On Zero Magic Sums of Integer Magic Graphs

Chia-Ming Lin and Tao-Ming Wang*
Department of Mathematics
Tunghai University
Taichung, Taiwan, 40704

Abstract

For a positive integer k, let $\mathbb{Z}_k = (\mathbb{Z}_k, +, 0)$ be the additive group of congruences modulo k with identity 0, and $\mathbb{Z}_k$ is the usual group of integers $\mathbb{Z}$ when $k = 1$. We call a finite simple graph $G = (V(G), E(G))$ to be $\mathbb{Z}_k$-**magic** if it admits an edge labeling $\ell : E(G) \to \mathbb{Z}_k \backslash \{0\}$ such that the induced vertex sum labeling $\ell^+ : V(G) \to \mathbb{Z}_k$ defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$ is constant. The constant is called a **magic sum index**, or an **index** for short, of G under the labeling ℓ, which follows R. Stanley. The **null set** of G, which is defined by E. Salehi as the set of all k such that G is $\mathbb{Z}_k$-magic with zero magic sum index, and is denoted by Null(G). For fix integer k, we consider the set of all possible magic sum indices r such that G is $\mathbb{Z}_k$-magic with a magic sum index r, and denote it by $I_k(G)$. We call $I_k(G)$ the **index set** of G with respect to $\mathbb{Z}_k$. In this paper, we investigate the properties and relations of the index sets $I_k(G)$ and the null sets Null(G) for $\mathbb{Z}_k$-magic graphs. Among others, we determine the null sets of generalized wheels and generalized fans, and also construct infinitely many examples of $\mathbb{Z}_k$-magic graphs with magic sum zero. Some open problems are presented.

Keywords: $\mathbb{Z}_k$-magic, magic sum index, null set, index set.

AMS Subject Classification: 05C15, 05C78

---

*Corresponding author with contact address wang@thu.edu.tw. Research supported partially by the National Science Council under Grant No. NSC 97-2115-M-029-002-MY2

1 Introduction and Terminology

For any additive abelian group A, let $A^* = A - \{0\}$ where 0 is the additive identity element. Given a graph G, any mapping $\ell : E(G) \to A^*$ is called an edge labeling of G. A graph G is said to be ***A-magic*** if there exists an edge labeling such that the induced vertex labeling $\ell^+ : V(G) \to A$ defined by

$$\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$$

is a constant map. We call the constant a ***magic sum index*** of G, an ***index*** for short, and $I_A(G) = \{r : G \text{ is } A\text{-magic with an index } r\}$ the ***index set*** of G with respect to A. In this article we focus on $A = \mathbb{Z}_k$ and denote $I_A(G)$ by $I_k(G)$. The notion of index sets is first introduced and studied by C-M Lin and T-M Wang in [3]. A related notion is the ***null set*** of G, which is defined as the set of all k such that G is $\mathbb{Z}_k$-magic with index 0. The problems related to the null sets was studied by E. Salehi in [13, 12]. E. Salehi also studied a particular class of graphs called ***uniformly null***, which is defined as the graphs G with the property that, if G is $\mathbb{Z}_k$-magic, then the magic sum is zero only. He identified the class of complete bipartite graphs $K_{n,n+1}$ to be uniformly null in [13].

In general, a graph may admit more than one edge labeling to become an A-magic graph. At present, no generally efficient algorithm is known for finding magic labeling for general graphs. It is well-known [2, 11, 20] that a graph G is $\mathbb{N}$-magic if and only if every edge of G is contained in a $\{1,2\}$-factor. For a list of properties of $\mathbb{N}$-magic graphs, see [1, 4, 6, 18, 19]. Stanley studied $\mathbb{Z}$-magic graphs in [16, 17]; he demonstrated that magic labelings can be found by solving a system of linear diophantine equations. Being $\mathbb{Z}$-magic is a weaker condition than being $\mathbb{N}$-magic. Given a graph G, the set of all k such that G is $\mathbb{Z}_k$-magic is defined as the ***integer-magic spectrum*** of G, and is denoted by IM(G). The integer-magic spectra of some families of graphs can be found in [5, 7, 8, 9]. The concepts of the index sets, the null sets, and the integer magic spectra of graphs are closely related. Note that the case of $\mathbb{Z}_2$-magicness is easy to settle. Since every edge must be labeled with 1, the magic sum is the degree of any vertex modulo 2. Therefore the degrees of the vertices must have the same parity. This leads to the following result.

Lemma 1.1 *A graph G is $\mathbb{Z}_2$-magic if and only if its degrees are all even or all odd.*

However the discussion of $\mathbb{Z}_2$-magic graphs is completely different from that of $\mathbb{Z}_k$-magic graphs for $k \geq 3$ or $k = 1$. It is quite challenging to obtain similar characterizations of $\mathbb{Z}_k$-magic graphs for $k \geq 3$ or $k = 1$.

