Two kinds of equicoverable paths and cycles *

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Abstract

Packing and covering are dual problems in graph theory. A graph G is called $H-$equipackable if every maximal $H-$packing in G is also a maximum $H-$packing in G. Dually, a graph G is called $H-$equicoverable if every minimal $H-$covering in G is also a minimum $H-$covering in G. In 2012, Zhang characterized two kinds of equipackable paths and cycles: P_k-equipackable paths and cycles, M_k-equipackable paths and cycles. In this paper, P_k-equicoverable $(k > 3)$ paths and cycles, M_k-equicoverable $(k > 2)$paths and cycles are characterized.

Keywords: Equicoverable, coverable, path, matching.

1 Introduction

Packing and covering are dual problems in graph theory. The problem that we study stems from research of H-decomposable graphs and equipackable graphs. The path and cycle on n vertices are denoted by P_n and C_n, respectively. In this paper, Denote the edges of P_n by $e_1, e_2, \cdots, e_{n-1}$. Denote the edges of C_n by $e_1, e_2, \cdots, e_n$. A vertex with degree 1 of a path is called an end vertex of the path. A matching in the graph G is a set of independent edges in G. A matching with $k(k \geq 1)$ edges is denoted by M_k. Let H be a subgraph of G. By $G - H$, we denote the graph left after we delete from G the edges of H and any resulting isolated vertices.

A collection of edge disjoint copies of H, say $H_1, H_2, \cdots, H_l$, where each $H_i(i = 1, 2, \cdots, l)$ is a subgraph of G, is called an $H-$packing in G. A graph

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G is called $H-$decomposable if there exists an $H-$packing of G which uses all edges in G. An $H-$packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximal if $G - \bigcup_{i=1}^{l} E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with l copies $H_1, H_2, \cdots, H_l$ of H is called maximum if no more than l edge disjoint copies of H can be packed into G. A graph G is called randomly $H-$decomposable if every maximal $H-$packing in G uses all edges in G. A graph G is called $H-$equipackable if every maximal $H-$packing in G is also a maximum $H-$packing in G. There have been many results on randomly $H-$decomposable and $H-$equipackable graphs: L. W. Beineke, P. Hamberger and W. D. Goddard ([1]) characterized all randomly M_k-decomposable graphs, all randomly K_n-decomposable graphs and all randomly P_k-decomposable for $k = 4, 5, 6$; B. Randerath and P. D. Vestergaard ([2]) characterized all P_3-equipackable graphs; Zhang and Fan([3]) characterized all M_2-equipackable graphs; Zhang([6]) characterized two kinds of equipackable paths and cycles.

An H-covering of G is a set $L = \{H_1, H_2, \cdots, H_l\}$ of subgraphs of G, where each subgraph H_i is isomorphic to H and every edge of G appears in at least one member of L. If G has an H-covering, G is called H-coverable. An H-covering of G with l copies $H_1, H_2, \cdots, H_l$ of H is called minimal if, for any H_j, $\bigcup_{i=1}^{l} H_i - H_j$ is not an $H-$covering of G. An H-covering of G with l copies $H_1, H_2, \cdots, H_l$ of H is called minimum if there exists no H-covering with less than l copies H. Let $c(G; H)$ denote the number of H in the minimum H-covering of G. In 2008, Zhang([4]) gave the dual definition of H-equipackable: H-equicoverable. A graph is called H-equicoverable if every minimal H-covering in G is also a minimum H-covering in G. And Zhang characterized all P_3-equicoverable graphs. The path P_n is P_3-equicoverable if and only if $n = 3, 4, 5, 6, 8$. The cycle C_n is P_3-equicoverable if and only if $n = 3, 4, 5, 7$. Later, Zhang and Lan([5]) gave some results on M_2-equicoverable graphs, and characterized some kinds of special M_2-equicoverable graphs. The path P_n is M_2-equicoverable if and only if $n = 5, 6$. The cycle C_n is M_2-equicoverable if and only if $n = 4, 5$.

In this paper, we investigate P_k-equicoverable $(k > 3)$ paths and cycles, M_k-equicoverable $(k > 2)$ paths and cycles.

