $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ denote the orbit of subspaces of type $(k_1,k_2,\ldots,k_t)$. Clearly, $\{0\}$ and $\{\mathbb{F}_q^{n_1+n_2+\cdots+n_t}\}$

are two trivial orbits in $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$. The set of subspaces which are intersections of subspaces in $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ denoted by $\mathcal{L}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ and call $\mathcal{L}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ the set of subspaces generated by $\mathcal{M}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$. We agree that the intersection of an empty set of subspaces is $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$. Then $\mathbb{F}_q^{n_1+n_2+\cdots+n_t} \in \mathcal{L}(k_1,k_2,\ldots,k_t;\, n_1,n_2,\ldots,n_t)$.

Partially ordered $\mathcal{L}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ by ordinary or reverse inclusion, we get two posets and denote them by $\mathcal{L}_O(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ or $\mathcal{L}_R(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ respectively. Clearly, for any two elements $P,Q \in \mathcal{L}_O(k_1,k_2,\ldots,k_t;\, n_1,n_2,\ldots,n_t)$,

$$P\wedge Q = P\cap Q,\;\; P\vee Q = \bigcap\{R\in \mathcal{L}_O(k_1,\ldots,k_t;n_1,\ldots,n_t) : R\supseteq \langle P,Q\rangle\}$$

where $\langle P,Q\rangle$ is the subspace spanned by P and Q. Therefore, $\mathcal{L}_O(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ is a finite lattice. Similarly, for any two elements $P,Q \in \mathcal{L}_R(k_1,\ldots,k_t;n_1,\,n_2,\ldots,n_t)$,

$$P\wedge Q = \bigcap\{R\in \mathcal{L}_R(k_1,\ldots,k_t;n_1,\ldots,n_t) : R\supseteq \langle P,Q\rangle\},\;\; P\vee Q = P\cap Q.$$

So $\mathcal{L}_R(k_1,k_2,\ldots,\,k_t;n_1,n_2,\ldots,n_t)$ is also a finite lattice. Both $\mathcal{L}_O(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ and $\mathcal{L}_R(k_1,k_2,\ldots,k_t;n_1,n_2,\,\ldots,n_t)$ are called the lattices generated by $\mathcal{M}(k_1,k_2,\,\ldots,k_t;n_1,n_2,\ldots,n_t)$.

If there exists some j with $1\le j\le t$ such that $n_j=0$, then $\mathcal{L}(k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_t;n_1,\ldots,n_{j-1},n_j,n_{j+1},\ldots,n_t) = \mathcal{L}(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_t;n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_t)$. If $k_t=n_t$, then $\mathcal{L}_O(k_1,k_2,\ldots,k_{t-1},n_t;n_1,n_2,\ldots,n_{t-1},n_t)$ (resp. $\mathcal{L}_R(k_1,k_2,\ldots,k_{t-1},n_t;n_1,n_2,\ldots,n_{t-1},n_t)$) is isomorphic to $\mathcal{L}_O(k_1,k_2,\ldots,k_{t-1};n_1,n_2,\ldots,n_{t-1})$ (resp. $\mathcal{L}_R(k_1,k_2,\ldots,k_{t-1};n_1,\,n_2,\ldots,n_{t-1})$). If $k_1=0$, then $\mathcal{L}(k_1,\ldots,k_t;n_1,\ldots,n_t) = \{\{0\},\mathbb{F}_q^{n_1+\cdots+n_t}\}$. Therefore, in the rest of this paper we always assume that n_j is a positive integer for each j with $1\le j\le t$, $k_t<n_t$ and $k_1>0$.

Before discussing the geometricity of these two lattices $\mathcal{L}_O(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ and $\mathcal{L}_R\,(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ we first give a lemma and some useful Theorems.

Lemma 3.1. *Let* $0\le k_i-k_{i+1}\le n_i$ $(1\le i\le t-1)$ *and* $0\le k_t<n_t$. *Then for each* j *with* $1\le j\le t$, *we have* $\mathcal{L}(k_1-1,\ldots,k_j-1,k_{j+1},\ldots,k_t;n_1,n_2,\ldots,n_t)\subseteq \mathcal{L}(k_1,\ldots,k_j,k_{j+1},\ldots,k_t;n_1,n_2,\ldots,n_t)$.