In [13], E. Salehi introduced the null set of a graph, and obtained some interesting results regarding various classes of graphs. Formally we define the null set of a graph as follows.

Definition 1.2 *The null set of a graph G is the set of all possible positive integers k, such that G has a zero magic sum index under a $\mathbb{Z}_k$-magic edge labeling, and is denoted by* Null(G).

We define in [3] a related and more general notion index sets as follows:

Definition 1.3 *For a graph G, we define the set of all magic sum indices r such that G is $\mathbb{Z}_k$-magic with magic sum index r to be the index set of G with respect to $\mathbb{Z}_k$, and denote such a set by $I_k(G)$. That is, $I_k(G) = \{r : G \text{ is } \mathbb{Z}_k\text{-magic with magic sum index } r\}$.*

Remark. In terms of the above definition, we may see that $\text{Null}(G) = \{k : 0 \in I_k(G)\}$. Therefore if enough information of the index set $I_k(G)$ is provided, then one may completely determine the null set of G.

We have the following basic observations over the index sets in [3]:

Theorem 1.4 *Let $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ be the edge disjoint union of spanning subgraphs $G_1, G_2, \cdots$ and G_m. Suppose for fix k the graphs G_i is $\mathbb{Z}_k$-magic with index r_i for $i = 1, \cdots, k$. Then we have:*

(1) $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ *is $\mathbb{Z}_k$-magic with an index* $\sum_{i=1}^{m} r_i$.
(2) $I_k(G_1) + I_k(G_2) + \cdots\cdots + I_k(G_m) \subseteq I_k(G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m)$, *and in particular,* $I_k(G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m) = \mathbb{Z}_k$ *if* $I_k(G_i) = \mathbb{Z}_k$ *for some i.*
(3) Let nG = the vertex disjoint union of n copies of G. Then nG has the same index set as G, that is, $I_k(nG) = I_k(G)$.
(4) Let G and H be any two graphs, and both are $\mathbb{Z}_k$-magic with indices r_1 and r_2, respectively. Then the Cartesian product $G \times H$ has an index $r_1 + r_2$, and the index set $I_k(G \times H) = \mathbb{Z}_k$ whenever $I_k(G)$ or $I_k(H) = \mathbb{Z}_k$.
(5) Let G and H be any two graphs, and H is $\mathbb{Z}_k$-magic. Then the index set of their lexicographic product is $I_k(G \circ H) = \mathbb{Z}_k$.

Remark. In case the graphs $G_1, G_2, \cdots$ and G_m are $\mathbb{Z}_k$-magic with indices 0 for each G_i, where $i = 1, \cdots, k$. Then we have the resulting arbitrary union graph $\bigcup G_i$ by attaching these m graphs in any "edge disjoint" way is still $\mathbb{Z}_k$-magic with an index 0.

In general, the index set may not be the full $\mathbb{Z}_k$. We have the following fact for index sets of cycles in [3]:

Proposition 1.5 *Let C_n be an n-cycle, where $n \geq 3$, and k be a positive integer. We have the following:*
(1) $I_k(C_n) = 2\mathbb{Z}_k^* = \{2x : x \neq 0,\ x \in \mathbb{Z}_k\}$, for n odd.
(2) $I_k(C_n) = \mathbb{Z}_k$, for n even.

Note that we may have many examples of regular graphs with index sets $\mathbb{Z}_k$. More generally, we have the following theorem for index sets of regular graphs admitting a 1-factor in [3].

Theorem 1.6 *Let G be a r-regular graph ($r \geq 2$) which admits a 1-factor, then*

$$I_k(G) = \begin{cases} \mathbb{Z}_k, & k = 1,\ k \geq 3. \\ \mathbb{Z}_2 - \{0\}, & k = 2,\ r \text{ odd}. \\ \mathbb{Z}_2 - \{1\}, & k = 2,\ r \text{ even}. \end{cases}$$

In later sections, we study the index sets, the null sets of various classes of graphs, and relations between them. Among others we determine the null sets of generalized wheels and generalized fans and also construct infinitely many examples of uniformly null $\mathbb{Z}_k$-magic graphs, and mention some applications to the calculation of $\mathbb{Z}_k$-magicness. Some open problems are presented in the concluding remarks.

2 Null Sets of Generalized Fans and Generalized Wheels

In this section, the null sets of windmills, fans, wheels, and their variants and generalizations are discussed and determined completely.