We first give one lemma which is crucial to our work:

Lemma 1. *Let G be an F-coverable graph and H be an F-coverable subgraph of G which satisfy: (1) H is not F-equicoverable; (2) $G - H$ is F-decomposable. Then G is not F-equicoverable.*

Proof. Since H is F-coverable but not F-equicoverable, by the definitions of coverable and equicoverable, H has at least one minimal F-covering which is not minimum. And $G - H$ is F-decomposable, that is, $G - H$ has an F-covering which is also an F-packing. The union of the two F-covering mentioned above forms a minimal F-covering which is not minimum. So G is not F-equicoverable. □

2 Main results

2.1 P_k-equicoverable ($k > 3$) paths and cycles

Theorem 2. *A path P_n is P_k-equicoverable if and only if $k \leq n \leq 2k$ or $n = 3k - 1$.*

Proof. In each P_k-covering of P_n, e_1 must be covered by $H_1 = \{e_1, e_2, \cdots, e_{k-1}\}$ and e_{n-1} must be covered by $H_2 = \{e_{n-k+1}, e_{n-k+2}, \cdots, e_{n-1}\}$. For the P_k−covering of a path P_n, we have seven cases.

1. $n \leq k - 1$. Since P_n contains no copy of P_k, P_n can't be P_k-equicoverable.

2. $n = k$. It's easy to see the number of P_k in the minimal P_k-covering of P_n only can be 1. By the definition, P_n is P_k-equicoverable.

3. $k+1 \leq n \leq 2k-1$. It's easy to see $c(P_n; P_k)$ is 2. $L = \{H_1, H_2\}$ covers all edges of P_n. So the number of P_k in the minimal P_k-covering of P_n only can be 2. By the definition, P_n is P_k-equicoverable.

4. $n = 2k$. It's easy to see $c(P_n; P_k)$ is 3. Besides H_1 and H_2, only one edge has not been covered, and we need only one copy of P_k to cover it. So the number of P_k in the minimal P_k-covering of P_n only can be 3. By the definition, P_n is P_k-equicoverable.

5. $2k+1 \leq n \leq 3k-2$. Obviously, $c(P_n; P_k)$ is 3. There exists a minimal P_k-covering with 4 copies of P_k denoted by $L = \{H_1, H_2, H_3, H_4\}$, where $H_3 = \{e_2, e_3, \cdots, e_k\}$, $H_4 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}$. By the definition, P_n is not P_k-equicoverable.

6. $n = 3k - 1$. Besides H_1 and H_2, there must be one copy $H_i = \{e_i, e_{i+1}, \cdots, e_{i+k-2}\}$ $(2 \leq i \leq k)$ to cover the edge e_k. There also must be one copy $H_j^i = \{e_j, e_{j+1}, \cdots, e_{j+k-2}\}(k+1 \leq j \leq i+k-1)$ to cover e_{i+k-1}. Since $j \leq i+k-1 \leq 2k-1 \leq j+k-2$, H_j^i also covers the edges $e_{i+k}, \cdots, e_{2k-1}$. $L = \{H_1, H_2, H_i, H_j^i\}$ contains all possible minimal P_k-coverings of P_n. So the number of P_k in the minimal P_k-covering of P_n only can be 4. By the definition, P_n is P_k-equicoverable.

7. $n \geq 3k$. $n - (2k+1) \equiv r(mod\ k-1)(r = 0, 1, \cdots, k-2)$, $n - (2k + 1 + r) = (k-1)t(t \in Z, t \geq 1)$. $n - (2k+1) \geq k - 1$.

(a) $0 \le r \le k-3$. $P_n - P_{2k+1+r}$ has $(k-1)t(t \in Z, t \ge 1)$ edges, so $P_n - P_{2k+1+r}$ is P_k-decomposable. Since $2k+1 \le 2k+1+r \le 3k-2$, from case 5, P_{2k+1+r} is not P_k-equicoverable. By Lemma 1, P_n is not P_k-equicoverable.

(b) $r = k-2$. $P_n = P_{4k-2}$ or $P_n - P_{4k-2}$ is P_k-decomposable. It's easy to see $c(P_{4k-2}; P_k)$ is 5. There exists a minimal P_k-covering of P_{4k-2} with 6 copies of P_k denoted by $L = \{H_1, H_2, \cdots, H_6\}$, where $H_3 = \{e_2, e_3, \cdots, e_k\}$, $H_4 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}$, $H_5 = \{e_{2k-1}, e_{2k}, \cdots, e_{3k-3}\}$, $H_6 = \{e_{3k-2}, e_{3k-1}, \cdots, e_{4k-4}\}$, so P_{4k-2} is not P_k-equicoverable. By Lemma 1, P_n is not P_k-equicoverable.