Proof. If $k_j-k_{j+1}=0$, then the result is obvious. Suppose $k_j-k_{j+1}\ge 1$. For any $P\in\mathcal{M}(k_1-1,\ldots,k_j-1,k_{j+1},\ldots,k_t;n_1,n_2,\ldots,n_t)$, by Proposition 2.2 the number of subspaces of type $(k_1,k_2,\ldots,k_t)$ containing P is $|\mathcal{M}'(k_1-1,\ldots,k_j-1,k_{j+1},\ldots,k_t;k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)|\ge 2$, which implies $P\in\mathcal{L}(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$. So $\mathcal{L}(k_1-1,\ldots,k_j-1,k_{j+1},\ldots,k_t;n_1,\,n_2,\ldots,n_t)\subseteq\mathcal{L}(k_1,\ldots,k_j,k_{j+1},\ldots,k_t;n_1,n_2,\ldots,n_t)$. □

Theorem 3.2. *Let $0 \leq k_i - k_{i+1} \leq n_i$ $(1 \leq i \leq t-1)$ and $0 \leq k_t < n_t$. Then $\mathcal{L}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t) \subseteq \mathcal{L}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ if and only if*

$$0 \leq l_i - l_{i+1} \leq k_i - k_{i+1} \ (1 \leq i \leq t-1) \text{ and } 0 \leq l_t \leq k_t. \tag{3}$$

Proof. First, we suppose (3) holds. By Lemma 3.1, we have $\mathcal{L}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t) \supseteq \mathcal{L}(k_1 - 1, k_2 - 1, \ldots, k_t - 1; n_1, n_2, \ldots, n_t) \supseteq \cdots \supseteq \mathcal{L}(k_1 + l_t - k_t, k_2 + l_t - k_t, \ldots, k_{t-1} + l_t - k_t, l_t; n_1, n_2, \ldots, n_t)$. Next, we have $\mathcal{L}(k_1 + l_t - k_t, k_2 + l_t - k_t, \ldots, k_{t-1} + l_t - k_t, l_t; n_1, n_2, \ldots, n_t) \supseteq \mathcal{L}(k_1 + l_t - k_t - 1, k_2 + l_t - k_t - 1, \ldots, k_{t-1} + l_t - k_t - 1, l_t; n_1, n_2, \ldots, n_t) \supseteq \mathcal{L}(k_1 + l_{t-1} - k_{t-1}, k_2 + l_{t-1} - k_{t-1}, \ldots, k_{t-2} + l_{t-1} - k_{t-1}, l_{t-1}, l_t; n_1, n_2, \ldots, n_t) \supseteq \cdots \supseteq \mathcal{L}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t)$.

Conversely, suppose that $\mathcal{L}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t) \subseteq \mathcal{L}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$. Since $\mathcal{M}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t) \subseteq \mathcal{L}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t) \subseteq \mathcal{L}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$, for any $Q \in \mathcal{M}(l_1, l_2, \ldots, l_t; n_1, n_2, \ldots, n_t)$, there exists $P \in \mathcal{M}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ such that $Q \subseteq P$. By Proposition 2.2, the desired result follows. □

Theorem 3.3. *Let $0 \leq k_i - k_{i+1} \leq n_i$ $(1 \leq i \leq t-1)$ and $0 \leq k_t < n_t$. Then $\mathcal{L}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ consists of $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ and all subspaces of type $(l_1, l_2, \ldots, l_t)$ with $0 \leq l_i - l_{i+1} \leq k_i - k_{i+1}$ $(1 \leq i \leq t-1)$ and $0 \leq l_t \leq k_t$.*

Proof. By Theorem 3.2, it is straightforward. □

Theorem 3.4. *Let $0 \leq k_i - k_{i+1} \leq n_i$ $(1 \leq i \leq t-1)$ and $0 \leq k_t < n_t$. Then $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is a geometric lattice if and only if $k_1 = 1$, $k_1 = n_1 + n_2 + \cdots + n_t - 1$ or $k_1 = k_t = n_t - 1$.*

Proof. For any $X \in \mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$, define

$$r_R(X) = \begin{cases} 0, & \text{if } X = \mathbb{F}_q^{n_1+n_2+\cdots+n_t}; \\ k_1 + 1 - \dim(X), & \text{otherwise.} \end{cases}$$

Then r_R is the rank function on $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$.

Note that $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ is the minimum element of $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$, and $\mathcal{M}(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is the set of atoms of $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$. For any $U \in \mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t) \setminus \{\mathbb{F}_q^{n_1+n_2+\cdots+n_t}\}$, by definition of $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$, we have U is the least upper bound of some atoms, which implies that $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is an atomic lattice.