Note that at first for any $\mathbb{Z}_k$-magic labeling f of a graph G with index r, we have the following equation by summing all vertex sums:

$$2 \sum_{e \in E(G)} f(e) \equiv r \cdot |V(G)| \pmod{k}.$$

Hence we have $2\sum_{e\in E(G)} f(e) \equiv 0 \pmod{k}$ for any magic labeling with index 0, and also note that the sum of labels for all the incident edges with one single vertex is 0, therefore we have the following Lemma:

Lemma 2.1 *Let f be a $\mathbb{Z}_k$-magic labeling of G with an index 0, v be a vertex of G, and $G' = G - \{v\}$. Then we have:*

$$\sum_{e \in E(G)} f(e) = \begin{cases} \frac{k}{2} \text{ or } 0 \pmod{k}, & \text{for } k \text{ even}, \\ 0 \pmod{k}, & \text{for } k \text{ odd}. \end{cases}$$

and

$$\sum_{e\in E(G')} f(e) = \begin{cases} \frac{k}{2} \text{ or } 0 \pmod{k}, & \text{for } k \text{ even}, \\ 0 \pmod{k}, & \text{for } k \text{ odd}. \end{cases}$$

We will determine the null sets of generalized fan graphs, the null sets of generalized wheel graphs, and the null sets of generalized windmill graphs completely in this section.

Definition 2.2 *A* **fan** *graph $F_n = \{v\} + P_n$ is formed by adding a vertex v to the vertex set of P_n and joining this vertex to every vertex of P_n, and a* **wheel** *graph $W_n = \{v\} + C_n$ is formed by adding a vertex v to the vertex set of C_n and joining this vertex to every vertex of C_n, for $n \geq 3$. A* **generalized fan** *graph is formed by joining one vertex to each vertex of a disjoint union of paths, and a* **generalized wheel** *graph is formed by joining one vertex to each vertex of a disjoint union of cycles. We call the vertex v the* **center**, *and the edges connecting the center v and vertices of paths P_n or cycles C_n* **spokes**.

Therefore, by the above Lemma 2.1, we observe that for the zero-sum $\mathbb{Z}_k$-magic labeling of $\{v\} + H$ (In particular $H = W_n$, $H = F_n$, or $H =$ disjoint union of paths or cycles), the sum of edge labels on H (that is cycles C_n, and paths P_n respectively) must be 0 or $\frac{k}{2}$ modulo an even k, and also the induced vertex sum of every vertex on H must be non-zero since the edge labels on spokes are non-zero.

Remark. We observe that fans F_n and wheels W_n for $n \geq 3$ have no $\mathbb{Z}_2$-magic labeling with index 0. Therefore $2 \notin \text{Null}(F_n)$ and $2 \notin \text{Null}(W_n)$.

Note that the null sets of wheels and fans have already been determined and presented at the 40th Southeastern International Conference on Combinatorics, Graph Theory and Computing, March 2-6, 2009(Please see the abstract of the talk "Zero-Sum Magic and Null Sets of Planar Graphs", E. Salehi and S. Hansen, University of Nevada Las Vegas). We work independently regarding this subject, and obtain the null sets of fans, wheels, and other graphs, respectively. Therefore we omit our proof and just put the result for the null set of the fans F_n for $n \geq 2$ in the following for reference. Note that the proof is based upon the induction and a method of subdivision.

Theorem 2.3 *For $n \geq 2$,*

$$\text{Null}(F_n) = \begin{cases} 2\mathbb{N}, & n = 2, \\ 2\mathbb{N}\backslash\{2\}, & n = 3, \\ \mathbb{N}\backslash\{2\}, & n > 3,\ n \equiv 1\ (mod\ 3), \\ \mathbb{N}\backslash\{2,3\}, & n > 3,\ n \equiv 0,2\ (mod\ 3). \end{cases}$$

More precisely, we define the generalized fans as follows.

Definition 2.4 *A generalized fan* $F(n_1, n_2, \cdots, n_m) = \{v\} + (P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_m})$, *where* P_{n_i} *are disjoint paths on* $n_i \geq 2$ *vertices, for* $i = 1, \cdots, m$ *and* $m \geq 2$.

We first deal with the $\mathbb{Z}_3$ case for a generalized fan $F(n_1, n_2, \cdots, n_m)$ where $n_i \geq 2$ vertices, for $i = 1, \cdots, m$:

Lemma 2.5 *For* $n_i \geq 2$, $\forall i = 1, \cdots, m$, *let* $S = \{n_i : n_i \equiv 0 \text{ or } 2\ (mod\ 3)\}$. *Then we have that* $0 \in I_3(F(n_1, n_2, \cdots, n_m))$ *if and only if* $|S| \neq 1$.