From the above, a path P_n is P_k-equicoverable if and only if $k \le n \le 2k$ or $n = 3k-1$. □

Theorem 3. *A cycle C_n is P_k-equicoverable if and only if $k \le n < \lceil \frac{3k}{2} \rceil$ or $n = 2k-1$.*

Proof. By the symmetry of the cycle, we can choose the first copy of P_k to be $H_1 = \{e_1, e_2, \cdots, e_{k-1}\}$ in this proof. For the P_k-covering of a cycle C_n, we have seven cases.

1. $n \le k-1$. Since C_n contains no copy of P_k, C_n can't be P_k-equicoverable.

2. $n = k$. It's easy to see $c(C_n; P_k)$ is 2. Besides H_1, only one edge has not been covered, and we need only one copy of P_k to cover it. So the number of P_k in the minimal P_k-covering of C_n only can be 2. By the definition, C_n is P_k-equicoverable.

3. $k+1 \le n \le 2k-2$. It's easy to see $c(C_n; P_k)$ is 2.

   In the covering, besides the copy H_1 , there must be another copy $H_i = \{e_i, e_{i+1}, \cdots, e_{i+k-2}\}(2 \le i \le k)$ to cover the edge e_k, where for $\forall e_x, x \leftarrow x \mod n$.

   (a) $i+k-2 \ge n$, $i-1 \le k-1$. Then $\{H_1, H_i\}$ is the only possible minimal P_k-covering of C_n with 2 copies.

   (b) $i+k-2 \le n-1$, since the edge e_{i+k-1} has not been covered, there must be the third copy $H_j^i = \{e_j, e_{j+1}, \cdots, e_{j+k-2}\}(k+1 \le j \le i+k-1)$ to cover it.

      - When $i+k-1 \le n < \lceil \frac{3k}{2} \rceil$, $(n+i-1)-(j+k-2) = n+i-j-k+1 \le n+(n-k+1)-(k+1)-k+1 = 2n-3k+1 < 2 * \frac{3k}{2} - 3k+1 = 1$. That is, $n+i-1 \le j+k-2$. So$\{H_j^i, H_i\}$ can cover all edges of C_n, H_1 is redundant. So when $n < \lceil \frac{3k}{2} \rceil$, there exists no minimal P_k-covering with 3 copies, C_n is P_k-equicoverable.

- When $n \geq \lceil \frac{3k}{2} \rceil$, $(n+i-1)-(j+k-2) = n+i-j-k+1 = n+(n-k+1)-(k+1)-k+1 = 2n-3k+1 > 2*\frac{3k}{2}-3k+1 = 1$. That is, $n+i-1 > j+k-2$. there exists a minimal P_k−covering with 3 copies of P_k denoted by $H = \{H_1, H_j^i, H_i\}$. so C_n isn't P_k-equicoverable.

4. $n = 2k-1$. C_n is P_k-equicoverable.

   In the covering, besides the copy H_1 , there must be another copy $H_i = \{e_i, e_{i+1}, \cdots, e_{i+k-2}\}(2 \leq i \leq k)$ to cover the edge e_k. Since the edge e_{i+k-1} has not been covered, there must be the third copy $H_j^i = \{e_j, e_{j+1}, \cdots, e_{j+k-2}\}(k+1 \leq j \leq i+k-1)$ to cover it. where for $\forall e_x, x \leftarrow x \mod n$. Since $j \leq i+k-1 < 2k-1 \leq j+k-2$, H_j^i always covers the edges $e_{i+k-1}, e_{i+k}, \cdots, e_{2k-1}$. $\{H_1, H_i, H_j^i\}$ contains all possible minimal P_k-coverings of C_n. So the number of P_k in the minimal P_k-covering of C_n only can be 3. By the definition, C_n is P_k-equicoverable.

5. $2k \leq n \leq 3k-3$. It's easy to see $c(C_n; P_k)$ is 3. There exists a minimal P_k-covering with 4 copies of P_k denoted by $L = \{H_1, H_2, H_3, H_4\}$, where $H_2 = \{e_2, e_3, \cdots, e_k\}$, $H_3 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}$, $H_4 = \{e_{n-k+2}, e_{n-k+3}, \cdots, e_n\}$. So C_n is not P_k-equicoverable.