If $k_1 = 1$, then $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is a lattice of rank 2, which implies that $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is a geometric lattice. Suppose $k_1 = n_1 + n_2 + \cdots + n_t - 1$. For any U, $V \in \mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2,$

$\ldots, n_t)$ we have $U \vee V = U \cap V$ and $U \wedge V \supseteq \langle U, V\rangle$, which implies that $\dim(U \wedge V) \geq \dim(\langle U, V\rangle)$. Therefore, $r_R(U \wedge V) + r_R(U \vee V) = (k_1+1-\dim(U \wedge V)) + (k_1+1-\dim(U \vee V)) \leq (k_1+1-\dim(\langle U,V\rangle)) + (k_1+1-\dim(U\cap V)) = (k_1+1-\dim U)+(k_1+1-\dim V) \leq r_R(U)+r_R(V)$, which implies that $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is a geometric lattice. If $k_1 = k_t \leq n_t - 1$, then $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is isomorphic to $\mathcal{L}_R(k_t; n_t)$, Theorem 5 in [11] tells us that $\mathcal{L}_R(k_t; n_t)$ is a geometric lattice if and only if $k_t = n_t - 1$.

Suppose $2 \leq k_1 \leq n_1 + n_2 + \cdots + n_t - 2$ and $k_1 > k_t$. Then there exist two k_1-dimensional subspaces U, V in $\mathcal{L}_R(k_1, \ldots, k_t; n_1, \ldots, n_t)$ such that $U \wedge V = \mathbb{F}_q^{n_1+\cdots+n_t}$ and $\dim(U \vee V) = k_1 - 2$, which implies that $r_R(U \wedge V) + r_R(U \vee V) > r_R(U) + r_R(V)$. Therefore $\mathcal{L}_R(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is an atomic lattice but not a geometric lattice. □

Theorem 3.5. *Let $0 \leq k_i - k_{i+1} \leq n_i$ $(1 \leq i \leq t-1)$ and $0 \leq k_t < n_t$. Then $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ is a geometric lattice if and only if $k_1 = 1$ or $k_1 = k_t$.*

Proof. For any $X \in \mathcal{L}_O(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$, define

$$r_O(X) = \begin{cases} k_1 + 1, & \text{if } X = \mathbb{F}_q^{n_1+n_2+\cdots+n_t}; \\ \dim(X), & \text{otherwise.} \end{cases}$$

Then r_O is the rank function on $\mathcal{L}_O(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$.

Note that $\{0\}$ is the minimum element of $\mathcal{L}_O(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$. For any $U \in \mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t) \backslash \{\{0\}, \mathbb{F}_q^{n_1+n_2+\cdots+n_t}\}$, let $\alpha_1, \ldots, \alpha_{\dim U}$ be a basis for U. By Theorem 3.3, each $\langle \alpha_j \rangle$ with $1 \leq j \leq \dim U$ is an atom of $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$, which implies that $U = \bigvee_{j=1}^{\dim U} \langle \alpha_j \rangle$. Hence $\mathcal{L}_O(k_1, k_2, \ldots, k_t; n_1, n_2, \ldots, n_t)$ is an atomic lattice.

If $k_1 = 1$, then $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ is a lattice of rank 2, which implies that $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ is a geometric lattice. If $k_1 = k_t$, then $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ is isomorphic to $\mathcal{L}_O(k_t; n_t)$, by Theorem 4 in [11] $\mathcal{L}_O(k_t; n_t)$ is a geometric lattice.