Proof. Suppose $0 \in I_3(F(n_1, n_2, \cdots, n_m))$. By Lemma 2.1, we note that every label restricted on each path is either 1 or -1 for any zero sum $\mathbb{Z}_3$-magic labeling of the generalized fan. If we label all 1s on some path P_{n_i}, then there are $(n_i - 2)$ 1's and two (-1)'s on the spokes incident with each vertex of such path. Therefore, the partial vertex sum of the center with respect to P_{n_i} is $n_i - 4 \equiv n_i - 1$ (mod 3). On the other hand, if we choose to label all (-1)'s on the path P_{n_i}, the partial vertex sum of the center with respect to P_{n_i} is $4 - n_i \equiv 1 - n_i$ (mod 3).

Let the total vertex sum of the center be T, we have the following four cases:

Case 1: $|S| = 1$.

Then T is never zero (mod 3).

Case 2: $|S| = 3k,\ k \in \mathbb{N} \cup \{0\}$.

Label 1 on all the edges, then $T \equiv 3k \equiv 0$ (mod 3).

Case 3: $|S| = 3k + 1,\ k \in \mathbb{N}$.

Label two fans such that the partial vertex sums of the center with respect to them are -1, and $3k - 1$ fans such that the partial vertex sums with respect to them are 1, then $T \equiv (3k - 1) - 2 \equiv 0$ (mod 3).

Case 4: $|S| = 3k + 2, k \in \mathbb{N}$.

Label one fans such that the partial vertex sum of the center is -1, and $3k + 1$ fans such that the partial vertex sums of the center with respect to them is 1, then $T \equiv (3k + 1) - 1 \equiv 0$ (mod 3).

The converse is clear from the given labeling. □

Note that we may view $F(n_1, n_2, \cdots, n_m)$ as the one vertex union of F_{n_i}, for $i = 1, \cdots, m$, and write it as $F(n_1, n_2, \cdots, n_m) = F_{n_1} \odot F_{n_2} \cdots \odot F_{n_m}$.

Since $N(F_n) \supset N\backslash\{2,3\}$, for all $n \geq 4$, that is F_n $(n \geq 4)$ is $\mathbb{Z}_k$-magic with index 0 for all $k \geq 4$, therefore if we can show that $F(n_1, n_2, \cdots, n_m)$ with certain $n_i \leq 3$ admits a $\mathbb{Z}_k$-magic labeling with zero sum, for all $k \geq 4$, then the null set of $F(n_1, n_2, \cdots, n_m)$ is completely determined. We proceed with the following steps.

Lemma 2.6 *The double fans* $F(2,m)$ *and* $F(3,m)$ *admit a* $\mathbb{Z}_k$*-magic labeling with* 0 *index, for all* $m \geq 4$ *and* $k \geq 4$.

Proof. Clearly, $F(2,4)$, $F(2,5)$, $F(3,4)$, and $F(3,5)$ admit $\mathbb{Z}_k$-magic labeling with 0 index, for all $k \geq 4$, as shown in the Figure 1 and Figure 2.

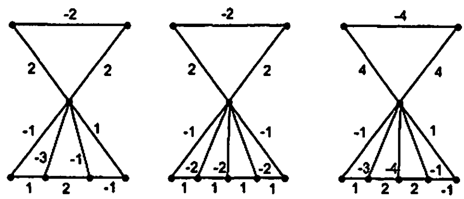

Figure 1: $F(2,4)$ and $F(2,5)$

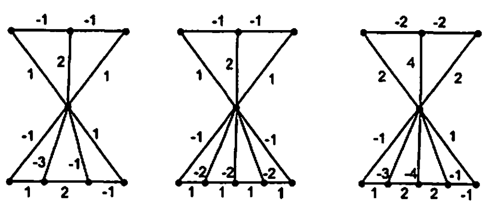

Figure 2: $F(3,4)$ and $F(3,5)$

Note that the given labeling has 1-edge and (-1)-edge over the F_4 and F_5 sides, then by inserting spokes labeled 2 and -2, we may get a $\mathbb{Z}_k$-magic labeling with 0 index for general $F(2, m+2)$ from $F(2,m)$, and get $F(3, m+2)$ from $F(3,m)$, respectively, for all $m \geq 4$ by induction. Please see the Figure 3 for the above mentioned method, which is also used in our proof in obtaining the null sets of fans. □

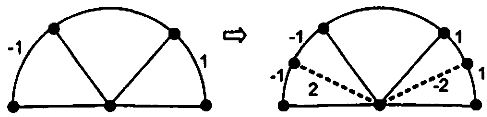

Figure 3: $F(*, m)$ to $F(*, m+2)$: a method of subdivision

Corollary 2.7 *For $k \geq 4$ and $m \geq 2$, suppose $\{n_1, n_2, \cdots, n_m\}$ contain only one 2 or only one 3. Then $F(n_1, n_2, \cdots, n_m)$ admits a $\mathbb{Z}_k$-magic labeling with zero sum index.*