6. $n = 3k-2$. It's easy to see $c(C_n; P_k)$ is 4. There exists a minimal P_k-covering with 5 copies of P_k denoted by $H = \{H_1, H_2, H_3, H_4, H_5\}$, where

   $$H_2 = \{e_3, e_4, \cdots, e_k, e_{k+1}\}, H_3 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-2}, e_{2k-1}\},$$
   $$H_4 = \{e_{k+3}, e_{k+4}, \cdots, e_{2k}, e_{2k+1}\}, H_5 = \{e_{2k+1}, e_{2k+2}, \cdots, e_{3k-2}, e_1\}.$$

   By the definition, so C_n is not P_k-equicoverable.

7. $n \geq 3k-1$, $n-2k \equiv r(mod\ k-1)$ $(r = 0, 1, \cdots, k-2)$.

   (a) $0 \leq r \leq k-3$.

   $C_n - P_{2k+1+r}$ has $(k-1)t(t \in Z, t \geq 1)$ edges, so $C_n - P_{2k+1+r}$ is P_k-decomposable. By Theorem 2, P_{2k+r+1} is not P_k-equicoverable. By Lemma 1, C_n is not P_k-equicoverable.

   (b) $r = k-2$.

   - When $n = 4k-3$, it's easy to see $c(C_{4k-3}; P_k)$ is 5. There exists a minimal P_k-covering with 6 copies of P_k denoted by $L = \{H_1, H_2, H_3, H_4, H_5, H_6\}$, where $H_2 = \{e_2, e_3, \cdots, e_k\}$, $H_3 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}$, $H_4 = \{e_{2k-1}, e_{2k}, \cdots, e_{3k-3}\}$, $H_5 = \{e_{3k-2}, e_{3k-1}, \cdots, e_{4k-4}\}$, $H_6 = \{e_{3k-1}, e_{3k}, \cdots, e_{4k-3}\}$, so C_{4k-3} is not P_k-equicoverable.

- When $n \neq 4k-3$, $C_n - P_{4k-2}$ is P_k-decomposable. By Theorem 2, P_{4k-2} is not P_k-equipackable. By Lemma 1, C_n is not P_k-equicoverable.

From the above, C_n is P_k–equicoverable if and only if $k \leq n < \lceil \frac{3k}{2} \rceil$ or $n = 2k-1$.

□

2.2 M_k-equicoverable ($k > 2$) paths and cycles

To get the results, we first give several lemmas.

Lemma 4. *Let P_n be an M_k-coverable path, then $c(P_n; M_k) = \lceil \frac{n-1}{k} \rceil$.*

Proof. Since P_n is M_k-coverable, by the definition of minimal covering, it is easy to see $c(P_n; M_k)$ is at least $\lceil \frac{n-1}{k} \rceil$. To get the desired result, clearly it suffices to find a minimal M_k-covering of P_n with $\lceil \frac{n-1}{k} \rceil$ copies.

Let $E(P_n) = A \cup B$, where $A = \{e_1, e_3, e_5, \cdots, e_{2p-1}\}$, $B = \{e_2, e_4, e_6, \cdots, e_{2q}\}$. Let $L = \{H_1, H_2, \cdots, H_{\lceil \frac{n-1}{k} \rceil}\}$ be a set of subgraphs of P_n, where H is shown in Fig.1, and let $t = n-1 \mod k$.

$$\underbrace{e_1 \; e_3 \; e_5 \; \cdots \; e_{2k-1}}_{H_1} \quad \underbrace{e_{2k+1} \; e_{2k+3} \; \cdots \; e_{4k-1}}_{H_2}$$

$$\cdots \quad \underbrace{e_{2k(i-1)+1} \; e_{2k(i-1)+3} \; e_{2k(i-1)+5} \; \cdots \; e_{2ki-1}}_{H_i}$$

$$\underbrace{e_{2ki+1} \; e_{2ki+3} \; \cdots \; e_{2p-1} \; e_2 \; e_4 \; \cdots \; e_{2(k-(p-ki))}}_{H_{i+1}}$$

$$\cdots \quad \underbrace{e_{2(q+1-t)} \; e_{2(q+1-t)} \; \cdots \; e_{2q} \; e_2 \; e_4 \; \cdots \; e_{2(k-t)}}_{H_{\lceil \frac{n-1}{k} \rceil}}$$

Fig.1 L of P_n

We claim that each subgraph H_i is isomorphic to M_k. For example, $L = \{H_1, H_2, H_3, H_4\}$ is a collection of subgraphs of P_{17}, whose each subgraph is isomorphic to M_4, which is illustrated in Fig.2.