Suppose $2 \leq k_1 \leq n_1 + n_2 + \cdots + n_t - 1$ and $k_1 > k_t$. If $k_t > 0$, by Theorem 3.3 $\mathcal{M}(k_t, \ldots, k_t; n_1, \ldots, n_t) \subseteq \mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$. Therefore, there exist two k_t-dimensional subspaces U, V in $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ such that $\dim(U \wedge V) = k_t - 1$ and $U \vee V = \mathbb{F}_q^{n_1+\cdots+n_t}$, which implies that $r_O(U \wedge V) + r_O(U \vee V) > r_O(U) + r_O(V)$. If $k_t = 0$, then there exists some j with $1 \leq j \leq t-1$ such that $k_j > 0$. Without loss of generality, assume that $j := \max\{l : k_l > 0\ (1 \leq l \leq t-1)\}$. By Theorem 3.3 $\mathcal{M}(k_j, \ldots, k_j, k_t, \ldots, k_t; n_1, \ldots, n_t) \subseteq \mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$. Therefore, there exist two k_j-dimensional subspaces U, V in $\mathcal{L}_O(k_1, \ldots, k_t; n_1, \ldots, n_t)$ such that $\dim(U \wedge V) = k_j - 1$ and $U \vee V = \mathbb{F}_q^{n_1+\cdots+n_t}$, which implies that $r_O(U \wedge V) + r_O(U \vee V) > r_O(U) + r_O(V)$ for $k_j < k_1$. Assume that $k_j = k_1$. By Theorem 3.3 $\mathcal{M}(k_j-1,\ldots,k_j-1,k_t,\ldots,k_t;n_1,\ldots,n_t) \subseteq \mathcal{L}_O(k_1,\ldots,k_t;n_1,\ldots,n_t)$. Therefore, there exist two (k_j-1)-dimensional subspaces U,V in $\mathcal{L}_O(k_1,\ldots,k_t;n_1,\ldots,n_t)$ such that $\dim(U\wedge V)=k_j-2$ and $U\vee V=\mathbb{F}_q^{n_1+\cdots+n_t}$, which implies that $r_O(U\wedge V)+r_O(U\vee V) > r_O(U)+r_O(V)$. Therefore $\mathcal{L}_O(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$ is an atomic lattice but not a geometric lattice. □

4 Characteristic polynomials

Theorem 4.1. *Let $0\le k_i-k_{i+1}\le n_i$ $(1\le i\le t-1)$ and $0\le k_t<n_t$. Then the characteristic polynomial of $\mathcal{L}_R(k_1,k_2,\ \ldots,k_t;n_1,n_2,\ldots,n_t)$ is* $\chi(\mathcal{L}_R,x)=x^{k_1+1}-\sum\limits_{l_t=0}^{k_t}\sum\limits_{l_{t-1}=l_t}^{k_{t-1}-k_t+l_t}\cdots\sum\limits_{l_2=l_3}^{k_2-k_3+l_3}\sum\limits_{l_1=l_2}^{k_1-k_2+l_2}|\mathcal{M}(l_1,l_2\ldots,l_t;n_1,n_2,$ $\ldots,n_t)|g_{l_1}(x)$, *where* $g_{l_1}(x)=\prod\limits_{i=0}^{l_1-1}(x-q^i)$, $g_0(x)=0$, *where* $\mathcal{L}_R=\mathcal{L}_R(k_1,k_2,$ $\ldots,k_t;n_1,n_2,\ldots,n_t)$.

Proof. For $U\in\mathcal{L}_R(k_1,k_2,\ldots,k_t;n_1,n_2,\ldots,n_t)$, let $\mathcal{L}^U=\{W\in\mathcal{L}_R: W\ge U\}$, then $\mathcal{L}^{\mathbb{F}_q^{n_1+n_2+\cdots+n_t}}=\mathcal{L}_R$. Since $\{0\}$ is the maximum element and $\mathbb{F}_q^{n_1+n_2+\cdots+n_t}$ is the minimum element in $\mathcal{L}_R$, the characteristic polynomial of $\mathcal{L}_R$ is

$$\chi(\mathcal{L}_R,x)=\sum_{U\in\mathcal{L}_R}\mu(\mathbb{F}_q^{n_1+n_2+\cdots+n_t},U)x^{k_1+1-r_R(U)}.$$

By the Möbius inversion formula $x^{k_1+1}=\sum\limits_{U\in\mathcal{L}}\chi(\mathcal{L}^U,x)$. By Theorem 3.3 and Lemma 3.1,

$$\chi(\mathcal{L}_R,x)=x^{k_1+1}-\sum_{U\in\mathcal{L}_R\setminus\mathbb{F}_q^{n_1+n_2+\cdots+n_t}}\chi(\mathcal{L}^U,x)=x^{k_1+1}-\sum_{l_t=0}^{k_t}\sum_{l_{t-1}=l_t}^{k_{t-1}-k_t+l_t}\cdots\sum_{l_2=l_3}^{k_2-k_3+l_3}\sum_{l_1=l_2}^{k_1-k_2+l_2}|\mathcal{M}(l_1,l_2,\ldots,l_t;n_1,n_2,\ldots,n_t)|\times g_{l_1}(x).$$ □

Acknowledgements This research is supported by National Natural Science Foundation of China (Grant No. 11302148).

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