Proof. If 2 (or 3) appears once in $\{n_1, n_2, \cdots, n_m\}$, then $F(n_1, n_2, \cdots, n_m)$ is a vertex union of $F(2, t)$ (or $F(3, t)$) for $t \geq 4$ and other fans F_j with $j \geq 4$. □

Theorem 2.8 *The generalized fan $F(n_1, n_2, \cdots, n_m)$, $n_i \geq 2$, $\forall i = 1, \cdots, m$ and $m \geq 2$, admits a $\mathbb{Z}_k$-magic labeling with zero index, for all $k \geq 4$.*

Proof. For convenience and without loss of generality, we may express the generalized fan as the one vertex union of a copies of F, b copies of F_2, and c copies of F_3, where F is an a vertex union of fans F_j, $j \geq 4$. That is, we assume that $F(n_1, n_2, \cdots, n_m) = aF \odot bF_2 \odot cF_3$ where b, c non-negative and a is 0 or 1. Note that F_j, $j \geq 4$ admits $\mathbb{Z}_k$-magic labeling with zero sum for $k \geq 4$. Then we have the following cases:

Case 1. b, c are both even.

Note that this case can be reduced to $F(2,2)$ and $F(3,3)$. For $F(2,2)$, it admits a $\mathbb{Z}_k$-magic labeling with zero index, for $k \geq 4$, since it is Eulerian of even size. For $F(3,3)$, it admits a $\mathbb{Z}_k$-magic labeling with zero index, for $k \geq 4$, see the Figure 4. Therefore, $bF_2 \odot cF_3 = (\frac{b}{2}F_2 \odot \frac{b}{2}F_2) \odot (\frac{c}{2}F_3 \odot \frac{c}{2}F_3) = \frac{b}{2}F(2,2) \odot \frac{c}{2}F(3,3)$ admits a $\mathbb{Z}_k$-magic labeling with zero index, $k \geq 4$.

Case 2. b even, c odd.

In case $b \geq 2$ it reduces to the case $F(3,2,2)$, see the Figure 5. In case $b = 0$ and $c = 1$, since $a \neq 0$, that is $F \neq \phi$, it reduces to the case $F(3,p)$, $p \geq 4$. In case $b = 0$ and $c \geq 3$ odd, it reduces to the case $F(3,3,3)$, see Figure 6.

Case 3. b odd, c even.

In case $b = 1$, $c = 0$. Then $a \neq 0$, that is $F \neq \phi$, it reduces to the case $F(2,p)$, $p \geq 4$. In case $b \geq 3$ odd, $c = 0$, it reduces to the case $F(2,2,2)$, see Figure 6. In case $c \geq 2$, it reduces to the case $F(2,3,3)$, see the Figure 5.

Case 4. b, c are both odd.

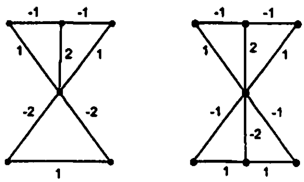

Figure 4: $F(2,3)$ and $F(3,3)$

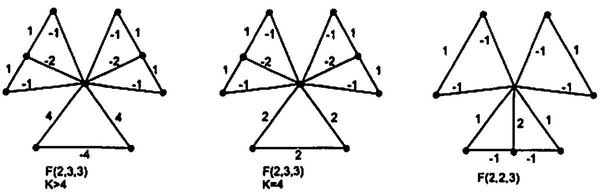

Figure 5: $F(2,3,3)$ and $F(2,2,3)$

In this case it reduces to the case $F(2,3)$, see the Figure 4.

□

Summarizing all up,

Theorem 2.9 *Let $F(n_1, n_2, \cdots, n_m)$ be a generalized fan, where $n_i \geq 2$, for $i = 1, \cdots, m$ and $m \geq 2$. The null set is*

$$\text{Null}(F(n_1, n_2, \cdots, n_m)) = \begin{cases} \mathbb{N}, & \textit{if } n_1 = n_2 = \cdots = n_m = 2, \\ \mathbb{N} \backslash \{2, 3\}, & \exists \textit{ unique } n_i \equiv 0 \textit{ or } 2 \ (\textit{mod } 3), \\ \mathbb{N} \backslash \{2\}, & \textit{otherwise.} \end{cases}$$

Remark. In particular, we have shown in the above Theorem, assuming $n_1 = n_2 = \cdots = n_m = 2$, that the **windmill graphs** (see the Figure 7) $WM_n = F(2, 2, \cdots, 2) = \{v\} + nP_2$, $n \geq 2$, admits $\mathbb{Z}_k$-magic zero sum labeling for all $k \in \mathbb{N}$, that is, the null set $\text{Null}(WM_n) = \mathbb{N}$, $n \geq 2$.