$$\underbrace{e_1 \; e_3 \; e_5 \; e_7}_{H_1} \quad \underbrace{e_9 \; e_{11} \; e_{13} \; e_{15}}_{H_2} \quad \underbrace{e17 \; e_2 \; e_4 \; e_6}_{H_3}$$

$$\underbrace{e_8 \; e_{10} \; e_{12} \; e_{14}}_{H_4} \quad \underbrace{e_{16} \; e_2 \; e_4 \; e_6}_{H_5}$$

Fig.2 L of P_{17}

Now we prove the above claim. Obviously $H_j (j \neq i+1, \lceil \frac{n-1}{k} \rceil)$ is isomorphic to M_k, we only need to prove H_{i+1} and $H_{\lceil \frac{n-1}{k} \rceil}$ is isomorphic to M_k, respectively. In H_{i+1}, comparing the subscript of e_{2ki+1} and $e_{2(k-(p-ki))}$,

$$2ki + 1 - 2(k - (p - ki)) = 2p - 2k + 1 \geq 2 + 1 > 2 (p > k),$$

e_{2ki+1} and $e_{2(k-(p-ki))}$ are not adjacent, and H_{i+1} has k edges, thus H_{i+1} is a copy of M_k. In $H_{\lceil \frac{n-1}{k} \rceil}$, comparing the subscript of $e_{2(q+1-t)}$ and $e_{2(k-t)}$,

$$2(q + 1 - t) - 2(k - t)) = 2(q - k) + 2 \geq 2$$

(since P_n is M_k-coverable, $q \geq k$ holds). That is, $e_{2(q+1-t)}$ and $e_{2(k-t)}$ are not adjacent, and $H_{\lceil \frac{n-1}{k} \rceil}$ has k edges, thus $H_{\lceil \frac{n-1}{k} \rceil}$ is also a copy of M_k.

From the above, we know that L is an M_k-covering of P_n with $\lceil \frac{n-1}{k} \rceil$ copies. More specifically, L is a minimal M_k-covering of P_n. This completes the proof. □

Lemma 5. *In a path P_n, if $n - 2k + 1 > \lceil \frac{n-1}{k} \rceil$, then P_n is not M_k-equicoverable.*

Proof. First we give a minimal M_k-covering of P_n, say $L = \{H_1, H_2, \cdots, H_{n-2k+1}\}$, where

$$\begin{cases} H_1 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k-1}\} \\ H_2 = \{e_2, e_4, \cdots, e_{2k-2}, e_{2k}\} \\ H_3 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k+1}\} \\ H_4 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k+2}\} \\ \cdots \\ H_{n-2k} = \{e_1, e_3, \cdots, e_{2k-3}, e_{n-2}\} \\ H_{n-2k+1} = \{e_1, e_3, \cdots, e_{2k-3}, e_{n-1}\} \end{cases}$$

By Lemma 4, we know $c(P_n; M_k)$ is $\lceil \frac{n-1}{k} \rceil$. Since $n - 2k + 1 > \lceil \frac{n-1}{k} \rceil$, L is a minimal M_k-covering of P_n which is not minimum. Thus P_n is not M_k-equicoverable. □

Lemma 6. *Let C_n be an M_k-coverable cycle, then $c(C_n; M_k) = \lceil \frac{n}{k} \rceil$.*

We omit the proof, which is similar to the proof of Lemma 4.

Lemma 7. *In a cycle C_n, if $n - 2k + 2 > \lceil \frac{n}{k} \rceil$, then C_n is not M_k-equicoverable.*

Proof. There is a minimal M_k-covering of C_n, say $L = \{H_1, H_2, \cdots, H_{n-2k+2}\}$, say

$$\begin{cases} H_1 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k-1}\} \\ H_2 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k}\} \\ H_3 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k+1}\} \\ \cdots \\ H_{n-2k+1} = \{e_1, e_3, \cdots, e_{2k-3}, e_{n-1}\} \\ H_{n-2k+2} = \{e_2, e_4, \cdots, e_{2k-2}, e_n\} \end{cases}$$

By Lemma 6, we know $c(C_n; M_k)$ is $\lceil \frac{n}{k} \rceil$. Since $n - 2k + 2 > \lceil \frac{n}{k} \rceil$, L is a minimal M_k-covering of C_n which is not minimum. Thus C_n is not M_k-equicoverable. □

Theorem 8. *A path P_n is M_k-equicoverable if and only if $n = 2k + 1$.*

Proof. For the M_k−covering of a path P_n, we have four cases.