Now we proceed to determine the null sets of generalized wheel graphs. First we put the results here without proof for the null set of the wheel graphs W_n, $n \geq 3$, for reference. Note that the proof is also based upon the

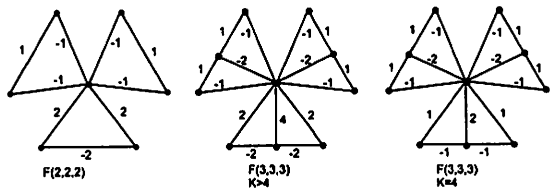

Figure 6: $F(2,2,2)$ and $F(3,3,3)$

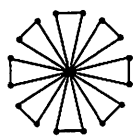

Figure 7: Windmill WM_n

induction and a method of subdivision as in the case of fans and generalized fans.

Theorem 2.10 *For $n \geq 3$,*

$$\text{Null}(W_n) = \begin{cases} \mathbb{N}\backslash\{2\}, & n \equiv 0 \ (mod\ 3), \\ \mathbb{N}\backslash\{2,3\}, & n \equiv 1,2 \ (mod\ 3). \end{cases}$$

To be more precise about the generalized wheel graphs, we define as follows:

Definition 2.11 *A generalized wheel graph $W(n_1, n_2, \cdots, n_m) = \{v\} + (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_m})$, where C_{n_i} are disjoint cycles on $n_i \geq 3$ vertices, for $i = 1, \cdots, m$ and $m \geq 2$.*

Similar to the situation in the $\mathbb{Z}_3$-magic case of generalized fans, we have the following Lemma for the generalized wheels $W(n_1, n_2, \cdots, n_m)$. The proof is straightforward and similar to the one in Lemma 2.5, hence it is left to the reader.

Lemma 2.12 *Let $n_i \geq 3$, $\forall i = 1, \cdots, m$, and $S = \{n_i : n_i \equiv 1 \text{ or } 2 \ (mod\, 3)\}$. Then we have that $0 \in I_3(W(n_1, n_2, \cdots, n_m))$ if and only if $|S| \neq 1$.*

Theorem 2.13 *Let $W(n_1, n_2, \cdots, n_m)$ be a generalized wheel, where $n_i \geq 3$ for $i = 1, \cdots, m$, and $m \geq 2$. The null set is*

$$\text{Null}(W(n_1, n_2, \cdots, n_m)) = \begin{cases} \mathbb{N}\backslash\{2,3\}, & \exists \text{ unique } n_i \equiv 1 \text{ or } 2 \pmod 3, \\ \mathbb{N}\backslash\{2\}, & \text{otherwise.} \end{cases}$$

Proof. Directly from Lemma 2.12 and Theorem 2.10. □

3 New Classes of Uniformly Null Graphs

In this section we study another class of graphs related to the null sets. A graph is **uniformly null** if every $\mathbb{Z}_k$-magic labeling induces 0 magic sum index, which was studied by E. Salehi in [13, 12]. Note that this definition implies all non-magic (that is non-$\mathbb{Z}_k$-magic for all k) graphs are uniformly null in Salehi's sense. He identified the class of complete bipartite graphs $K_{n,n+1}$ to be uniformly null.

Definition 3.1 *We call G an* **almost equi-bipartite graph** *if G is a bipartite graph (without isolated vertices) with bipartition (X, Y) and $||X| - |Y|| = 1$.*

We have the following observation for the index sets of almost equi-bipartite graphs:

Proposition 3.2 *Let G be an almost equi-bipartite graph with bipartition (X, Y), and $||X| - |Y|| = 1$. If G is $\mathbb{Z}_k$-magic, then it is uniformly null, that is, $I_k(G) = \{0\}$, $\forall k \geq 3$.*

Proof. Suppose G admits a $\mathbb{Z}_k$-magic labeling f with index r, and $|X| = m$, $|Y| = m + 1$. By adding all the vertex sums in X, and adding all the vertex sums in Y respectively, we have

$$mr = \sum_{e \in E(G)} f(e) = (m+1)r,$$

which implies $r \equiv 0 \pmod k$. Thus the proof is complete.

□

Remark. If G is an almost equi-bipartite graph, and if moreover G is an even graph (that is in G every vertex is of even degree), then G is $\mathbb{Z}_k$-magic with an index 0 since it is a disjoint union of Eulerian graphs of even size. However conversely we have examples of uniformly null graphs which are not even graphs, namely, the complete almost equi-bipartite graphs $K_{n,n+1}$ for $n \geq 3$, as E. Salehi pointed out in [13].

3.1 C_4-Construction of Almost Equi-bipartite Graphs

The following is a construction with C_4 of an infinite family of almost equi-bipartite graphs G whose degrees are all even and for which $I_k(G) = \{0\}$, $\forall k \geq 3$. In fact, $I_k(G) = \{0\}$, for $k = 1, 2$ as well.

Note that since almost equi-bipartite graphs G with degree one vertices are not $\mathbb{Z}_k$-magic with zero sum, the minimum degree $\delta(G) \geq 2$ and hence G contains cycles and only even cycles. Therefore, we may have that the order $|V(G)| \geq 7$ and is odd. So the minimum order of such graphs is 7 as shown in the graph of Figure 8, which is isomorphic to the dumbbell graph $D(4,4)$, one vertex union of two four-cycles. We denote it by B_7, and clearly $I_k(B_7) = 0$, $\forall k \geq 3$. Let $B_7 \in \beta_7$ be the first family of almost equi-bipartite graphs with the least order 7 and B_7 is the one with the least number of edges.

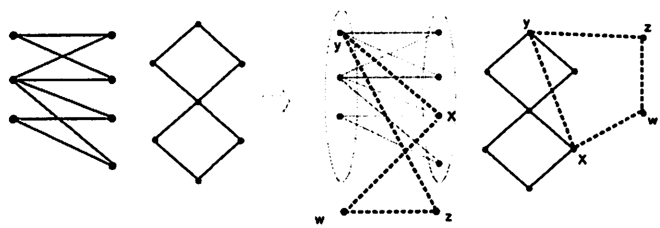

Figure 8: Construction from $B_7 \cong D(4,4)$ to $B_9 \in \beta_9$

Construct families β_{n+2} from β_n using the following steps to obtain $B_{n+2} \in \beta_{n+2}$ from $B_n \in \beta_n$ for $n \geq 1$:

Step 1. Choose vertices x, y in $B_n \in \beta_n$ such that their distance $d(x,y)$ is odd and strictly greater than 1, that is $d(x,y) \in \{3, 5, 7, \cdots\}$. Hence x, y are in different partite sets and non-adjacent.

Step 2. Then we add two new vertices w, z such that x, z and y, w are in the same partite sets, respectively, and join the edges to get xy, xw, yz, wz to create a graph $B_{n+2} \in \beta_{n+2}$.

In such a way we attach a C_4 to the chosen vertices x, y to create new graphs $B_{n+2} \in \beta_{n+2}$. Note that the new graphs are still Eulerian of even size and, in fact, are of the smallest size in β_{n+2} if the construction starts from $B_7 \cong D(4,4)$. Therefore, we obtain an infinitely family β of uniformly null almost equi-bipartite graphs via the above constructions.

3.2 One Point Union Construction of Almost Equibipartite Graphs

We have another infinite set of examples of uniformly null graphs, which is constructed by attaching even cycles in a particular way. The construction is as follows.

Step 1. Pick a path of even length, and make each edge to be two parallel edges between every pair of adjacent vertices.

Step 2. Insert even number (including none) of vertices of degree 2 in each edge so that the resulting graph is simple and a one point union of even cycles.

Then it is routine to check that the resulting graph is an almost equibipartite graph, therefore it is a uniformly null graph by the above Proposition 3.2. Please see Figure 9.

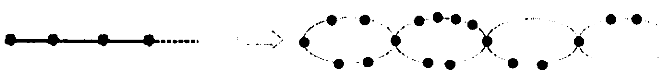

Figure 9: One Point Union of Even Cycles

3.3 Application to Integer Magic Spectrum of Corona Product

Let G and H be two graphs, where $|V(G)| = n$. Take one copy of G and n copies of H, for each i from 1 to n, join the ith vertex of G to each vertex in the ith copy of H. The resulting graph is called the ***corona product*** of G with H, which we shall denote $G©H$. On the other hand, given a graph G, the set of all k such that G is $\mathbb{Z}_k$-magic is defined as the ***integer-magic spectrum*** of G, and is denoted by $\mathrm{IM}(G)$. Please see [5, 7, 8, 14, 15].

We obtain the following criterion to get the integer magic spectra of the corona $G©N_m$ using the information of index sets of G in [3], where N_m is the null graph (empty graph) with m isolated points.

Proposition 3.3 *For fixed m and $k \geq 2$, let $d = gcd(k, 1-m)$. We analyze $\mathrm{IM}(G©N_m)$ in the following cases:*

Case 1. *$d > 1$, then $k \in \mathrm{IM}(G©N_m)$ if and only if $d|r_i$, for some $r_i \in I_k(G)$.*

Case 2. $d = 1$, *then* $k \in \mathrm{IM}(G©N_m)$ *if and only if* $I_k(G)$ *has an non-zero element.*

Therefore,

1. If $I_k(G)$ contains both 0 and another non-zero element, then $k \in \mathrm{IM}(G©N_m)$.
2. In particular if $0 \in I_k(G)$, then $k \in \mathrm{IM}(G©N_m)$.
3. Moreover, for all non-negative integer n, $G©N_{nk+1}$ is $\mathbb{Z}_k$-magic if and only if $0 \in I_k(G)$.