1. $n \leq 2k$. Since P_n is not M_k-coverable, P_n is not M_k-equicoverable.

2. $n = 2k + 1$. There must be one copy $H_1 = \{e_2, e_4, \cdots, e_{2k}\}$ to cover e_2. And there also must be another copy $H_2 = \{e_1, e_3, \cdots, e_{2k-1}\}$ to cover e_{2k-1}. Then $L = \{H_1, H_2\}$ covers all edges of the path P_n, so $L = \{H_1, H_2\}$ is the unique minimal M_k-covering of P_n. The number of M_k in the minimal M_k-covering of P_n only can be 2, so P_n is M_k-equicoverable.

3. When $n = 2k+2$, it's easy to see $c(P_n; M_k)$ is 3. There exists a minimal M_k-covering with 4 copies of M_k denoted by $H = \{H_1, H_2, H_3, H_4\}$, where

$$\begin{aligned} H_1 &= \{e_{2k+1}, e_{2k-3}, e_{2k-5}, e_{2k-7}, e_{2k-9}, \cdots, e_1\}, \\ H_2 &= \{e_{2k}, e_{2k-2}, e_{2k-5}, e_{2k-7}, e_{2k-9}, \cdots, e_1\}, \\ H_3 &= \{e_{2k+1}, e_{2k-1}, e_{2k-5}, e_{2k-7}, e_{2k-9}, \cdots, e_1\}, \\ H_4 &= \{e_{2k+1}, e_{2k-2}, e_{2k-4}, e_{2k-6}, e_{2k-8}, \cdots, e_2\}. \end{aligned}$$

   By the definition, so P_n is not M_k-equicoverable.

4. When $n \geq 2k + 3$, it's easy to verify that $n - 2k + 1 > \lceil \frac{n-1}{k} \rceil$, by Lemma 5, P_n is not M_k-equicoverable, a contradiction.

From the above, a path P_n is M_k-equicoverable if and only if $n = 2k + 1$. □

Theorem 9. *A cycle C_n is M_k-equicoverable if and only if $n = 2k$ or $n = 2k + 1$.*

Proof. For the M_k-covering of a cycle C_n, we have four cases.

1. $n \leq 2k-1$. Since C_n is not M_k-coverable, C_n is not M_k-equicoverable.

2. $n = 2k$. $H_1 = \{e_2, e_4, \cdots, e_{2k}\}$ is the unique copy of M_k to cover e_2. $H_2 = \{e_1, e_3, \cdots, e_{2k-1}\}$ is the unique copy of M_k to cover e_{2k-1}. And $L = \{H_1, H_2\}$ covers all edges of the cycle C_n, so $L = \{H_1, H_2\}$ is the unique minimal M_k-covering of C_n. The number of M_k in the minimal M_k-covering of C_n only can be 2, so C_n is M_k-equicoverable.

3. $n = 2k+1$. We use induction on k to prove C_n is M_k-equicoverable. For $k = 2$, it's easy to verify that C_5 is M_2-equicoverable. For $k > 2$, we suppose that the claim is true for $k-1$. In the following, we prove the claim is also true for k.

   For any M_k-covering of C_{2k+1}, say $L = \{H_1, H_2, \cdots, H_l\}(l > 3)$, where the elements of H_i are labled in increasing order. Let H_i^* denote the set of the former $k-1$ elements of H_i. Let $L^* = \{H_1^*, H_2^*, \cdots, H_l^*\}$.

   (a) e_{2k-1} is not covered by L^*. Thus L^* is an M_{k-1}-covering of P_{2k-1} or C_{2k-2}. Whatever P_{2k-1} or C_{2k-2}, there must be one copy $H_1^* = \{e_1, e_3, \cdots, e_{2k-5}, e_{2k-3}\}$ to cover e_{2k-3}. There must be another copy $H_2^* = \{e_2, e_4, \cdots, e_{2k-4}, e_{2k-2}\}$ to cover e_2. $H_1^* \cup H_2^*$ is the unique minimal M_{k-1}-covering of P_{2k-1} or C_{2k-2}. Since e_{2k-1},e_{2k} and e_{2k+1} have not been covered, $H_1 - H_1^*$ may be e_{2k-1} or e_{2k},$H_2 - H_2^*$ may be e_{2k} or e_{2k+1}. We have the following possibilities.