Hence we have the following observation for calculating the integer magic spectra of the corona products of uniformly null graphs with null graphs:

Proposition 3.4 *If* $I_k(G) = \{0\}$ *for all* $k \geq 3$, *then the integer magic spectrum*

$$\mathrm{IM}(G©N_m) = \{k : \ gcd(1-m, k) > 1\} = \bigcup_{i=1}^{t} p_i\mathbb{N},$$

where $m - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ *the prime divisor decomposition.*

Therefore we have obtained the integer magic spectra of infinitely many examples of corona products of G with null graphs N_m by the above Proposition 3.4, where G could be any of previously constructed uniformly null graphs, say $K_{n,n+1}$, graphs in the family β by C_4-construction, or graphs constructed by one point union of even cycles.

4 Concluding Remarks

Note that we have obtained the null sets of generalized wheels and generalized fans in this article. Also we have created infinitely many examples of uniformly null graphs using the different even cycle constructions. Therefore in particular we answer the open problems posted by E. Salehi in [13], which are finding the null sets of wheels and fans, and identifying families of uniformly null graphs other than the complete bipartite graphs $K_{n,n+1}$.

We conclude this paper by posting the following open problems left out of the discussion over these related topics:

1. Determine the index sets of the generalized fans and the generalized wheels.

2. Characterize the class of graphs G for which $I_k(G) = \{0\}$, $\forall k \geq 3$.
3. Characterize the class of almost equi-bipartite graphs G which are uniformly null.

Acknowledgment

We would like to thank one of the referees' reminder for the previously presented results of the null sets of wheels and fans which we were not aware of while working on the topic. We also thank for referees' comments and suggestions which result in a lot of improvements for the organization of the paper.

References

[1] R.H. Jeurissen, Disconnected graphs with magic labelings, *Discrete Math.* **43** (1983), 47-53.

[2] S. Jezny and M. Trenkler, Characterization of magic graphs, *Czechoslovak Math. J.* **33** **(108)** (1983), 435-438.

[3] C.M. Lin and T.M. Wang, Magic Sums of Integer Magic Graphs, Submitted.

[4] M.C. Kong, S.M. Lee and H. Sun, On magic strength of graphs, *Ars Combin.* **45** (1997), 193-200.

[5] S.M. Lee, A.N.T. Lee, H. Sun and I. Wen, On the integer-magic spectra of graphs, *J. Combin. Math. Combin. Comput.* **42** (2002), 77-86.

[6] S.M. Lee, F. Saba and G.C. Sun, Magic strength of the k-th power of paths, *Congr. Numer.* **92** (1993), 177-184.

[7] S.M. Lee, and E.Salehi, On integer-magic spectra of amalgamation of stars and cycles, *Ars Combin.* **97** (2003), 199-212.

[8] S.M. Lee, E.Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, *J. Combin. Math. Combin. Comput.* **50** (2004), 3-15.

[9] S.M. Lee and H. Wong, On the integer-magic spectra or the power of paths, The *J. Combin. Math. Combin. Comput.* **42** (2002),187-194.

[10] S.M. Lee, L. Valdes and Y.S. Ho, On group-magic of trees, double trees and abbreviated double trees, *J. Combin. Math. Combin. Comput.* **46** (2003), 85-95.

[11] L. Sadorovand M. Trenkler, On a generalization of magic graphs, in *Combinatorics (Eger 1987)*, pp. 447–452, *Colloquia Mathematica Societatis Janos Bolyai* **52**, North-Holland, Amsterdam, 1988.

[12] E. Salehi, On Zero-Sum Magic Graphs and Their Null Sets, Bulletin of the Institute of Mathematics, Academia Sinica 3 (2008), 255-264.

[13] E. Salehi, Zero-Sum Magic Graphs and Their Null Sets, Ars Combinatoria 82 (2007), 41-53.

[14] E. Salehi, Integer-Magic Spectra of Cycle Related Graphs, Iranian Journal of Mathematical Sciences and Informatics 2 (2006), 53-63.

[15] E. Salehi, On Integer-Magic Spectra of Caterpillars, Journal of Combinatorial Mathematics and Combinatorial Computing 61 (2007) 65-71.

[16] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632.

[17] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.* **43** (1976), 511-531.

[18] B.M. Stewart, Magic graphs, *Canad. J. Math.* **18** (1966), 1031–1059.

[19] B.M. Stewart, Supermagic complete graphs, *Canad. J. Math.* **19** (1967), 427-438.

[20] M. Trenkler, Some results on magic graphs, in *Graphs and Other Combinatorial Topics (Prague, 1982)*, pp. 328–332, edited by M. Fieldler, *Teubner-Texte Math.* **59**, Teubner, Leipzig, 1983.