      - If $H_1 - H_1^* = \{e_{2k-1}\}$ and $H_2 - H_2^* = \{e_{2k}\}$, e_{2k+1} has not been covered, there needs only one copy of M_k denoted by H_3 to cover e_{2k+1}. So $H_1 \cup H_2 \cup H_3$ is a minimal M_k-covering of C_{2k+1}. In the same way, if $H_1 - H_1^* = \{e_{2k-1}\}$ and $H_2 - H_2^* = \{e_{2k+1}\}$, or if $H_1 - H_1^* = \{e_{2k}\}$ and $H_2 - H_2^* = \{e_{2k+1}\}$, $H_1 \cup H_2 \cup H_3$ is a minimal M_k-covering of C_{2k+1}.
      - If $H_1 - H_1^* = H_2 - H_2^* = \{e_{2k}\}$, e_{2k-1} and e_{2k+1} have not been covered. Since e_{2k-1} is not covered by H^*, there must be the unique copy $H_3 = \{e_1, e_3, \cdots, e_{2k-3}, e_{2k-1}\}$ to cover e_{2k-1}. Since $H_1^* \subset H_3$, $H_1 - H_1^* \subset H_2$, $H_2 \cup H_3$ covers the edge $e_1, e_2, \cdots, e_{2k-1}, e_{2k}$. There needs only one copy of M_k denoted by H_4 to cover e_{2k+1}. Thus $H_2 \cup H_3 \cup H_4$ is a minimal M_k-covering of C_{2k+1}.

   (b) e_{2k-1} is covered by L^*. Thus L^* is an M_{k-1}-covering of C_{2k-1}. By the induction hypothesis, C_{2k-1} is M_{k-1}-equicoverable. So the number of M_{k-1} in every minimal M_{k-1}-covering of C_{2k-1} is 3. We arbitrarily select a minimal M_{k-1}-covering of C_{2k-1}

denoted by H_1^*, H_2^*, H_3^* from L^*. Suppose $e_{2k-1} \in H_3^*$, then $H_3 - H_3^* = \{e_{2k+1}\}$. Let $E = (H_1 - H_1^*) \cup (H_2 - H_2^*)$, there are two possibilities.

- If $e_{2k} \in E$, e_{2k},e_{2k-1}, e_{2k+1} and all the former edges are all covered by $H_1 \cup H_2 \cup H_3$, so $L = \{H_1, H_2, H_3\}$ is a minimal M_k-covering of C_{2k+1}.
- If $e_{2k} \notin E$, since the copy of M_k covering e_1 doesn't contain e_{2k+1}, e_{2k+1} can not belong to H_1, H_2, H_3 at the same time. Thus, we suppose $e_{2k+1} \notin H_1$, and $e_{2k} \notin H_1$, so $H_1 - H_1^* = \{e_{2k-1}\}$, then H_1 can only be $\{e_1, e_3, \cdots, e_{2k-3}, e_{2k-1}\}$. So $H_2 - H_2^* = \{e_{2k+1}\}$. Otherwise,$H_2 = H_1$. H_2^* may contain e_{2k-1} or e_{2k-2}. If H_2^* contains e_{2k-1}, then e_{2k-2} is not covered by $H_1^* \cup H_2^* \cup H_3^*$, which contracts to the fact that $H_1^* \cup H_2^* \cup H_3^*$ is an M_{k-1}-covering of C_{2k-1}. Therefore H_2^* contains e_{2k-2}. H_2 can only be $\{e_2, e_4, \cdots, e_{2k-2}, e_{2k+1}\}$. $H_1 \cup H_2$ covers the edges $e_1, e_2, \cdots, e_{2k-3}, e_{2k-2}, e_{2k-1}, e_{2k+1}$. There needs only one copy of M_k to cover e_{2k} denoted by H_4. $H_1 \cup H_2 \cup H_4$ is a minimal M_k-covering of C_{2k+1}.

4. $n \geq 2k+2$. It's easy to verify that $n - 2k + 2 > \lceil \frac{n}{k} \rceil$, by Lemma 7, C_n is not M_k-equicoverable.

From above, we can get the conclusion that the number of M_k in every minimal M_k-covering of C_{2k+1} is 3. Thus C_n is M_k-equicoverable.

So a cycle C_n is M_k-equicoverable if and only if $n = 2k$ or $n = 2k+1$. □

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